REGULAR CROSS SECTIONS OF BOREL FLOWS

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ABSTRACT. Any free Borel flow is shown to admit a cross section with only two possible distances between adjacent points, answering an old question of Nadkarni. As an application of this result, we derive a short proof of the classification of Borel flows up to Lebesgue orbit equivalence.

1. INTRODUCTION

This paper is a contribution to the theory of Borel flows, that is Borel actions of the real line on a standard Borel space. An important tool in studying Borel flows is the concept of a cross section of the action. A cross section for a flow $\mathbb{R} \curvearrowright X$ is a Borel set $\mathcal{C} \subseteq X$ which intersects every orbit in a non-empty countable set.

A "sufficiently nice"[1] cross section $\mathcal{C}$ of a flow can be endowed with an induced automorphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which sends a point in $\mathcal{C}$ to the next one within the same orbit. This gives a representation of the flow $\mathbb{R} \curvearrowright X$ as a flow under a function depicted in Figure 1 (We refer the reader to Subsection 2.2 for the formal treatment). It has been known since the work of V. M. Wagh [Wag88] that any Borel flow admits a cross section. The representation as in Figure 1 has two parameters — the "gap" function f, and the base automorphism $\phi_{\mathcal{C}}$. Since there are many ways of presenting a given flow as such under a function, one naturally wants to understand the flexibility of these parameters. We concentrate here on the gap function. Given a flow $\mathbb{R} \curvearrowright X$ we aim at finding the simplest function f such that $\mathbb{R} \curvearrowright X$ can be represented as a flow under f. But before we state our main results, let us make a short detour and describe this problem from the point of view of ergodic theory.

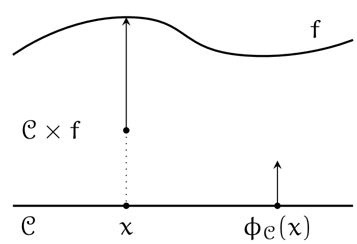

FIG. 1.

Borel dynamics and ergodic theory are mathematical siblings — they study dynamical systems from similar perspectives and share a large portion of methods. Yet they also have some fundamental differences. A slightly non-standard (but equivalent to the classical) way of setting up an ergodic theoretical system is to consider a Borel action of a locally compact group[2] G on a standard Borel space X which is moreover equipped with a probability measure μ. We assume that the action $G \curvearrowright X$ preserves μ. So one difference from Borel dynamics is that we consider only actions which admit an invariant measure. Much more importantly, we disregard sets of measure zero, in the sense that we are satisfied if the result holds true *almost* everywhere with respect to μ, and may throw away a set of points of zero measure if necessary. The latter is a significant luxury, which is not available in the context of Borel dynamics. We shall elaborate on this below, but first let us describe the relevant theorems that have been proved in ergodic theory.

1.1. **Known results.** The idea of constructing cross sections and reducing the analysis of the flow to the analysis of the induced automorphism goes back to H. Poincaré, and has been very useful in many areas of dynamical systems. In the generality of measure preserving flows on standard Lebesgue spaces, existence of cross sections was proved by W. Ambrose and S. Kakutani [Amb41, AK42].

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[1]Meaning that each point should have a successor and a predecessor.

[2]The relevant case here is $G = \mathbb{R}$.

Theorem (Ambrose–Kakutani). *Any measure preserving flow $\mathbb{R} \curvearrowright X$ on a standard Lebesgue space admits a cross section on an invariant subset of full measure.*

Ambrose also established a criterion for a flow to admit a cross section with constant gaps, i.e., to admit a representation under a constant function. Flows admitting such cross sections turn out to be very special, and a typical flow is not a flow under a constant function. It came as a surprise that, as proved by D. Rudolph [Rud76], any free flow can be represented, at least as far as an ergodic theorist is concerned, as a flow under a two-valued function.

Theorem (Rudolph). *Let α and β be positive rationally independent reals. When restricted to an invariant subset of full measure, any free measure preserving flow $\mathbb{R} \curvearrowright X$ on a standard Lebesgue space admits a cross section with α and β being the only possible distances between adjacent points; i.e., after throwing away a set of zero measure, any flow can be represented as a flow under a two-valued function.*

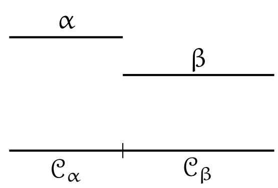

FIG. 2.

So given α and β, for any free measure preserving flow $\mathbb{R} \curvearrowright X$, one may find a Borel invariant subset $Y \subseteq X$ and a cross section $\mathcal{C} \subseteq Y$ for the restriction $\mathbb{R} \curvearrowright Y$ such that $\operatorname{dist}\bigl(x, \phi_{\mathcal{C}}(x)\bigr) \in \{\alpha, \beta\}$ for all $x \in \mathcal{C}$ (see Figure 2). Let

$$\mathcal{C}_\alpha = \bigl\{x \in \mathcal{C} : \operatorname{dist}\bigl(x, \phi_{\mathcal{C}}(x)\bigr) = \alpha\bigr\}.$$

As proved by Ambrose [Amb41], the measure μ on X can be disintegrated into $\nu \times \lambda$, where ν is a $\phi_{\mathcal{C}}$-invariant finite measure on $\mathcal{C}$ and λ is the restriction of the Lebesgue measure. In fact, there is a one-to-one correspondence between finite measures on X invariant with respect to the flow $\mathbb{R} \curvearrowright X$ and finite measures on $\mathcal{C}$ invariant under the induced automorphism $\phi_{\mathcal{C}}$. U. Krengel [Kre76] strengthened Rudolph's result by showing that for any $\rho \in (0, 1)$, one may always find a cross section $\mathcal{C}$ as in Rudolph's Theorem such that moreover $\nu(\mathcal{C}_\alpha) = \rho\nu(\mathcal{C})$; i.e., the proportion of α-points is exactly ρ. Analogs of the above theorems for semiflows were investigated by D. McClendon in [McC09].

This summarizes the relevant ergodic theoretical results. All the arguments employed in the aforementioned works require throwing certain sets (of zero measure) away. The first result in the purely descriptive set theoretical context was obtained by Wagh [Wag88], where he proved the analog of Ambrose's result for Borel flows.

Theorem (Wagh). *Any Borel flow $\mathbb{R} \curvearrowright X$ on a standard Borel space admits a Borel cross section.*

1.2. **Main Theorem.** The question of finding the simplest possible gap function for a flow, and in particular whether the analogs of Rudolph's Theorem and its refinement due to Krengel hold true in the Borel context remained open, and was explicitly posed by M. G. Nadkarni [Nad98, Remark 2 after Theorem 12.37]. The Borel theoretic version of Rudolph's original method was worked out by Nadkarni and Wagh in [NW]. They have constructed cross sections with only two possible gaps for flows that satisfy a certain technical condition, which, in particular, implies that the flow must be sparse (a flow is sparse if it admits a cross section with arbitrarily large gaps within each orbit, see Section 3 below). Though presented differently, their method is essentially equivalent to the argument outlined at the beginning of Subsection 4.2.

The property of being sparse is a significant restriction (see, for example, Proposition 3.2), and the main result of this work is the affirmative answer to Nadkarni's question in the full generality: Every free Borel flow admits a cross section with only two gaps between adjacent points. Moreover, we also provide a Borel strengthening of Krengel's result.

Main Theorem (see Theorem 9.1). *Let α and β be positive rationally independent reals and let $\rho \in (0, 1)$. Any free Borel flow $\mathbb{R} \curvearrowright X$ on a standard Borel space X admits a cross section $\mathcal{C} \subseteq X$ such that*

$$\operatorname{dist}\bigl(x, \phi_{\mathcal{C}}(x)\bigr) \in \{\alpha, \beta\} \quad \textit{for all } x \in \mathcal{C},$$

and moreover for any $\eta > 0$ there is $N \in \mathbb{N}$ such that for all $n \geqslant N$ and all $x \in \mathcal{C}$ one has

$$\Bigl|\rho - \frac{1}{n}\sum_{i=0}^{n-1} \chi_{\mathcal{C}_\alpha}\bigl(\phi_{\mathcal{C}}^{i}(x)\bigr)\Bigr| < \eta,$$

where $\chi_{\mathcal{C}_\alpha}$ *is the characteristic function of* $\mathcal{C}_\alpha$*.*

As an application of the main theorem, we derive the classification of Borel flows up to Lebesgue orbit equivalence proved earlier in [Slu15].

Theorem (see Theorem 10.4)**.** *Non-smooth free Borel flows are Lebesgue orbit equivalent if and only if they have the same number of invariant ergodic probability measures.*

1.3. **Borel Dynamics versus Ergodic Theory.** A reader with background in ergodic theory and little experience in Borel dynamics may be puzzled by the following question. How big is the difference between proving a certain statement almost everywhere and achieving the same result on literally every orbit? We would like to take an opportunity and address this question now.

The short answer is that the difference between an everywhere and an almost everywhere argument is often huge. Of course, it is perfectly possible for a certain property to hold almost everywhere but to fail on some orbits. For instance, as shown in Section 3, every free flow admits a sparse cross section on an invariant subset of (uniformly) full measure, i.e., every free flow is sparse as far as an ergodic theorist would care, but any cross section for a free minimal continuous flow on a compact metrizable space has bounded gaps on a comeager set, Proposition 3.2. So from the topological notion of largeness such a flow is the opposite of being sparse.

When an ergodic theoretical result happens to be true everywhere (as opposed to just almost everywhere) it usually does so for a non-trivial reason and the proof frequently relies upon a different set of ideas. Let us give some examples. One of the high points in ergodic theory is the theorem of D. S. Ornstein and B. Weiss [OW87] which shows that any action of an amenable group is necessarily hyperfinite. In the Borel context current state of the art is the work of S. Schneider and B. Seward [SS13] based on the methods developed by S. Gao and S. Jackson [GJ15]. Schneider and Seward proved that all Borel actions of countable locally nilpotent groups are hyperfinite. What happens beyond this class of groups is currently widely open.

The arguments in the works of Gao–Jackson and Schneider–Seward do not use directly the fact that the corresponding statements are known to be true almost everywhere, instead a general ingenious construction witnessing hyperfiniteness everywhere is provided. But sometimes a different approach is more successful. Quite often an ergodic theoretical argument that works with respect to a given invariant measure can be run with mild additional effort uniformly over all invariant measures simultaneously. This makes it possible to reduce the problem to an action with no finite invariant measures. The latter has a “positive” reformulation discovered by Nadkarni [Nad90] for the actions of $\mathbb{Z}$ and generalized to arbitrary countable equivalence relations by H. Becker and A. S. Kechris [BK96, Theorem 4.3.1]. It turns out that not having a finite invariant measure is equivalent to being compressible (see Section 10). This suggests a different strategy for a proof in the Borel world: First run a uniform version of the ergodic theoretical argument and then complete the proof by giving a different argument for the compressible case. The pivotal example of the power of this approach is the classification of hyperfinite equivalence relations by R. Dougherty, S. Jackson, and A. S. Kechris [DJK94]. A baby version of this idea is also used in the proof of Theorem 10.3 in Section 10. When the proof follows this ambivalent path, one effectively provides two reasons for the statement to be true — an ergodic theoretical reason and another one based on compressibility. The second part exists in the Borel context only and is usually the main reason for the step up in the complexity of the proof.

The proof of our main theorem also splits into two parts, but the interaction between the ergodic theoretical and “compressible” cases is more intricate, and unlike the previous examples we start with a “compressible” argument which is then complemented by an “ergodic theoretical” method. We first run a construction which aims at a weaker goal than the one prescribed in the Main Theorem: Instead of constructing a regular cross section, we construct a cross section with arbitrarily large regular blocks within each orbit. Despite the a priori weaker goal, the algorithm may accidentally achieve the result of the Main Theorem on certain parts of the space. On the complementary part, on the other hand, its failure will manifest existence of a sparse cross section which will be enough for another construction, inspired by the ergodic theoretical technique of Rudolph [Rud76], to succeed. We believe that the general approach of a sparse/co-sparse decomposition used in this paper can be of value for attacking other problems in Borel dynamics.

We hope that the reader has been convinced by now that an everywhere case may be noticeably different from an almost everywhere one. Let us now offer some reasons for working in the Borel context. One reason comes from topological dynamics where it may be unnatural to disregard sets of measure zero even in the presence of an invariant measure. And as is shown, for instance, by a simple argument in Section 3, topological and measurable genericities may be completely different. Another motivation comes from the theory of Borel equivalence relations. An important subclass of Borel equivalence relations consists of the equivalence relations coming from group actions. Those coming from the actions of $\mathbb{R}$ are the orbit equivalence relations of Borel flows. The main theorem of this paper implies a short proof of the classification of Borel flows up to Lebesgue orbit equivalence from [Slu15]. This is the content of Section 10. In fact, most of the results which prove existence of regular cross sections in ergodic theory and Borel dynamics were tools developed to solve some concrete problems. Rudolph [Rud76] used his construction of regular cross sections to settle in the case of finite entropy the problem of Sinai about equivalence between two definitions of K-flows. Krengel applied his strengthening of Rudolph's construction to prove a version of Dye's Theorem for flows. The classification up to Lebesgue orbit equivalence given in Theorem 10.4 is analogous to the Dougherty–Jackson–Kechris classification of hyperfinite equivalence relations.

The paper is structured as follows. Section 2 covers the basics of the theory of Borel flows. Section 3 introduces the family of sparse flows; they form the right class of flows for which ergodic theoretical methods can be applied. In Section 4 we give an overview of the proof of the main theorem motivating the further analysis. Sections 5, 6, and 7 form the technical core of the paper. In Section 8 we prove the main theorem under the additional assumption that the flow is sparse, and Section 9 provides the complementary argument proving the general case. Finally, Section 10 gives an application of the tiling result to the classification problem of Borel flows up to Lebesgue orbit equivalence.

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2. Basic concepts of Borel flows

This section will serve as a foundation for our study. We recall some well-known results and establish notations to be used throughout the paper.

A **Borel flow** is a Borel measurable action $\mathfrak{F} : \mathbb{R} \curvearrowright \Omega$ of the group of reals on a standard Borel space $(\Omega, \mathcal{B}_\Omega)$. For $r \in \mathbb{R}$ and $\omega \in \Omega$ we use $\omega + r$ as a shortcut for a more formal $\mathfrak{F}(r, \omega)$, provided the flow $\mathfrak{F}$ is unambiguous from the context. The orbit of a point $\omega \in \Omega$ is denoted by $\mathrm{Orb}_{\mathfrak{F}}(\omega)$ or just by $\mathrm{Orb}(\omega)$. With any flow $\mathfrak{F}$ we associate an **orbit equivalence relation** $\mathsf{E}_\Omega^{\mathfrak{F}}$ (or just E_Ω when the flow in understood) on Ω defined by $\omega_1 \mathsf{E}_\Omega \omega_2$ whenever $\mathrm{Orb}(\omega_1) = \mathrm{Orb}(\omega_2)$. This equivalence relation is Borel as a subset of $\Omega \times \Omega$. A flow is **free** if $\omega + r_1 \neq \omega + r_2$ for any $r_1 \neq r_2$ and all $\omega \in \Omega$.

For subsets $S \subseteq \mathbb{R}$ and $\mathcal{C} \subseteq \Omega$ expression $\mathcal{C} + S$ denotes the union of translates

$$\mathcal{C} + S = \bigcup_{r \in S} \mathcal{C} + r.$$

We shall frequently use the following fact: If $\mathcal{C}$ is Borel and its intersection with any orbit is countable, then $\mathcal{C} + S$ is Borel for any Borel $S \subseteq \mathbb{R}$. In essence, this follows from Luzin-Novikov's Theorem (see, for example, [Kec95, 18.10]). Here is a detailed explanation. Recall that by Miller's Theorem [Kec95, 9.17] stabilizers of Borel actions of Polish groups are necessarily closed subgroups. In the case of flows, a stabilizer of $\omega \in \Omega$ can therefore be either the whole real line (whenever ω is a fixed point) or a subgroup $\lambda\mathbb{Z}$ for some $\lambda \geqslant 0$. In particular, the set of fixed points

$$\mathrm{Fix}(\mathfrak{F}) = \{\, \omega \in \Omega \mid \omega + r = \omega \text{ for all } r \in \mathbb{R} \,\}$$

is Borel, since it is enough to quantify over the rationals:

$$\mathrm{Fix}(\mathfrak{F}) = \{\, \omega \in \Omega \mid \omega + q = \omega \text{ for all } q \in \mathbb{Q} \,\}.$$

If $\mathcal{C} \subseteq \Omega$ has a countable intersection with any orbit of $\mathfrak{F}$, and if $S \subseteq \mathbb{R}$ is Borel, then the map

$$\mathcal{C} \times S \ni (\omega, r) \mapsto \omega + r \in \Omega$$

is countable-to-one for $\omega \in \mathcal{C} \setminus \mathrm{Fix}(\mathfrak{F})$ and $r \in S$. By Luzin-Novikov's Theorem its image is Borel, hence so is $\mathcal{C} + S$. Borelness of sets of this form will be routinely used throughout the paper.

We established that the set of fixed points $\mathrm{Fix}(\mathfrak{F})$ is Borel. In fact, so is the set of periodic points

$$\mathrm{Per}(\mathfrak{F}) = \{\, \omega \in \Omega \mid \omega + r = \omega \text{ for some } r \in \mathbb{R}^{>0} \,\}.$$

It is immediate to see that this set is analytic. Borelness is established using the following fairly general trick: Pick a discrete cross section $\mathcal{C}$, enlarge it by adding small intervals around each point, $\mathcal{C} + [0, \epsilon]$, and then express $\mathrm{Per}(\mathfrak{F})$ by quantifying over the rationals. Here is a more formal argument. First of all, it is enough to show that $\mathrm{Per}_{>0}(\mathfrak{F}) = \mathrm{Per}(\mathfrak{F}) \setminus \mathrm{Fix}(\mathfrak{F})$ is Borel. We may therefore restrict our flow to $\Omega \setminus \mathrm{Fix}(\mathfrak{F})$ and assume that it has no fixed points. Under this assumption, Wagh's Theorem [Wag88] claims existence of a Borel set $\mathcal{C}$ intersecting every orbit, and such that $\{\, r \in \mathbb{R} \mid \omega + r \in \mathcal{C} \,\}$ is a separated[3] subset of $\mathbb{R}$ for any $\omega \in \mathcal{C}$. In particular, for any $\omega \in \mathcal{C}$ the set $\{\, y \in \mathcal{C} \mid \omega \mathsf{E}_\Omega y \,\}$ is countable, and if furthermore $\omega \in \mathrm{Per}_{>0}(\mathfrak{F})$, then it is necessarily finite. The orbit equivalence relation E_Ω induces a Borel relation $\mathsf{E}_\mathcal{C} = \mathsf{E}_\Omega \cap \mathcal{C} \times \mathcal{C}$ on $\mathcal{C}$, and the subset $\mathcal{C}' \subseteq \mathcal{C}$ consisting of points with finite $\mathsf{E}_\mathcal{C}$-equivalence classes is Borel. We saw that $\mathcal{C} \cap \mathrm{Per}_{>0}(\mathfrak{F}) \subseteq \mathcal{C}'$, but $\mathcal{C}'$ may also include some points from the free part $\mathrm{Free}(\mathfrak{F}) = \Omega \setminus \mathrm{Per}(\mathfrak{F})$. Since each $\mathsf{E}_\mathcal{C}$ class in $\mathcal{C}'$ is finite, it admits a **Borel transversal —** a Borel $\mathcal{D} \subseteq \mathcal{C}'$ which picks a representative from each class. Note that the **saturation**

$$[\mathcal{D}]_\mathfrak{F} = [\mathcal{D}]_{\mathsf{E}^\mathfrak{F}_\Omega} := \mathcal{D} + \mathbb{R} \quad \text{is Borel.}$$

Finally, Borelness of $\mathrm{Per}_{>0}(\mathfrak{F})$ is witnessed by the following equality:

$$\mathrm{Per}_{>0}(\mathfrak{F}) = \Big[\Big\{\, \omega \in \mathcal{D} \;\Big|\; \forall \epsilon \in \mathbb{Q} \cap (0,1)\ \exists r \in \mathbb{Q}^{>1} \text{ such that } \omega + r \in \mathcal{D} + [0, \epsilon] \,\Big\}\Big]_\mathfrak{F}.$$

To summarize, the decomposition of any flow into a fixed part, periodic part, and free part is Borel:

$$\Omega = \mathrm{Fix}(\mathfrak{F}) \sqcup \mathrm{Per}_{>0}(\mathfrak{F}) \sqcup \mathrm{Free}(\mathfrak{F}).$$

It may also be convenient to know that the function $\mathrm{per} : \mathrm{Per}_{>0}(\mathfrak{F}) \to \mathbb{R}^{>0}$, which assigns to a point $\omega \in \mathrm{Per}_{>0}(\mathfrak{F})$ its period, i.e., the unique $\lambda > 0$ such that $[0, \lambda) \ni r \mapsto \omega + r \in \mathrm{Orb}(\omega)$ is a bijection, is Borel. Indeed, in the notation above let $\mathcal{D}' = \mathcal{D} \cap \mathrm{Per}_{>0}(\mathfrak{F})$, and let $s : \mathrm{Per}_{>0}(\mathfrak{F}) \to \mathcal{D}'$ be the selector function: $s(\omega) = x$ if and only if $\omega \mathsf{E}_\Omega x$ and $x \in \mathcal{D}'$. The graph of per is Borel, since it can be written as

$$\Big\{\, (\omega, \lambda) \;\Big|\; \omega + \lambda = \omega \text{ and } \forall \epsilon \in \mathbb{Q}^{>0}\ \forall \delta \in (\epsilon, \lambda) \cap \mathbb{Q} \quad s(\omega) + \delta \notin \mathcal{D}' + [0, \epsilon] \,\Big\},$$

and thus $\mathrm{per} : \mathrm{Per}_{>0}(\mathfrak{F}) \to \mathbb{R}^{>0}$ is Borel.

From now on all flows are assumed to be free unless stated otherwise.

2.1. **Cross sections.** When a flow $\mathfrak{F}$ is free, any orbit can be "identified" with a copy of the real line. Concrete identification cannot be done in a Borel way throughout all orbits, unless the flow is smooth. But we may nonetheless unambiguously transfer translation invariant notions from $\mathbb{R}$ to each orbit. In particular, we shall talk about:

- Distances between points within an orbit: $\mathrm{dist}(\omega_1, \omega_2) = r \in \mathbb{R}^{\geqslant 0}$ if $\omega_1 + r = \omega_2$ or $\omega_1 - r = \omega_2$.
- Lebesgue measure on an orbit: for a Borel $A \subseteq \Omega$ we set
$$\lambda_x(A) = \lambda\big(\{r \in \mathbb{R} : x + r \in A\}\big),$$
where λ is the Lebesgue measure on $\mathbb{R}$. Note that $\lambda_x = \lambda_y$ whenever $x \mathsf{E}_\Omega y$ and λ_x is supported on $\mathrm{Orb}(x)$.
- Linear order within orbits: $\omega_1 < \omega_2$ if $\omega_1 + r = \omega_2$ for some $r > 0$.

A **countable cross section**, or just **cross section**, for a flow $\mathfrak{F}$ on Ω is a Borel set $\mathcal{C} \subseteq \Omega$ which intersects every orbit of $\mathfrak{F}$ in a non-empty countable set. A cross section $\mathcal{C}$ is **lacunary** if the gaps in $\mathcal{C}$ are bounded away from zero: there exists $\epsilon > 0$ such that $\mathrm{dist}(x, y) \geqslant \epsilon$ for all $x, y \in \mathcal{C}$ with $x \mathsf{E}_\mathcal{C} y$. A lacunary cross section is **bi-infinite** if for any $\omega \in \Omega$ there are $y_1, y_2 \in \mathcal{C}$ such that $y_1 < \omega < y_2$.

Unless stated otherwise all cross sections are assumed to be lacunary and bi-infinite.

[3]A subset $S \subseteq \mathbb{R}$ is **separated** if there is $\epsilon > 0$ such that $|x - y| > \epsilon$ for all distinct $x, y \in S$.

With such a cross section $\mathcal{C}$ one can associate an **induced automorphism** $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that sends $x \in \mathcal{C}$ to the next point from $\mathcal{C}$ within the same orbit: $\phi_{\mathcal{C}}(x) = y$ whenever $x, y \in \mathcal{C}$, $x < y$ and for no $z \in \mathcal{C}$ one has $x < z < y$. Finally, with any cross section we also associate a **gap function** $\vec{\mathrm{gap}}_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}^{>0}$ which measures distance to the next point: $\vec{\mathrm{gap}}_{\mathcal{C}}(x) = \mathrm{dist}\big(x, \phi_{\mathcal{C}}(x)\big)$. The gap function $\vec{\mathrm{gap}}_{\mathcal{C}}$ and the induced automorphism $\phi_{\mathcal{C}}$ are Borel.

The concept of a cross section also makes sense for the actions of $\mathbb{Z}$, i.e., for Borel automorphisms. In this case the distance between any two points within an orbit is a natural number and the condition of lacunarity is therefore automatic.

2.2. **Flow under a function.** An important concept in the theory of flows is the notion of a **flow under a function** (also known as a **suspension flow**). Given a standard Borel space $\mathcal{C}$, a free Borel automorphism $\phi : \mathcal{C} \to \mathcal{C}$ and a bounded away from zero Borel function $f : \mathcal{C} \to \mathbb{R}^{>0}$, $f(x) \geqslant \epsilon > 0$ for all $x \in \mathcal{C}$, one defines the space $\mathcal{C} \times f$ of points "under the graph of f:"

$$\mathcal{C} \times f = \big\{ (x, s) \in \mathcal{C} \times \mathbb{R}^{\geqslant 0} \bigm| 0 \leqslant s < f(x) \big\},$$

and a flow on $\mathcal{C} \times f$ by letting points flow upward until they reach the graph of f and then jump to the next fiber as determined by ϕ (see Figure 3). In symbols, for $\omega = (x, s) \in \mathcal{C} \times f$ and $r \in \mathbb{R}^{\geqslant 0}$

$$\omega + r = \Big(\phi^{n}(x), s + r - \sum_{i=0}^{n-1} f\big(\phi^{i}(x)\big)\Big)$$

for the unique $n \in \mathbb{N}$ such that $0 \leqslant s + r - \sum_{i=0}^{n-1} f\big(\phi^{i}(x)\big) < f\big(\phi^{n}(x)\big)$; and for $r \in \mathbb{R}^{<0}$

$$\omega + r = \Big(\phi^{-n}(x), s + r + \sum_{i=1}^{n} f\big(\phi^{-i}(x)\big)\Big)$$

for the unique $n \in \mathbb{N}$ such that $0 \leqslant s + r + \sum_{i=1}^{n} f\big(\phi^{-i}(x)\big) < f\big(\phi^{-n}(x)\big)$.

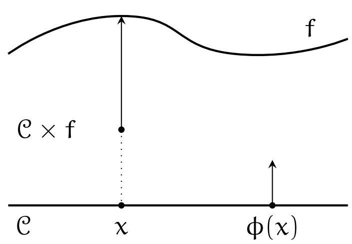


FIG. 3. Suspension flow

Note that $\mathcal{C}$, when identified with the subset $\{ (x, 0) \mid x \in \mathcal{C} \}$ of $\mathcal{C} \times f$, is a (lacunary bi-infinite) cross section, the function f is the gap function $\vec{\mathrm{gap}}_{\mathcal{C}}$ of this cross section and ϕ coincides with the induced automorphism $\phi_{\mathcal{C}}$. Conversely, if $\mathfrak{F}$ is a free Borel flow on Ω, and $\mathcal{C} \subseteq \Omega$ is a (lacunary bi-infinite) cross section, then the flow under the function $\vec{\mathrm{gap}}_{\mathcal{C}}$ and the induced automorphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is naturally isomorphic to $\mathfrak{F}$. Our point here is that *realizing a free $\mathfrak{F}$ as a flow under a (bounded away from zero) function is the same as finding a (lacunary bi-infinite) cross section.* In this terminology Wagh's Theorem [Wag88] implies the following.

Theorem 2.1 (Wagh)**.** *Any free Borel flow is isomorphic to a flow under a (bounded away from zero) function, which one may moreover assume to be bounded from above.*

While the notion of a cross section makes sense for actions of any Polish group, its simple geometric interpretation as in Figure 3 seems to be specific to actions of the real line. In the language of cross sections Theorem 2.1 is valid for all locally compact groups as showed by Kechris in [Kec92].

One can refine the formulation of Theorem 2.1 by specifying bounds on the gap function, see Corollary 2.3 below. First of all we recall a simple marker lemma for aperiodic Borel automorphisms. It follows easily from the observation that for any natural $d \geqslant 1$ any sufficiently large integer is of the form $md + n(d+1)$ for some $m, n \in \mathbb{N}$. The following proposition is a particular case of [GJ15, Lemma 2.9].

Proposition 2.2. *For any free Borel automorphism $T : \mathcal{C} \to \mathcal{C}$ on a standard Borel space*[4] *$\mathcal{C}$ and for any natural $d \geqslant 1$ there exists a bi-infinite Borel cross section $\mathcal{D} \subseteq \mathcal{C}$ with gaps of size d or $d+1$: for all $x \in \mathcal{D}$*

$$\min\{i \geqslant 1 : T^{i}(x) \in \mathcal{D}\} \in \{d, d+1\}.$$

[4]This proposition will typically be applied to the induced automorphism associated with a cross section of a flow, hence the notation $\mathcal{C}$ for the phase space.

Proof. To begin with, note that for any $d \geqslant 1$ and any $N \geqslant d^2$ there are $m, n \in \mathbb{N}$ such that $N = md + n(d+1)$. Indeed, let $N = qd + r$, where $q, r \in \mathbb{N}$ and $r \leqslant d-1$. Since $N \geqslant d^2$, $q \geqslant d$, hence

$$N = qd + r = (q-r)d + r(d+1).$$

We may select a sub cross section $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathrm{ga}\vec{\mathrm{p}}_{\mathcal{C}'}(x) \geqslant d^2$ for all $x \in \mathcal{C}'$ (see, for instance, [Nad98, 7.25]). For each $N \geqslant d^2$ fix a decomposition $N = m_N d + n_N(d+1)$. The cross section $\mathcal{D}$ is given by

$$\mathcal{D} = \left\{ T^{id}(x) : 0 \leqslant i \leqslant m_N,\ N = \mathrm{ga}\vec{\mathrm{p}}_{\mathcal{C}'}(x) \right\} \cup \left\{ T^{m_N d + i(d+1)}(x) : 0 \leqslant i \leqslant n_N,\ N = \mathrm{ga}\vec{\mathrm{p}}_{\mathcal{C}'}(x) \right\}. \qquad \square$$

Corollary 2.3. *Let $k, K \in \mathbb{R}^{>0}$ be positive reals, $k < K$. For any free Borel flow $\mathbb{R} \curvearrowright \Omega$ there exists a cross section $\mathcal{C} \subset \Omega$ such that $\mathrm{ga}\vec{\mathrm{p}}_{\mathcal{C}}(x) \in [k, K]$ for all $x \in \mathcal{C}$.*

Proof. First of all note that any sufficiently large real $x \in \mathbb{R}$, $x \geqslant N$, can be partitioned into pieces $x = \sum_i \tilde{x}_i$ such that $\tilde{x}_i \in [k, K]$. Fix such an $N \in \mathbb{R}$ and a Borel map[5] $\zeta : \mathbb{R}^{\geqslant N} \to \left[\mathbb{R}^{>0}\right]^{\mathrm{fin}}$ such that $\zeta(x) = \{z_0, z_1, z_2, \ldots, z_n\}$, where $z_0 = 0$, $z_n = x$ and $z_i - z_{i-1} \in [k, K]$ for all $1 \leqslant i \leqslant n$.

By Theorem 2.1 we may pick a cross section $\widetilde{\mathcal{C}} \subset \Omega$ and let $c \in \mathbb{R}^{>0}$ be so small that $\mathrm{ga}\vec{\mathrm{p}}_{\widetilde{\mathcal{C}}}(x) \geqslant c$ for all $x \in \widetilde{\mathcal{C}}$. Set $d = \lceil N/c \rceil$ and apply Proposition 2.2 to the induced flow $\phi_{\widetilde{\mathcal{C}}} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$. This results in a cross section $\mathcal{D}$ of $\phi_{\widetilde{\mathcal{C}}}$ which when viewed as a cross section of the flow $\mathbb{R} \curvearrowright \Omega$ has gaps bounded from below by $dc \geqslant N$. The cross section

$$\mathcal{C} = \left\{ x + z : x \in \mathcal{D} \text{ and } z \in \zeta\big(\mathrm{ga}\vec{\mathrm{p}}_{\mathcal{D}}(x)\big) \right\}$$

has all its gaps in $[k, K]$. $\square$

Remark 2.4. We remark for future reference that given a cross section $\mathcal{C}$ and a real $N \in \mathbb{R}^{>0}$ one may always find a sub cross section $\mathcal{C}' \subseteq \mathcal{C}$ with $\mathrm{ga}\vec{\mathrm{p}}_{\mathcal{C}'}(x) > N$ for all $x \in \mathcal{C}'$. This follows easily from, say, [Nad98, 7.25].

2.3. **Flows under constant functions.** In this paper we are concerned with the following question: How simple the function in the Wagh's Theorem 2.1 can be? The simplest case would be to have a constant function. This turns out to be impossible in many examples. A criterion for a flow to admit such a cross section was found by Ambrose [Amb41, Theorem 3]. While Ambrose's original argument is carried in the measure-theoretical context, it immediately adapts to our setting as well. For the reader's convenience we include the short proof of the characterization.

Proposition 2.5 (Ambrose)**.** *A Borel flow $\mathfrak{F}$ on Ω can be written as a flow under a constant function*

$$\Omega = \mathcal{C} \times \lambda, \quad \lambda \in \mathbb{R}^{>0},$$

if and only if $\mathfrak{F}$ has a nowhere zero eigenfunction with eigenvalue $2\pi/\lambda$, i.e., if and only if there is a Borel function $h : \Omega \to \mathbb{C} \setminus \{0\}$ such that

$$h(\omega + r) = e^{\frac{2\pi i r}{\lambda}} h(\omega)$$

for all $\omega \in \Omega$ and $r \in \mathbb{R}$.

Proof. $\Rightarrow$ If $\Omega = \mathcal{C} \times \lambda$, we may set $h(\omega) = e^{\frac{2\pi i r}{\lambda}}$, where $\omega = (x, r) \in \mathcal{C} \times \lambda$.

$\Leftarrow$ Let $h : \Omega \to \mathbb{C} \setminus \{0\}$ be an eigenfunction with an eigenvalue $2\pi/\lambda$. By considering $\frac{h(\omega)}{|h(\omega)|}$ instead of $h(\omega)$ we may assume without loss of generality that $|h(\omega)| = 1$ for all $\omega \in \Omega$. Let $\mathcal{C} = h^{-1}(1)$. The identity

$$h(\omega + r) = e^{\frac{2\pi i r}{\lambda}} h(\omega)$$

together with $|h(\omega)| = 1$ imply that $\mathcal{C}$ is a cross section and $\mathrm{ga}\vec{\mathrm{p}}_{\mathcal{C}}(x) = \lambda$ for all $x \in \mathcal{C}$. $\square$

This proposition has a more natural interpretation in the measure theoretical context. If the flow $\mathfrak{F}$ preserves a finite measure μ on Ω, we may associate a one-parameter subgroup of unitary operators on $L^2(\Omega, \mu)$ via the Koopman representation: $\big(U_r h\big)(\omega) = h(\omega + r)$ for $r \in \mathbb{R}$, $h \in L^2(\Omega, \mu)$, and $\omega \in \Omega$. In this framework Proposition 2.5 asserts equivalence between admitting a cross section with constant gaps and having a nowhere vanishing eigenfunction for the associated one-parameter subgroup of unitary operators.

[5]We use $\left[\mathbb{R}^{>0}\right]^{\mathrm{fin}}$ to denote the standard Borel space of finite subsets of $\mathbb{R}^{>0}$.

2.4. **Regular cross sections.** Let $\mathcal{C}$ be a cross section. For a subset $S \subseteq \mathbb{R}^{>0}$ we say that $\mathcal{C}$ is S-**regular** if $\vec{\mathrm{gap}}_{\mathcal{C}}(x) \in S$ for all $x \in \mathcal{C}$. For $r \in \mathbb{R}$ we may say that $x \in \mathcal{C}$ is an r-point if $\vec{\mathrm{gap}}_{\mathcal{C}}(x) = r$.

We shall need to consider equivalence relations on $\mathcal{C}$ that are finer than $E_{\mathcal{C}}$. One example is the K-**chain equivalence relation** $E_{\mathcal{C}}^{\leqslant K}$, where $K \in \mathbb{R}^{>0}$, defined as follows. Two points $x, y \in \mathcal{C}$ are $E_{\mathcal{C}}^{\leqslant K}$ related if one can get from one of the points to the other via jumps of size at most K: there exists $n \in \mathbb{N}$ such that $\phi_{\mathcal{C}}^{n}(x) = y$ and $\vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi^{i}(x)\big) \leqslant K$ for all $0 \leqslant i < n$, or the same condition holds with roles of x and y interchanged. Evidently, $E_{\mathcal{C}}^{\leqslant K}$ is Borel and is finer than $E_{\mathcal{C}}$, i.e., $E_{\mathcal{C}}^{\leqslant K} \subseteq E_{\mathcal{C}}$. We also note that $E_{\mathcal{C}}^{\leqslant K} \subseteq E_{\mathcal{C}}^{\leqslant L}$ whenever $K \leqslant L$. If $\mathcal{C}$ is sparse, that is if $\mathcal{C}$ has "bi-infinitely" unbounded gaps on each orbit (see the next section for the rigorous definition), then $E_{\mathcal{C}}^{\leqslant K}$ is a finite equivalence relation.

More generally, given a set $S \subseteq \mathbb{R}^{>0}$ we let $E_{\mathcal{C}}^{S}$ to relate points connected by jumps of sizes in S: $x \, E_{\mathcal{C}}^{S} \, y$ if and only if there exists $n \in \mathbb{N}$ such that $\phi_{\mathcal{C}}^{n}(x) = y$ and $\vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi^{i}(x)\big) \in S$ for all $0 \leqslant i < n$ or the same condition with roles of x and y interchanged. In this notation $E_{\mathcal{C}}^{\leqslant K} = E_{\mathcal{C}}^{(0,K]}$. Note that a cross section is S-regular if and only if $E_{\mathcal{C}} = E_{\mathcal{C}}^{S}$.

Of the utmost importance for us will be the relation $E_{\mathcal{C}}^{S}$ when $S = \{\alpha, \beta\}$ is a pair of rationally independent positive reals $\alpha, \beta \in \mathbb{R}^{>0}$. We shall abuse the notation slightly and denote it by $E_{\mathcal{C}}^{\alpha,\beta}$ omitting the curly brackets.

2.5. **Invariant measures.** An important invariant of a flow is its set of invariant measures. Recall that a probability measure μ on the phase space Ω is said to be **ergodic** if for any invariant Borel set $Z \subseteq \Omega$ either $\mu(Z) = 0$ or $\mu(Z) = 1$. Given a flow $\mathfrak{F}$ its set of ergodic invariant probability measures is denoted by $\mathcal{E}(\mathfrak{F})$.

Let $\mathcal{C} \subset \Omega$ be a cross section of $\mathfrak{F}$. Ambrose [Amb41, Theorem 1] showed that for any finite $\mathfrak{F}$-invariant measure μ on Ω there exists a $\phi_{\mathcal{C}}$-invariant measure ν_μ on $\mathcal{C}$ such that μ is the product of ν_μ with the Lebesgue measure λ on $\mathbb{R}$. More formally, when Ω is viewed as subset of $\mathcal{C} \times \mathbb{R}$ via the identification with

$$\big\{(x, r) \in \mathcal{C} \times \mathbb{R} : 0 \leqslant r < \vec{\mathrm{gap}}_{\mathcal{C}}(x)\big\},$$

then $\mu = (\nu_\mu \times \lambda)|_{\Omega}$.

The definition of ν_μ is simple. If $c \in \mathbb{R}^{>0}$ is such that $\vec{\mathrm{gap}}_{\mathcal{C}}(x) \geqslant c$ for all $x \in \mathcal{C}$, then for any Borel $A \subseteq \mathcal{C}$

$$\nu_\mu(A) = \frac{\mu(A \times [0, c])}{c}.$$

The definition is independent of the choice of c.

The above construction of ν_μ is valid for any cross section $\mathcal{C}$. When $\mathcal{C}$ moreover admits an upper bound on its gap function, we also have a map in the other direction. For any $\phi_{\mathcal{C}}$-invariant finite measure ν we define an $\mathfrak{F}$-invariant μ_ν on Ω by setting for $A \subseteq \Omega$

$$\mu_\nu(A) = \int_{\mathcal{C}} \tilde{\lambda}_x(A) \, d\nu(x),$$

where

$$\tilde{\lambda}_x(A) = \lambda\big(\{r \in \mathbb{R} : 0 \leqslant r \leqslant \vec{\mathrm{gap}}_{\mathcal{C}}(x) \text{ and } x + r \in A\}\big).$$

Boundedness of gaps from above is needed to ensure that the integral is finite.

The maps $\mu \mapsto \nu_\mu$ and $\nu \mapsto \mu_\nu$ are inverses of each other and provide a bijection between finite $\mathfrak{F}$-invariant measure on Ω and finite $\phi_{\mathcal{C}}$-invariant measures on a cross section $\mathcal{C} \subset \Omega$ with bounded gaps. These maps preserve ergodicity, but do not, in general, preserve normalization: $\mu(\Omega)$ is generally not equal to $\nu_\mu(\mathcal{C})$. When normalized manually, $\mu \mapsto \nu_\mu/\nu_\mu(\mathcal{C})$ is a bijection between $\mathcal{E}(\mathfrak{F})$ and $\mathcal{E}(\phi_{\mathcal{C}})$.

Theorem 2.6. *Let $\mathfrak{F} : \mathbb{R} \curvearrowright \Omega$ be a free Borel flow and $\mathcal{C} \subset \Omega$ be a cross section with bounded gaps. The sets of ergodic invariant probability measures $\mathcal{E}(\mathfrak{F})$ and $\mathcal{E}(\phi_{\mathcal{C}})$ have the same cardinalities.*

This correspondence is valid, in fact, for all unimodular locally compact groups, see [Slu15, Proposition 4.4].

3. SPARSE FLOWS

Definition 3.1. Let $\mathfrak{F} : \mathbb{R} \curvearrowright \Omega$ be a free flow. A cross section $\mathcal{C} \subseteq \Omega$ is said to be **sparse** if it has gaps bi-infinitely unbounded on each orbit: for each $N \in \mathbb{R}$ and $x \in \mathcal{C}$ there are integers $n_1 \geqslant 0$ and $n_2 < 0$ such that $\vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi_{\mathcal{C}}^{n_1}(x)\big) \geqslant N$ and $\vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi_{\mathcal{C}}^{n_2}(x)\big) \geqslant N$. We say that a flow is sparse if it admits a sparse cross section. The definition of a sparse cross section for a Borel automorphism is analogous.

The requirement of having unbounded gaps in the bi-infinite fashion is a matter of convenience only. For if $\mathcal{C}$ is any cross section, the set of orbits where gaps are unbounded, but are not bi-infinitely unbounded, is smooth. Indeed, for any real N the set

$$\mathcal{D}_N^l = \{x \in \mathcal{C} : \vec{\mathrm{gap}}_{\mathcal{C}}(x) \geqslant N \text{ and } \vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi_{\mathcal{C}}^n(x)\big) < N \text{ for all } n \in \mathbb{Z}^{<0}\}$$

selects the minimal point with the gap of size at least N, whenever such a point exists. Similarly,

$$\mathcal{D}_N^r = \{x \in \mathcal{C} : \vec{\mathrm{gap}}_{\mathcal{C}}(x) \geqslant N \text{ and } \vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi_{\mathcal{C}}^n(x)\big) < N \text{ for all } n \in \mathbb{Z}^{>0}\}$$

selects the maximal such point. For any $\mathcal{C}$ the set of orbits where $\mathcal{C}$ has unbounded but not bi-infinitely unbounded gaps is the saturation of the set

$$\bigcup_{N \in \mathbb{N}} (\mathcal{D}_N^l \cup \mathcal{D}_N^r)$$

and is therefore smooth. We shall thus always assume that our sparse cross sections have bi-infinitely unbounded gaps and this implies that in the notation of Subsection 2.4 each $E_{\mathcal{C}}^{\leqslant K}$-class is finite for any $K \in \mathbb{R}$.

The notion of a sparse automorphism appeared for the first time (under the name of gapped automorphisms) in the lecture notes of B. D. Miller [Mil07]. This class of automorphisms and flows seems to isolate a purely Borel property for which some ergodic theoretical methods can be applied.

In general, a Borel flow may not admit a sparse cross section. We start by exhibiting an important class of flows which never admit sparse cross sections. Recall that a continuous flow is said to be **minimal** if every orbit is dense.

Proposition 3.2. *Free continuous flows on compact metrizable spaces do not admit sparse cross sections. If such a flow is moreover minimal, then any cross section has bounded gaps on a Borel invariant comeager set.*

Proof. Let $\mathfrak{F}$ be a minimal flow on a compact metrizable space Ω and let $\mathcal{C}$ be a Borel cross section; put $\mathcal{I} = \mathcal{C} + (-1, 1)$. Note that $\mathcal{I}$ has unbounded gaps between intervals within every orbit of $\mathfrak{F}$ if and only if $\mathcal{C}$ is sparse. Since

$$\bigcup_{n \in \mathbb{Z}} (\mathcal{I} + n) = \Omega,$$

it follows that $\mathcal{I}$ is non meager, therefore it is comeager in a non-empty open subset $U \subseteq \Omega$. By minimality of the action we get $U + \mathbb{Q} = \Omega$, and therefore compactness ensures existence of $q_1, \ldots, q_N \in \mathbb{Q}$ such that $\Omega = \bigcup_{n=1}^N (U + q_n)$. Set $\mathcal{J} = \bigcup_{n=1}^N (\mathcal{I} + q_n)$ and note that gaps between intervals in $\mathcal{J}$ are bounded within an orbit if and only if they are bounded within the same orbit for $\mathcal{I}$. Note also that $\mathcal{J}$ is comeager in Ω, hence so is $\bigcap_{q \in \mathbb{Q}} (\mathcal{J} + q)$, which is moreover invariant under the flow. Since each $\omega \in \mathcal{J}$ is a bounded distance away from a point in $\mathcal{C}$, we conclude that gaps in

$$\mathcal{C} \cap \bigcap_{q \in \mathbb{Q}} (\mathcal{J} + q)$$

are bounded by $2 \max\big\{|q_i| + 2 : 1 \leqslant i \leqslant N\big\}$. This proves the proposition for minimal flows.

The general case follows from the minimal one, since any flow has a minimal subflow: by Zorn's lemma the family of non-empty invariant closed subsets of Ω ordered by inclusion has a minimal element $M \subseteq \Omega$. The restriction $\mathfrak{F}|_M$ is minimal, and therefore does not admit a sparse cross section by the above argument. □

A similar argument shows that free homeomorphisms of compact metrizable spaces never admit sparse cross sections. We note also that if $\mathcal{C}$ is a cross section with bounded gaps, then the flow $\mathfrak{F}$ is sparse if and only if the induced automorphism $\phi_{\mathcal{C}}$ is sparse. Indeed, if $\mathcal{D}$ is a sparse cross section for $\phi_{\mathcal{C}}$, then it is also a sparse cross section for the flow $\mathfrak{F}$ when viewed as a subset of Ω. For the other direction, if $\mathcal{D} \subseteq \Omega = \mathcal{C} \times \vec{\mathrm{gap}}_{\mathcal{C}}$ is sparse for $\mathfrak{F}$, then $\mathrm{proj}_{\mathcal{C}}(\mathcal{D})$ is sparse for $\phi_{\mathcal{C}}$.

While from the topological point of view minimal flows on compact spaces are far from sparse, any flow is sparse in the measure theoretical context.

Theorem 3.3. *Let $\mathfrak{F}$ be a free Borel flow on a standard Borel space Ω. There exists a Borel invariant subset $\Omega_{sp} \subseteq \Omega$ such that $\mathfrak{F}|_{\Omega_{sp}}$ is sparse and $\mu(\Omega_{sp}) = 1$ for any $\mathfrak{F}$-invariant probability measure μ.*

Proof. The theorem follows from a sequence of applications of the descriptive Rokhlin Lemma. More formally, we may use [Slu15, Theorem 6.3] to construct a Borel $\mathfrak{F}$-invariant subset $\Omega_{sp} \subseteq \Omega$ of full measure for any $\mathfrak{F}$-invariant finite measure on Ω, a sequence $(l_n)_{n=1}^{\infty}$, $l_n \geqslant n$, and sets $\mathcal{C}_n \subseteq \Omega_{sp}$ satisfying the following properties for all n.

(i) $\big(c + [-l_{n+1}, l_{n+1})\big) \cap \mathcal{C}_n \neq \varnothing$ for all $c \in \mathcal{C}_{n+1}$;
(ii) $\Omega_{sp} = \bigcup_n \big(\mathcal{C}_n + [-l_n, l_n)\big)$;
(iii) $\big(c + [-l_n, l_n)\big) \cap \big(c' + [-l_n, l_n)\big) = \varnothing$ for all distinct $c, c' \in \mathcal{C}_n$;
(iv) $\mathcal{C}_n + [-l_n, l_n) \subseteq \mathcal{C}_{n+1} + [-l_{n+1} + n + 1, l_{n+1} - n - 1)$.

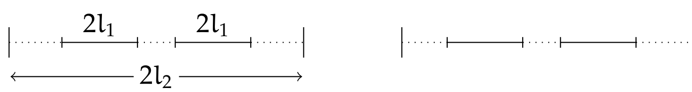


FIG. 4. The intervals in $\mathcal{C}_1 + [-l_1, l_1)$ are inside the intervals of $\mathcal{C}_2 + [-l_2, l_2)$, and are far from their boundary.

By item (iii), $\mathcal{C}_n$ is lacunary, and by (i) and (ii) each $\mathcal{C}_n$ intersects every orbit of $\mathfrak{F}$ in Ω_{sp}. Using also (iv), it is not hard to see that each of $\mathcal{C}_n$ must be a (lacunary bi-infinite) cross section for the restriction of $\mathfrak{F}$ on Ω_{sp}. One can think of $\mathcal{C}_n + [-l_n, l_n)$ as an "interval cross section," where each point has been fattened into an interval of length $2l_n$. The main property here is that each interval in $\mathcal{C}_n + [-l_n, l_n)$ lies inside an interval of $\mathcal{C}_{n+1} + [-l_{n+1}, l_{n+1})$ and, moreover, it lies at a distance at least $n+1$ from its boundary. Figure 4 shows how intervals in $\mathcal{C}_1 + [-l_1, l_1)$ (solid lines) may lie relative to intervals of $\mathcal{C}_2 + [-l_2, l_2)$ (dotted lines).

The proof is completed by showing that $\mathcal{C}_1$ is sparse. Pick some $x \in \mathcal{C}_1$, we show that $(x + \mathbb{R}) \cap \mathcal{C}_1$ has arbitrarily large gaps. Items (i) and (iv) guarantee that for any $c \in \mathcal{C}_n$ the set $\big(c + [-l_n, l_n)\big) \cap \mathcal{C}_1$ is non-empty, and we set

$$d = \max\big(c + [-l_n, l_n)\big) \cap \mathcal{C}_1,$$
$$d' = \min\big(c' + [-l_n, l_n)\big) \cap \mathcal{C}_1,$$

where $c, c' \in \mathcal{C}_n \cap (x + \mathbb{R})$ are any adjacent points, $c < c'$. Elements d and d' must be adjacent in $\mathcal{C}_1$, and $\mathrm{dist}(d, d') \geqslant 2n$ by (iv). Since n was arbitrary, $\mathcal{C}_1$ is sparse. □

4. Overview of the Proof

This section contains a rough sketch of the proof of the Main Theorem. During the first reading we suggest skimming over and returning to the relevant parts as the reader goes through the next chapters.

4.1. **The big picture.** The main argument of Theorem 9.1 requires a certain amount of technical preparation, and we would like to take this opportunity and outline the big picture of the proof. The argument splits into two parts. First, we construct regular cross sections under the additional assumption that the flow is sparse. This is done in Theorem 8.1. Now suppose $\mathcal{C}$ is a cross section which has arbitrarily large $\{\alpha, \beta\}$-regular blocks within each orbit in the sense that for any $x \in \mathcal{C}$ and any $N \in \mathbb{N}$ there exists $y \in \mathcal{C}$ coming from the same orbit, $x \mathsf{E}_{\mathcal{C}} y$, such that the cardinality of the $\mathsf{E}_{\mathcal{C}}^{\alpha,\beta}$-class of y is at least N: $\big|[y]_{\mathsf{E}_{\mathcal{C}}^{\alpha,\beta}}\big| \geqslant N$. Orbits in such a cross section $\mathcal{C}$ necessarily fall into three categories:

- Some orbits may be tiled completely, i.e., the whole orbit may constitute a single $E^{\alpha,\beta}_{\mathcal{C}}$-class. On the part of the space which consists of these orbits the cross section $\mathcal{C}$ is $\{\alpha, \beta\}$-regular.
- It is possible that "half" of an orbit is tiled, meaning that $E^{\alpha,\beta}_{\mathcal{C}}$ may have at least two equivalence classes within the orbit, one of which is infinite. Restriction of the flow onto the set of such orbits is smooth for we may pick the finite endpoints of the infinite classes to get a Borel transversal. Constructing a regular cross section on this part of the space will therefore be trivial.
- Finally, in a typical orbit each $E^{\alpha,\beta}_{\mathcal{C}}$-class will be finite. Consider the sub cross section that consists of the endpoints of $E^{\alpha,\beta}_{\mathcal{C}}$-classes:
$$\mathcal{C}' = \big\{x \in \mathcal{C} : x = \min[x]_{E^{\alpha,\beta}_{\mathcal{C}}} \text{ or } x = \max[x]_{E^{\alpha,\beta}_{\mathcal{C}}}\big\}.$$
A simple but crucial observation is that the assumption of having arbitrarily large $E^{\alpha,\beta}_{\mathcal{C}}$-classes within each orbit implies that the cross section $\mathcal{C}'$ is sparse.

To summarize, given a cross section with unbounded $E^{\alpha,\beta}_{\mathcal{C}}$-classes within each orbit, the phase space splits into three invariant Borel parts: a sparse piece where Theorem 8.1 applies, a smooth piece, and a piece where $\mathcal{C}$ is already an $\{\alpha, \beta\}$-regular cross section. Once Theorem 8.1 is proved, the problem of constructing a regular cross section for a general Borel flow is therefore reduced to the problem of constructing a cross section with arbitrarily large $E^{\alpha,\beta}_{\mathcal{C}}$-classes. Theorem 9.1 achieves just that.

4.2. **Sparse case.** Let us now explain how sparsity of the flow is helpful in constructing a regular cross section. If α and β are positive rationally independent reals, then the set $\mathcal{T}$ of reals of the form $p\alpha + q\beta$, $p, q \in \mathbb{N}$, (we call such reals **tileable**) is **asymptotically dense** in $\mathbb{R}$ in the sense that for any $\epsilon > 0$ this set is ϵ-dense in $[K, \infty)$ for a sufficiently large K (see Definition 6.2). We may therefore pick a sequence K_n growing so fast that $\mathcal{T}$ is ϵ_{n+1}-dense in[6] $[K_n, \infty)$, where $\epsilon_n = 2^{-n}$. Given a sparse cross section $\mathcal{C}$ we now argue as follows.

By passing to a sub cross section if necessary we may suppose without loss of generality that $\vec{\mathrm{gap}}_{\mathcal{C}}(x) \geqslant K_0$ for all $x \in \mathcal{C}$. Consider the relation $E^{\leqslant K_1}_{\mathcal{C}}$ on $\mathcal{C}$. Sparsity ensures that each $E^{\leqslant K_1}_{\mathcal{C}}$-class is finite. Take a single $E^{\leqslant K_1}_{\mathcal{C}}$-class. Let us say it consists of points $x_0, x_1, \ldots, x_N$ listed in the increasing order (Figure 5). By construction gaps between adjacent points within this class are between K_0 and K_1. By the choice of K_0 we may shift x_1 by at most $\epsilon_1 = 1/2$ to a new position x_1' such that $\mathrm{dist}(x_0, x_1') \in \mathcal{T}$. Since $\mathrm{dist}(x_0, x_1')$ is of the form $p\alpha + q\beta$, where p and q are non-negative integers, we may add $p + q - 1$ points to the interval $[x_0, x_1']$ to make every gap between adjacent points be either α or β. We call the processing of adding such points **tiling the gap**. One now proceeds in the same fashion with x_2 — the distance from x_1' to x_2 differs

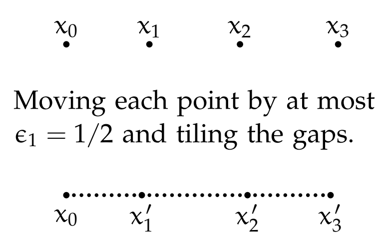


FIG. 5. Tiling a sparse cross section

from $\mathrm{dist}(x_1, x_2)$ by no more than ϵ_1, so it is still large enough, and we may therefore shift x_2 by at most ϵ_1 to a new position x_2' in such a way that $\mathrm{dist}(x_1', x_2') \in \mathcal{T}$. Once x_2 has been shifted, we tile the gap by adding extra points to $[x_1', x_2']$. This process is continued until we reach the maximal point x_N within the given $E^{\leqslant K_1}_{\mathcal{C}}$-class. Note that the amount of shift does not grow — each point is shifted by at most ϵ_1, and the distance $\mathrm{dist}(x_k', x_{k+1})$ is always at least $K_0 - \epsilon_1$ for all k, so it will always be possible to shift x_{k+1} to x_{k+1}' which makes $\mathrm{dist}(x_k', x_{k+1}') \in \mathcal{T}$ by the choice of K_0. This procedure is applied to all $E^{\leqslant K_1}_{\mathcal{C}}$-classes.

[6] In fact, we need $\mathcal{T}$ to be ϵ_{n+1}-dense in $[K_n - 4, \infty)$, because during our construction points will be shifted a bit, but the total shift of each point will never exceed 2, so if we start with two points which are distance K_n apart, then during the whole process they will remain at least $K_n - 4$ apart. In this sketch we ignore minor changes in distance because of point shifts.

After the first step of the construction, we arrive at a cross section $\mathcal{C}_1$ which differs from the cross section $\mathcal{C}$ in two aspects. Some of the points in $\mathcal{C}_1$ correspond to the points from $\mathcal{C}$ shifted by at most ϵ_1, and other points have been added to tile the gaps. Note that $\mathcal{C}_1$ is still sparse. Consider a $\mathsf{E}_{\mathcal{C}_1}^{\leqslant K_2}$-class. It consists of a number of $\mathsf{E}_{\mathcal{C}_1}^{\alpha,\beta}$-classes separated by gaps of size at least[7] K_1 (Figure 6). We may run the same process as

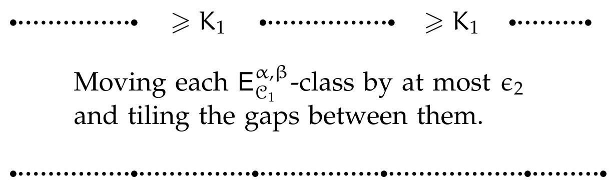


FIG. 6. Tiling a sparse cross section, step 2.

before, but now with smaller shifts applied to $\mathsf{E}_{\mathcal{C}_1}^{\alpha,\beta}$-blocks. By the choice of K_1, the second $\mathsf{E}_{\mathcal{C}_1}^{\alpha,\beta}$-class as a whole can be shifted by at most ϵ_2 in such a way that the distance between the first and the second class becomes a real in $\mathcal{T}$. Doing this with each $\mathsf{E}_{\mathcal{C}_1}^{\alpha,\beta}$-class one by one and tiling the gaps in the midst results in a cross section $\mathcal{C}_2$ in which $\mathsf{E}_{\mathcal{C}_2}^{\alpha,\beta}$-classes correspond to $\mathsf{E}_{\mathcal{C}_1}^{\leqslant K_1}$-classes in $\mathcal{C}_1$. Moreover, $\mathcal{C}_2$ is sparse, and the distance between distinct $\mathsf{E}_{\mathcal{C}_2}^{\alpha,\beta}$-classes is at least K_2. The construction continues.

To summarize, at step $n+1$ we shift points in $\mathcal{C}_n$ by at most ϵ_{n+1} and add a handful of new points between $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-classes. Since $\sum \epsilon_n$ converges, each point "converges" to a limit position, and the limit tiling is $\{\alpha, \beta\}$-regular.

This sketch can easily be turned into a rigorous argument which is just a different presentation of the one given in Section 2 of [Rud76]. Additional care seems to be necessary if one wants to control the frequency of $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-blocks during the construction. More precisely, let x and y be respectively the minimal and the maximal point of an $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-class, and let z denote the distance $\mathrm{dist}(x,y)$ which is necessarily of the form $p\alpha + q\beta$ for some $p, q \in \mathbb{N}$, i.e., z is the length of the $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-class. We consider the quantity $p/(p+q)$ which represents the frequency of gaps of size α in this class. This quantity is called the α-**frequency** of z and is denoted by $\mathrm{fr}_\alpha(z)$. We want to run our construction in a way ensuring this frequency converges to ρ as $n \to \infty$.

The main difficulty lies in the fact that $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-classes can be arbitrarily large and there is no upper bound on the size of z as above. Here is why this is a problem. Let us say we have an estimate on the α-frequencies of all $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-classes of the form $\left|\mathrm{fr}_\alpha(z) - \rho\right| < \eta$ for some small $\eta > 0$. We would like to tile the gaps in such a way that if w is the length of an $\mathsf{E}_{\mathcal{C}_{n+1}}^{\alpha,\beta}$-class, then $\left|\mathrm{fr}_\alpha(w) - \rho\right| < \eta/2$. In other words, we would like to tile the gaps between $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-classes inside $\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}$-classes in a way that improves the approximation of ρ by the α-frequencies of $\mathsf{E}_{\mathcal{C}_{n+1}}^{\alpha,\beta}$-classes. Having the estimate $\left|\mathrm{fr}_\alpha(z) - \rho\right| < \eta$ is not enough, for if z is huge compared to the size of the gap K_{n+1}, then $\mathrm{fr}_\alpha(z+y)$ can be arbitrarily close to $\mathrm{fr}_\alpha(z)$ for all tileable $y \leqslant K_{n+1}$ (this is quantified in Lemma 6.4).

To remedy this obstacle, we shall impose a stronger assumption on the lengths of $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-classes, which we call N_n-nearness (see Definition 7.8). We say that a tileable real z is N-near ρ, $N \in \mathbb{N}$, if either $\big(\mathrm{fr}_\alpha(z) \leqslant \rho$ and $\mathrm{fr}_\alpha(z + N\alpha) \geqslant \rho\big)$ or $\big(\mathrm{fr}_\alpha(z) \geqslant \rho$ and $\mathrm{fr}_\alpha(z + N\beta) \leqslant \rho\big)$. Less rigorously it means that by adding N tiles of the right type one can flip the α-frequency of z to the other side of ρ. The key observation here is that among those tileable reals z that are N-near ρ, the bigger z is the closer its α-frequency to ρ has to be. The quantitative estimate of this intuition is provided by item (vii) of Proposition 7.9. We shall run the tiling construction described above ensuring that all lengths z of $\mathsf{E}_{\mathcal{C}_n}^{\alpha,\beta}$-classes are N_n-near ρ for some natural number $N_n \in \mathbb{N}$ that depends only on n. This will allow us to tile the gaps between such class in a way that improves α-frequency approximation of ρ.

But we shall need even more. We want the convergence to ρ in the Main Theorem to be uniform over all $x \in \mathcal{C}$. Lemma 6.5 offers a witness for such a uniformity. By step n we shall pick real numbers L_k, $k \leqslant n$,

[7] In fact, gaps can be as small as $K_1 - \epsilon_1$, but we agreed to ignore shifts for now.

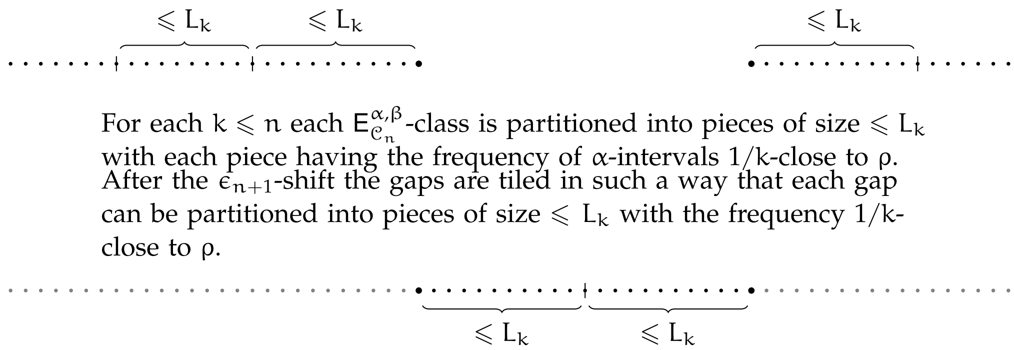


FIG. 7. Witnessing uniform convergence of the frequencies to ρ.

and for each $k \leqslant n$ we shall have a partition of each $E^{\alpha,\beta}_{\mathcal{C}_n}$-class into pieces of length at most L_k, each piece having the frequency of α-intervals $1/k$-close to ρ. Figure 7 illustrates this situation. What we need is to shift $E^{\alpha,\beta}_{\mathcal{C}_n}$-classes by at most ϵ_{n+1} and tile the gaps, but the tiling is supposed to be special: for each $k \leqslant n$ we want to be able to partition the gap into pieces of size $\leqslant L_k$ each having frequency $1/k$-close to ρ. The pieces from $E^{\alpha,\beta}_{\mathcal{C}_n}$-classes together with the pieces from the tiled gaps will constitute the partition of $E^{\alpha,\beta}_{\mathcal{C}_{n+1}}$-class. Given L_k, $k \leqslant n$, Lemma 7.4 computes how large K_{n+1} needs to be so as to make the construction possible.

To finish the inductive step it remains to pick L_{n+1}. Remember that we have no control on how many $E^{\alpha,\beta}_{\mathcal{C}_n}$-classes fall into a single $E^{\leqslant K_{n+1}}_{\mathcal{C}_n}$-class, so the resulting $E^{\alpha,\beta}_{\mathcal{C}_{n+1}}$-class may be arbitrarily long, but we need to argue that there exists a single L_{n+1} such that any $E^{\alpha,\beta}_{\mathcal{C}_{n+1}}$-class can be partitioned into blocks of size $\leqslant L_{n+1}$ each having the α-frequency $1/(n+1)$-close to ρ. This is the content of Lemmas 7.6 and 7.12 with the latter one encompassing precisely the set up of the induction step in the sparse case.

This finishes a rough summary of the content of Section 7. We have mentioned on a number of occasions that each $E^{\leqslant K_{n+1}}_{\mathcal{C}_n}$-class may consist of many $E^{\leqslant K_n}_{\mathcal{C}_n}$-classes, but for technical reasons it is desirable to know that it always consists of at least two such classes. Construction of such a cross section is the content of Section 5 and Lemma 5.2 specifically. Moreover, the cross section constructed therein has that property in a stable way: even if each point in $\mathcal{C}$ is perturbed by at most $\sum \epsilon_k$, each $E^{\leqslant K_{n+1}}_{\mathcal{C}_n}$-class will still consist of at least two $E^{\leqslant K_n}_{\mathcal{C}_n}$-classes. This allows us not to worry about the minor change in the sizes of the gaps as the inductive construction unfolds.

4.3. **Co-sparse case.** As we have speculated in Subsection 4.1 above, to boost the sparse argument to the general situation it is enough to show how given a free Borel flow to construct a cross section $\mathcal{C}$ with arbitrarily large $E^{\alpha,\beta}_{\mathcal{C}}$-classes within each orbit. Our basic approach is similar to the sparse case — we construct a sequence of cross sections $\mathcal{C}_n$, which "converge" to a limit cross section $\mathcal{C}_\infty$. The next cross section in the sequence is obtained from the previous one by shifting some of its elements and adding extra points to tile a few gaps. Convergence of $\mathcal{C}_n$ relies upon moving the points by smaller and smaller amounts, and here lies the first important difference. In the sparse argument there was the following uniformity in the size of jumps: when constructing $\mathcal{C}_{n+1}$ out of $\mathcal{C}_n$ any point was moved by no more than ϵ_n. This time there will be no uniformity of that sort. It will still be the case that the first jump of any point is at most ϵ_1, the second one will be bounded by ϵ_2, etc., but each point will have its own sequence (n_k) of indices when the jumps are made. When passing from $\mathcal{C}_n$ to $\mathcal{C}_{n+1}$, some of the points will make their first jump, which can be as large as ϵ_1, others will jump for the second time, etc., and there will also be points that were jumping all the time from $\mathcal{C}_0$ to $\mathcal{C}_{n+1}$.

The construction starts with a cross section $\mathcal{C}_0$ with bounded gaps. When building $\mathcal{C}_1$, we shall take sufficiently distant pairs of adjacent points in $\mathcal{C}_0$, within each pair we shall move the right point making the gap to the left one tileable, and finally we shall add points tiling these gaps. Figure 8 tries to illustrate the process.

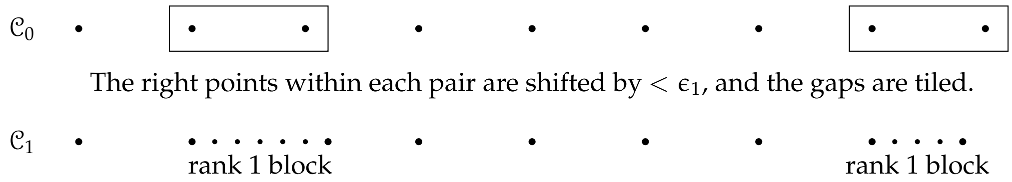


FIG. 8. Constructing $\mathcal{C}_1$.

The relation $\mathsf{E}^{\alpha,\beta}_{\mathcal{C}_1}$ will have classes of two sorts. Some of them will be just isolated points, we call them rank 0 classes, while others will be given by the selected pairs together with the new points added in the midst. These are rank 1 blocks.

What we would like to do at the second step is to take sufficiently distant pairs of adjacent rank 1 blocks, move the right block by at most ϵ_2, move the rank 0 points inside by no more than ϵ_1 making all the gaps tileable, and tile these gaps creating a rank 2 block. See Figure 9.

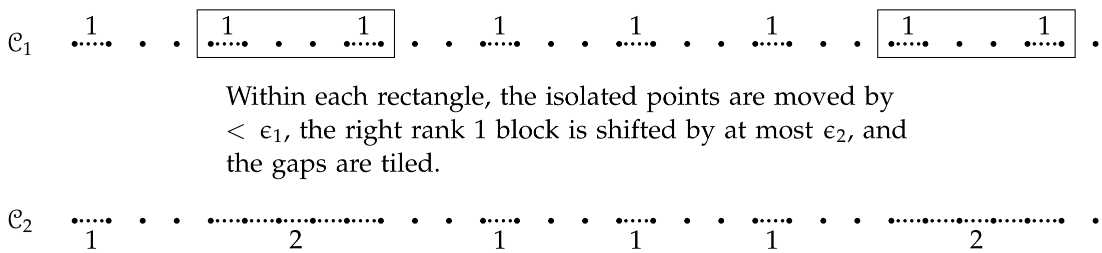


FIG. 9. Constructing $\mathcal{C}_2$

The key technicality lies in ensuring that each rank 1 block can be moved by at most ϵ_2 rather than ϵ_1. We don't have any problems in constructing $\mathcal{C}_1$ — we may choose $\mathcal{C}_0$ to have all the gaps sufficiently large as to allow for turning each gap into a tileable real after an ϵ_1 perturbation. But imagine now that in $\mathcal{C}_0$ we always have several ways of moving points by at most ϵ_1 to make gaps tileable; i.e., in the notation of Figure 5, we have several ways of moving x_1. For *each* such shift we have *several* ways to shift x_2, and for each shift of x_2 we have several possibilities for x_3, etc. While some of the arrangements may coincide, it is nonetheless natural to expect that the number of possible terminal positions for x_N, as each x_k, $k \leqslant N$, is shifted by $\leqslant \epsilon_1$, will grow with N, and in fact, will eventually densely pack ϵ_1-neighborhood of x_N. Formalization of this intuition is called "propagation of freedom". In a short form it can be summarized as follows. Let $\epsilon > 0$, let $N \in \mathbb{N}$, and let $x_0, x_1, \ldots, x_N$ be a family of points such that for each $i < N$ there are "sufficiently many" ways to shift x_{i+1} to x'_{i+1} by at most ϵ to make $\mathrm{dist}(x_i, x'_{i+1})$ tileable. Let $\mathcal{A}_N$ denote the set of all possible[8] $\tilde{z} \in \mathcal{U}_\epsilon(x_N)$ for which there exists $x'_i \in \mathcal{U}_\epsilon(x_i)$ satisfying $\mathrm{dist}(x'_i, x'_{i+1}) \in \mathcal{T}$ for all $i \leqslant N$, $x'_0 = x_0$, and $x'_N = \tilde{z}$. In the notation of Figure 5 this is the set of all possible terminal positions for x'_N. The "propagation of freedom" principle states that $\mathcal{A}_N$ is δ-dense in $\mathcal{U}_\epsilon(x_N)$, where $\delta \to 0$ as $N \to \infty$. In other words, the cumulative shift of x_N can be made arbitrarily small by picking N sufficiently large.

The second step of the inductive construction corresponds to picking $\delta = \epsilon_2$ in the propagation of freedom principle. In fact, to run the construction further we shall need to take δ so small that we have "sufficiently many" shifts of x_N by at most ϵ_2. Section 6 formalizes this approach and Lemma 6.12 encapsulates the inductive step of the construction. There are certain degenerate ways of picking the admissible shifts for the points x_i, where the "freedom" does not "propagate", so to avoid these exceptional cases we need to control two things about sets $\mathcal{A}_N$. First, we need to know how dense they are in $\mathcal{U}_\epsilon(x_N)$, but also we need to control the variety of the α-frequencies of the elements in $\mathcal{A}_N$ (i.e., the α-frequencies of $\mathrm{dist}(x_0, x'_N)$).

It is worth mentioning the following difference from the sparse argument. As was explained at the beginning of Subsection 4.2, constructing an $\{\alpha, \beta\}$-regular cross section in the sparse case without the additional control on the distribution of α-intervals is much easier. This does not seem to be the case in the co-sparse

[8] $\mathcal{U}_\epsilon(x_N)$ denotes the ϵ-neighborhood of x_N, i.e., $\mathcal{U}_\epsilon(x_N) = (x_N - \epsilon, x_N + \epsilon)$.

piece of the argument. Controlling frequencies is a natural way of avoiding the aforementioned degenerate choices of the admissible shifts, and we are not aware of any substantial simplifications in the argument if one is not interested in the distribution of α-intervals in the Main Theorem.

Once the "propagation of freedom" lemma is available, the inductive construction continues in a way no different from the described step. We shall continue by selecting sufficiently distant pairs of adjacent rank 2 intervals and moving all the rank 2 blocks by at most ϵ_3 in an admissible way, i.e., in a way that results in moving each rank 1 block in between by at most ϵ_2 and each rank 0 point by no more than ϵ_1 and tiling all the resulting gaps, creating in this fashion rank 3 blocks. Since at each step of the construction we increase the size of the maximal block within any orbit, it is evident that in the limit the cross section will have arbitrarily large $\mathsf{E}^{\alpha,\beta}$-blocks within each orbit.

5. MODIFYING CROSS SECTIONS

Let $\mathcal{C}$ be a sparse cross section for a flow $\mathfrak{F}$. For any real K each $\mathsf{E}_{\mathcal{C}}^{\leqslant K}$-class is finite, and if $(K_n)_{n=0}^{\infty}$ is an increasing sequence of reals, $K_n \to \infty$, then $\mathsf{E}_{\mathcal{C}} = \bigcup_n \mathsf{E}_{\mathcal{C}}^{\leqslant K_n}$. In this section we present a construction of a cross section $\mathcal{C}$ such that for a given (K_n) each $\mathsf{E}_{\mathcal{C}}^{\leqslant K_{n+1}}$-class consists of at least two $\mathsf{E}_{\mathcal{C}}^{\leqslant K_n}$-classes.

We start with subsets $B(K_0, \ldots, K_n) \subset \mathbb{R}$ of the real line where each $\mathsf{E}^{\leqslant K_{n+1}}$-class consists of precisely two $\mathsf{E}^{\leqslant K_n}$-classes. For example, consider the subset $B(K_0, K_1, K_2, K_3)$ with the distances between adjacent points of the form $\frac{K_{n+1}+K_n}{2}$ depicted in Figure 10. It is easy to see that each $\mathsf{E}^{\leqslant K_n}$-class consists of two $\mathsf{E}^{\leqslant K_{n-1}}$-

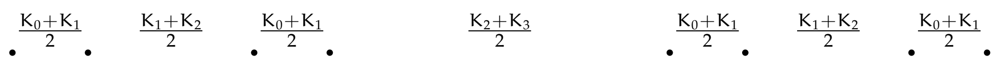


FIG. 10. The set $B(K_0, K_1, K_2, K_3)$. The unique $\mathsf{E}^{\leqslant K_3}$-class has two $\mathsf{E}^{\leqslant K_2}$-classes, each having two $\mathsf{E}^{\leqslant K_1}$-classes.

classes for n up to three. We shall modify a sparse cross section by adding sets of the form $B(K_0, \ldots, K_n)$ into sufficiently large gaps.

Now, in a more formal language, given a finite or infinite increasing sequence $(K_n)_{n=0}^{\infty}$ we define inductively finite subsets $B(K_0, \ldots, K_n) \subset \mathbb{R}^{\geqslant 0}$ and reals $b(K_0, \ldots, K_n) \in \mathbb{R}^{\geqslant 0}$ for $n \in \mathbb{N}$ as follows. For $n = 0$ let $B(K_0) = \{0\}$ and $b(K_0) = 0$, and set

$$B(K_0, \ldots, K_{n+1}) = B(K_0, \ldots, K_n) \cup \Big(B(K_0, \ldots, K_n) + b(K_0, \ldots, K_n) + \frac{K_{n+1} + K_n}{2}\Big),$$

$$b(K_0, \ldots, K_{n+1}) = 2b(K_0, \ldots, K_n) + \frac{K_{n+1} + K_n}{2}.$$

The number $b(K_0, \ldots, K_n)$ is just the length of the block $B(K_0, \ldots, K_n)$. Blocks $B(K_0, \ldots, K_n)$ also have the property that gaps between adjacent $\mathsf{E}^{\leqslant K_i}$-classes within a $\mathsf{E}^{\leqslant K_{i+1}}$-class are in the interior of the intervals $[K_i, K_{i+1}]$. This will imply stability of $\mathsf{E}^{\leqslant K_i}$-classes under small perturbations of points.

Lemma 5.1. *Let $\mathfrak{F}$ be a sparse flow on a standard Borel space Ω. For any sequence $(M_n)_{n=0}^{\infty}$ there is an increasing unbounded sequence $(N_n)_{n=0}^{\infty}$ and a sparse cross section $\mathcal{C} \subset \Omega$ such that*

(i) $\vec{\mathrm{gap}}_{\mathcal{C}}(x) > M_0$ *for all* $x \in \mathcal{C}$.
(ii) $N_n \geqslant M_n$ *for all* n *and* $N_0 = M_0$.
(iii) Each $\mathsf{E}_{\mathcal{C}}^{\leqslant N_{n+1}}$*-class consists of at least two* $\mathsf{E}_{\mathcal{C}}^{\leqslant N_n}$*-classes.*

Proof. Set $N_0 = M_0$ and

$$N_{n+1} = \max\big\{M_{n+1}, b(N_0, \ldots, N_n) + 2N_n + 2\big\}.$$

Let $\mathcal{D}_0$ be any sparse section such that $\vec{\mathrm{gap}}_{\mathcal{D}_0}(x) > M_0$ for all $x \in \mathcal{D}_0$ (it exists by Remark 2.4) and define $\mathcal{D}_n$ inductively by

$$\mathcal{D}_{n+1} = \mathcal{D}_n \cup \big\{\, x + N_n + 1 + B(N_0, \ldots, N_n) \;\big|\; x \in \mathcal{D}_n,\ x = \max[x]_{\mathsf{E}_{\mathcal{D}_n}^{\leqslant N_n}} \text{ and } [x]_{\mathsf{E}_{\mathcal{D}_n}^{\leqslant N_n}} = [x]_{\mathsf{E}_{\mathcal{D}_n}^{\leqslant N_{n+1}}} \big\}.$$

Here is a less formal definition. If an $\mathsf{E}_{\mathcal{D}_n}^{\leqslant N_n}$-class happens to coincide with the ambient $\mathsf{E}_{\mathcal{D}_n}^{\leqslant N_{n+1}}$-class, then the gap to the next $\mathsf{E}_{\mathcal{D}_n}^{\leqslant N_n}$-class is more than N_{n+1} and into this gap we insert a $B(N_0, \ldots, N_n)$ block at the distance $N_n + 1$ from the $\mathsf{E}_{\mathcal{D}_n}^{\leqslant N_n}$-class; see Figure 11.

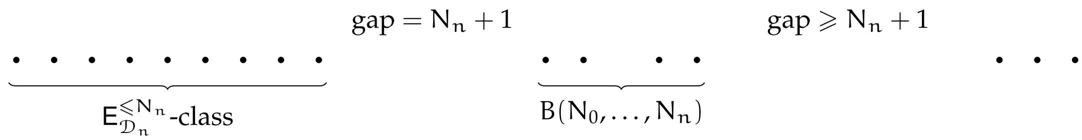


FIG. 11. Construction of $\mathcal{D}_{n+1}$. Adding a $B(N_0,\dots,N_n)$ block between two $E^{\leqslant N_n}_{\mathcal{D}_n}$-classes.

Cross sections $\mathcal{D}_n$ have the following properties.

- $\vec{\mathrm{gap}}_{\mathcal{D}_n}(x) > M_0$ for all $x \in \mathcal{D}_0$.
- $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$.
- $[x]_{E^{\leqslant N_k}_{\mathcal{D}_n}} = [x]_{E^{\leqslant N_k}_{\mathcal{D}_{n+1}}}$ for any $x \in \mathcal{D}_n$ and $k \leqslant n$. This is so, because the new blocks $B(N_0,\dots,N_n)$ that we added are at least $N_n + 1$ far from any point in $\mathcal{D}_n$, and therefore do not change $E^{\leqslant N_k}$-classes for $k \leqslant n$.
- Each block $B(N_0,\dots,N_n)$ added to $\mathcal{D}_n$ constitutes a single $E^{\leqslant N_n}_{\mathcal{D}_{n+1}}$-class.
- Each $E^{\leqslant N_{n+1}}_{\mathcal{D}_{n+1}}$-class consists of at least two $E^{\leqslant N_n}_{\mathcal{D}_{n+1}}$-classes. Indeed, consider an equivalence class $[x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_{n+1}}}$, $x \in \mathcal{D}_{n+1}$. If x belongs to some block $B(N_0,\dots,N_n)$ added during the construction of $\mathcal{D}_{n+1}$, then $[x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_{n+1}}}$ contains a block $B(N_0,\dots,N_n)$, which gives one $E^{\leqslant N_n}_{\mathcal{D}_{n+1}}$-class, and it also contains at least one $E^{\leqslant N_n}_{\mathcal{D}_{n+1}}$-class corresponding to a $E^{\leqslant N_n}_{\mathcal{D}_n}$-class. The same holds true if $x \in \mathcal{D}_{n+1}$ comes from $\mathcal{D}_n$ and $[x]_{E^{\leqslant N_n}_{\mathcal{D}_n}} = [x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_n}}$. Finally, if $x \in \mathcal{D}_n$, but $[x]_{E^{\leqslant N_n}_{\mathcal{D}_n}} \neq [x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_n}}$, then already $[x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_n}}$ consists of at least two $E^{\leqslant N_n}_{\mathcal{D}_n}$-classes, and therefore so does $[x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_{n+1}}}$, since $E^{\leqslant N_{n+1}}_{\mathcal{D}_{n+1}}$ restricted onto $\mathcal{D}_n$ is coarser than $E^{\leqslant N_{n+1}}_{\mathcal{D}_n}$.

One concludes that any $E^{\leqslant N_{k+1}}_{\mathcal{D}_{n+1}}$-class consists of at least two $E^{\leqslant N_k}_{\mathcal{D}_{n+1}}$-classes for $k \leqslant n$. Note also that all $\mathcal{D}_n$ are sparse cross sections. We claim that $\mathcal{C} = \bigcup_n \mathcal{D}_n$ is the desired cross section. To begin with, it is sparse, since blocks $B(N_0,\dots,N_n)$ have gaps of size at least N_{n-1}. Finally, any $E^{\leqslant N_{n+1}}_{\mathcal{C}}$-class consists of two or more $E^{\leqslant N_n}_{\mathcal{C}}$-classes. Indeed, take $x \in \mathcal{C}$ and let $m \geqslant n+1$ be such that $x \in \mathcal{D}_m$. We know that $[x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_m}}$ consists of at least two $E^{\leqslant N_n}_{\mathcal{D}_m}$-classes. Since $[x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_m}} = [x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_{m'}}}$ for any $m' \geqslant m$, it follows that $[x]_{E^{\leqslant N_{n+1}}_{\mathcal{D}_m}} = [x]_{E^{\leqslant N_{n+1}}_{\mathcal{C}}}$, and therefore $\mathcal{C}$ satisfies the conclusion of the lemma. □

Recall that for a real $x \in \mathbb{R}$ and $\epsilon > 0$ we let the ϵ-neighborhood $(x-\epsilon, x+\epsilon)$ of x to be denoted by $\mathcal{U}_\epsilon(x)$.

Lemma 5.2. *Let $\mathfrak{F}$ be a sparse flow on a standard Borel space. Given an $\epsilon > 0$ and a sequence of reals $(K_n)_{n=0}^{\infty}$, $K_{n+1} \geqslant K_n + 2\epsilon$, there exists a sparse cross section $\mathcal{C}$ such that*

(i) $E^{\leqslant K_0}_{\mathcal{C}}$ *is the trivial equivalence relation:* $x E^{\leqslant K_0}_{\mathcal{C}} y$ *if and only if* $x = y$*. In other words,* $\vec{\mathrm{gap}}_{\mathcal{C}}(x) > K_0$ *for all* $x \in \mathcal{C}$.
(ii) Any $E^{\leqslant K_{n+1}}_{\mathcal{C}}$*-class consists of at least two* $E^{\leqslant K_n}_{\mathcal{C}}$*-classes.*
(iii) Gaps between adjacent $E^{\leqslant K_n}_{\mathcal{C}}$*-classes within a* $E^{\leqslant K_{n+1}}_{\mathcal{C}}$*-class are in* $\mathcal{U}_\epsilon\big(\frac{K_n+K_{n+1}}{2}\big)$*:*

$$\textit{if } \vec{\mathrm{gap}}_{\mathcal{C}}(x) \in (K_n, K_{n+1}], \textit{ then } \vec{\mathrm{gap}}_{\mathcal{C}}(x) \in \mathcal{U}_\epsilon\Big(\frac{K_n + K_{n+1}}{2}\Big).$$

Proof. Let M_n be so large that into any gap of length at least M_n one can insert blocks $B(K_0,\dots,K_n)$ in such a way that the distance between two adjacent blocks is ϵ-close to $\frac{K_{n+1}+K_n}{2}$, see Figure 12. In other words, M_n is so large that one can find a Borel function $\xi_n : [M_n,\infty) \to \big[\mathbb{R}^{>0}\big]^{\mathrm{fin}}$ such that $\xi_n(z)$ corresponds to the set of left endpoint of blocks $B(K_0,\dots,K_n)$ in Figure 12, i.e., if $\xi_n(z) = \{y_1,\dots,y_m\}$ with $y_1 < y_2 < \dots < y_m$, then

- $y_1 \in \mathcal{U}_\epsilon\big(\frac{K_n+K_{n+1}}{2}\big)$;
- $y_{k+1} - y_k - b(K_0,\dots,K_n) \in \mathcal{U}_\epsilon\big(\frac{K_n+K_{n+1}}{2}\big)$ for all $k < m$;
- $z - y_m - b(K_0,\dots,K_n) \in \mathcal{U}_\epsilon\big(\frac{K_n+K_{n+1}}{2}\big)$.

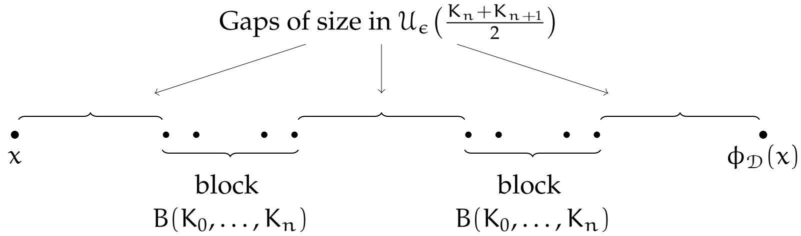


FIG. 12. Inserting blocks $B(K_0,\ldots,K_n)$ into the interval $\big[x,\phi_{\mathcal{D}}(x)\big]$.

A moment of thought will convince the reader that such an M_n exists.

By Lemma 5.1, we may find a sparse cross section $\mathcal{D}$, $\vec{\mathrm{gap}}_{\mathcal{D}}(x) > M_0$, and an increasing sequence $(N_n)_{n=0}^{\infty}$ such that $N_n \geqslant M_n$, $N_0 = M_0$, and each $E_{\mathcal{D}}^{\leqslant N_{n+1}}$-class has at least two $E_{\mathcal{D}}^{\leqslant N_n}$-classes. The cross section $\mathcal{C}$ is defined by

$$\mathcal{C} = \mathcal{D} \cup \Big\{\, x + r + B(K_0,\ldots,K_n) \;\Big|\; x \in \mathcal{D},\ \vec{\mathrm{gap}}_{\mathcal{D}}(x) \in (N_n, N_{n+1}],\ r \in \xi_n\big(\vec{\mathrm{gap}}_{\mathcal{D}}(x)\big) \,\Big\}.$$

For $x \in \mathcal{C}$ we need to show that $[x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}}$ consists of at least two $E_{\mathcal{C}}^{\leqslant K_n}$-classes. There are two cases.

Case I: $[x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}} \cap \mathcal{D} \neq \varnothing$. Let $y_1 = \min\big([x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}} \cap \mathcal{D}\big)$ and $y_2 = \max\big([x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}} \cap \mathcal{D}\big)$. Notice that:

- $\vec{\mathrm{gap}}_{\mathcal{D}}(y_2) > N_{n+1}$. Indeed, if $\vec{\mathrm{gap}}_{\mathcal{D}}(y_2) \leqslant N_{n+1}$, then by construction $y_2\, E_{\mathcal{C}}^{\leqslant K_{n+1}}\, \phi_{\mathcal{D}}(y_2)$, contradicting the choice of y_2. For a similar reason one also has
- $\vec{\mathrm{gap}}_{\mathcal{D}}\big(\phi_{\mathcal{D}}^{-1}(y_1)\big) > N_{n+1}$.
- $y_1 E_{\mathcal{D}}^{\leqslant N_{n+1}} y_2$. Suppose not. Let $y_1 \leqslant z < y_2$ be such that $\vec{\mathrm{gap}}_{\mathcal{D}}(z) > N_{n+1}$. By construction,

  $$\vec{\mathrm{gap}}_{\mathcal{C}}(z) \in \mathcal{U}_\epsilon\Big(\frac{K_{n+1}+K_{n+2}}{2}\Big).$$

  Since $K_{n+2} \geqslant K_{n+1} + 2\epsilon$, $\vec{\mathrm{gap}}_{\mathcal{C}}(z) > \frac{K_{n+1}+K_{n+1}+2\epsilon}{2} - \epsilon = K_{n+1}$, whence $\neg\big(y_1\, E_{\mathcal{C}}^{\leqslant K_{n+1}}\, y_2\big)$, contradicting the choice of y_1 and y_2.

The items above prove that

$$[x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}} \cap \mathcal{D} = [x]_{E_{\mathcal{D}}^{\leqslant N_{n+1}}} \text{ for all } n \in \mathbb{N} \text{ and } x \in \mathcal{D}.$$

By the choice of $\mathcal{D}$, there are some $x_1, x_2 \in [x]_{E_{\mathcal{D}}^{\leqslant N_{n+1}}}$ such that $[x_i]_{E_{\mathcal{D}}^{\leqslant N_n}} \subseteq [x]_{E_{\mathcal{D}}^{\leqslant N_{n+1}}}$ and $\neg\big(x_1 E_{\mathcal{D}}^{\leqslant N_n} x_2\big)$. But $[x_i]_{E_{\mathcal{D}}^{\leqslant N_n}} = [x_i]_{E_{\mathcal{C}}^{\leqslant K_n}} \cap \mathcal{D}$, therefore $[x_i]_{E_{\mathcal{C}}^{\leqslant K_n}}$, $i = 1,2$, are two distinct $E_{\mathcal{C}}^{\leqslant K_n}$-classes inside $[x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}}$.

Case II: $[x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}} \cap \mathcal{D} = \varnothing$. In this case x belongs to some block $B(K_0,\ldots,K_m)$ for some $m \geqslant n+1$ (if $m < n+1$, $[x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}} \cap \mathcal{D} \neq \varnothing$). Therefore, $[x]_{E_{\mathcal{C}}^{\leqslant K_{n+1}}}$ contains at least two $E_{\mathcal{C}}^{\leqslant K_n}$ classes by the properties of $B(K_0,\ldots,K_m)$.

To see (iii) we need to show that for any $x \in \mathcal{C}$, $\vec{\mathrm{gap}}_{\mathcal{C}}(x) \in \mathcal{U}_\epsilon(\frac{K_n+K_{n+1}}{2})$ for some n. If there is a block $B(K_0,\ldots,K_m)$ such that both x and $\phi_{\mathcal{C}}(x)$ belong to this block, then this is true by the properties of such a block. Otherwise, $\vec{\mathrm{gap}}_{\mathcal{C}}(x)$ corresponds to a gap between blocks, or between a point from $\mathcal{D}$ and a block. In both cases it is in $\mathcal{U}_\epsilon(\frac{K_n+K_{n+1}}{2})$ for some n by the choice of ξ_n. □

6. PROPAGATION OF FREEDOM

For the rest of the paper we fix two rationally independent reals $\alpha, \beta \in \mathbb{R}^{>0}$, $\alpha < \beta$, and a real $\rho \in (0,1)$.

6.1. **Tileable reals.**

Definition 6.1. A positive real number $x \in \mathbb{R}^{>0}$ is said to be **tileable** if there are $p, q \in \mathbb{N}$ such that $x = p\alpha + q\beta$. The term tileable signifies that an interval of length x can be tiled by intervals of length α and β. The set of tileable reals is denoted by $\mathcal{T}$. Note that $\mathcal{T}$ is just the subsemigroup of $\mathbb{R}$ generated by α and β.

Definition 6.2. Let $I \subseteq \mathbb{R}^{\geqslant 0}$ be a finite or infinite interval. Given $\epsilon > 0$, we say that a set $S \subseteq \mathbb{R}$ is **ϵ-dense** in I if it intersects any subinterval of I of length ϵ: $S \cap \mathcal{U}_{\epsilon/2}(x) \neq \varnothing$ for any $x \in I$ such that $\mathcal{U}_{\epsilon/2}(x) \subseteq I$.

A set $S \subseteq \mathbb{R}^{\geqslant 0}$ is **asymptotically dense** in $\mathbb{R}$ if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that S is ϵ-dense in $[N, \infty)$.

Remark 6.3. This definition of ϵ-density is equivalent up to a multiple of ϵ to an arguably more familiar

$$\forall x \in I\ \exists s \in S\ |x - s| < \epsilon.$$

Given a tileable $x \in \mathcal{T}$, $x = p\alpha + q\beta$, we define the **α-frequency** of x, denoted by $\mathrm{fr}_\alpha(x)$, to be the quotient $p/(p+q)$.

We shall routinely make use of the following observation. If $x, y \in \mathcal{T}$ satisfy $\mathrm{fr}_\alpha(x) \leqslant \rho$ and $\mathrm{fr}_\alpha(y) \leqslant \rho$, then $\mathrm{fr}_\alpha(x+y) \leqslant \rho$; moreover, if one of the inequalities in the assumption is strict, then $\mathrm{fr}_\alpha(x+y) < \rho$. Of course, a similar statement holds for all inequalities reversed.

Our first lemma quantifies the following rather obvious fact: if x and y are tileable reals, and if x is minuscule compared to y, then the α-frequency of $x + y$ is very close to the α-frequency of y.

Lemma 6.4. *Let $x, y \in \mathcal{T}$ be two tileable reals and let $\epsilon > 0$. If $\frac{x}{y} < \frac{\alpha\epsilon}{2\beta}$, then $\left|\mathrm{fr}_\alpha(x+y) - \mathrm{fr}_\alpha(y)\right| < \epsilon$.*

Proof. Let $x = p_x\alpha + q_x\beta$ and $y = p_y\alpha + q_y\beta$. Recall that $\alpha < \beta$ and therefore

$$\frac{\alpha\epsilon}{2\beta} > \frac{p_x\alpha + q_x\beta}{p_y\alpha + q_y\beta} \geqslant \frac{\alpha}{\beta} \cdot \frac{p_x + q_x}{p_y + q_y} \implies \frac{p_x + q_x}{p_y + q_y} < \frac{\epsilon}{2}.$$

Therefore also

$$\frac{p_x}{p_x + q_x + p_y + q_y} \leqslant \frac{p_x + q_x}{p_x + q_x + p_y + q_y} \leqslant \frac{p_x + q_x}{p_y + q_y} < \frac{\epsilon}{2}.$$

Since

$$\mathrm{fr}_\alpha(x+y) = \frac{p_x}{p_x + q_x + p_y + q_y} + \frac{p_y}{p_x + q_x + p_y + q_y},$$

we get

$$\begin{aligned}\left|\mathrm{fr}_\alpha(y) - \mathrm{fr}_\alpha(x+y)\right| &< \left|\frac{p_y}{p_y + q_y} - \frac{p_y}{p_x + q_x + p_y + q_y}\right| + \epsilon/2 \\ &= \frac{p_y}{p_y + q_y} \cdot \frac{p_x + q_x}{p_x + q_x + p_y + q_y} + \frac{\epsilon}{2} \\ &\leqslant \frac{p_x + q_x}{p_y + q_y} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

The lemma follows. □

Given an $\{\alpha, \beta\}$-regular cross section $\mathcal{C}$, we shall be interested in the frequency of occurrence of α-points within orbits. More precisely, we shall consider the sums of the form $\frac{1}{n}\sum_{k=0}^{n-1} \chi_{\mathcal{C}_\alpha}\big(\phi_{\mathcal{C}}^k(x)\big)$, where $x \in \mathcal{C}$ and $\chi_{\mathcal{C}_\alpha}$ is the characteristic function of $\mathcal{C}_\alpha$. Note that

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_{\mathcal{C}_\alpha}\big(\phi_{\mathcal{C}}^k(x)\big) = \mathrm{fr}_\alpha\Big(\mathrm{dist}\big(x, \phi_{\mathcal{C}}^n(x)\big)\Big).$$

Our next lemma gives a criterion of uniform convergence of these sums to $\rho \in (0, 1)$.

Lemma 6.5. *Let $\mathcal{C}$ be an $\{\alpha, \beta\}$-regular cross section. The following are equivalent.*

(i) For any $\eta > 0$ there is $N(\eta)$ such that

$$\Big|\rho - \frac{1}{n}\sum_{k=0}^{n-1} \chi_{\mathcal{C}_\alpha}\big(\phi_{\mathcal{C}}^k(x)\big)\Big| < \eta$$

for all $x \in \mathcal{C}$ and all $n \geqslant N(\eta)$.

(ii) For any $\eta > 0$ there is a bounded sub cross section $\mathcal{D}_\eta \subseteq \mathcal{C}$ with α-frequencies of gaps η-close to ρ: there exists $M(\eta) \in \mathbb{R}$ such that

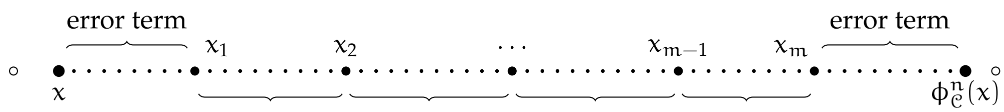
$$\vec{\mathrm{gap}}_{\mathcal{D}_\eta}(x) \leqslant M(\eta) \textit{ and } \left|\rho - \mathrm{fr}_\alpha\big(\vec{\mathrm{gap}}_{\mathcal{D}_\eta}(x)\big)\right| < \eta \quad \textit{holds for all } x \in \mathcal{D}_\eta.$$

Proof. It is clear that (i) $\implies$ (ii), since it is enough to take for $\mathcal{D}_\eta$ a bounded sub cross section with at least $N(\eta)$-many points from $\mathcal{C}$ between any two adjacent points in $\mathcal{D}_\eta$ (see Proposition 2.2). For instance, one can take $M(\eta) = \beta\big(N(\eta) + 2\big)$.

We need to show (ii) $\implies$ (i). Fix $\eta > 0$ and consider $\mathcal{D}_{\eta/2}$ and $M(\eta/2)$ guaranteed by (ii). In view of Lemma 6.4, there is $\widetilde{N} \in \mathbb{R}$ such that $|\mathrm{fr}_\alpha(z + x) - \mathrm{fr}_\alpha(z)| < \eta/2$ for all tileable $z \geqslant \widetilde{N} - 2M(\eta/2)$ and all $x \leqslant 2M(\eta/2)$, $x \in \mathcal{T}$. We claim that $N(\eta) = \widetilde{N}/\alpha$ works. Pick $x \in \mathcal{C}$ and some natural $n \geqslant N(\eta)$. Consider a block of $n+1$-points $x, \phi_{\mathcal{C}}(x), \ldots, \phi_{\mathcal{C}}^n(x)$. Let $k_1 < \cdots < k_m$ be all the indices for which $\phi_{\mathcal{C}}^{k_i}(x) \in \mathcal{D}_{\eta/2}$:

$$\{\phi_{\mathcal{C}}^i(x)\}_{i=0}^n \cap \mathcal{D}_{\eta/2} = \{\phi_{\mathcal{C}}^{k_i}(x)\}_{i=1}^m.$$

Set $x_i = \phi_{\mathcal{C}}^{k_i}(x)$.



Blocks between $\mathcal{D}_{\eta/2}$-points. Each block has length at most $M(\eta/2)$ and α-frequency $\eta/2$-close to ρ.

FIG. 13. $\mathcal{D}_{\eta/2}$-blocks within a segment of $N(\eta)$-many points.

Since $\vec{\mathrm{gap}}_{\mathcal{D}_{\eta/2}} \leqslant M(\eta/2)$, it follows that $\mathrm{dist}(x, x_1) \leqslant M(\eta/2)$ and $\mathrm{dist}\big(x_m, \phi_{\mathcal{C}}^n(x)\big) \leqslant M(\eta/2)$. Let $z = \mathrm{dist}(x_1, x_m)$. One has

$$z \geqslant \mathrm{dist}\big(x, \phi_{\mathcal{C}}^n(x)\big) - 2M(\eta/2) \geqslant n\alpha - 2M(\eta/2) \geqslant \widetilde{N} - 2M(\eta/2).$$

Let $y = \mathrm{dist}(x, x_1) + \mathrm{dist}\big(x_m, \phi_{\mathcal{C}}^n(x)\big)$ be the error term. By the choice of $\widetilde{N}$ we conclude that

$$|\mathrm{fr}_\alpha(z + y) - \mathrm{fr}_\alpha(z)| < \eta/2.$$

Since $|\mathrm{fr}_\alpha(z) - \rho| < \eta/2$, it follows that

$$\big|\mathrm{fr}_\alpha\big(\mathrm{dist}\big(x, \phi_{\mathcal{C}}^n(x)\big)\big) - \rho\big| < \eta/2 + \eta/2 = \eta. \qquad \square$$

6.2. **Propagation of Freedom.** Let $(d_k)_{k=1}^n$ be a sequence of positive reals, and let $\epsilon > 0$. Suppose that for each $1 \leqslant k \leqslant n$ we are given a set $R_k \subseteq \mathcal{U}_\epsilon(d_k)$. Define

$$\mathcal{A}_n = \mathcal{A}_n\big(\epsilon, (d_k)_{k=1}^n, (R_k)_{k=1}^n\big)$$

to consists of all $x \in \mathcal{U}_\epsilon\big(\sum_{k=1}^n d_k\big)$ for which there exist $y_k \in R_k$, $k \leqslant n$, such that $\big|\sum_{k=1}^r (d_k - y_k)\big| < \epsilon$ for all $r \leqslant n$, and $x = \sum_{k=1}^n y_k$. Properties of sets $\mathcal{A}_n$ are in the core of this section, so let us try to give a more geometrical explanation of the definition. Imagine a family of points $z_0, \ldots, z_n$ with $\mathrm{dist}(z_{k-1}, z_k) = d_k$, see Figure 14. We would like to move each z_k by at most ϵ to some new point z'_k so as to make distances

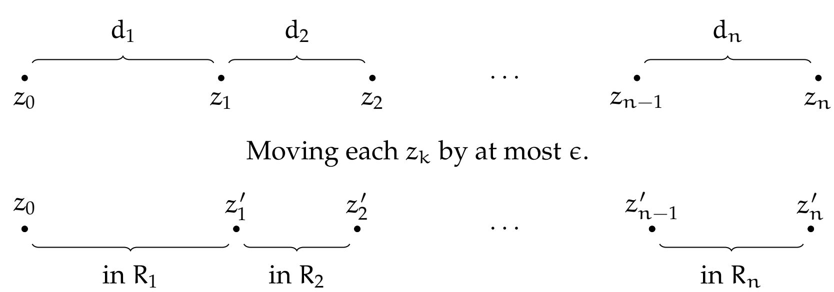


FIG. 14. The set $\mathcal{A}_n\big(\epsilon, (d_k), (R_k)\big)$ consists of all possible $x = \mathrm{dist}(z_0, z'_n)$.

from z'_{k-1} to z'_k lie in R_k. Let us say that a distance $\mathrm{dist}(z'_{k-1}, z'_k)$ is **admissible** if $\mathrm{dist}(z'_{k-1}, z'_k) \in R_k$. The leftmost point z_0 stays fixed, $z'_0 = z_0$. Given any $y_1 \in R_1$, we may take $z'_1 = z_0 + y_1$. Pick some $y_2 \in R_2$.

In general, $z_2' = z_1' + y_2$ may not work, since such z_2' may not be in the ϵ-neighborhood of z_2 anymore, but if, for example, R_2 contains elements both smaller and larger than $\mathrm{dist}(z_1, z_2)$, then one can always pick $y_2 \in R_2$ such that $z_2' = z_1' + y_2$ is within ϵ from z_2. Depending on the sets R_k, there may or may not be any ways to move points in the described way, but when R_k are diverse enough, there will necessarily be many such arrangements. We are specifically concerned with the set of possible positions for the last point z_n'. The set $\mathcal{A}_n$ is the collection of the possible distances from z_0 to z_n'.

The sets of admissible distances R_k will consist of tileable reals. All the elements in $\mathcal{A}_n$, being sums of elements of R_k, will therefore also be tileable. The main technical ingredient of our argument is the quantification of the richness of $\mathcal{A}_n$, in particular we expect the cardinality of $\mathcal{A}_n$ to grow with n and to pack densely the ϵ-neighborhood of z_n. The details are subtle. There are some degenerate ways to pick R_k and d_k resulting in the failure of the above expectations, with sets $\mathcal{A}_n$ not growing with n. The section thus concentrates on finding sufficient conditions to avoid such pathologies. As a matter of fact, we need to control two aspects of $\mathcal{A}_n$. First of all, we need to measure the density of $\mathcal{A}_n$ in $\mathcal{U}_\epsilon(z_n)$. Second, we need information about the diversity of α-frequencies of elements in $\mathcal{A}_n$. We are unable to do both items at once, but fortunately once $\mathcal{A}_n$ is known to be δ-dense, it will be easy to increase n improving the α-frequencies while keeping the δ-density because of the following Frequency Boost Lemma.

Lemma 6.6 (Frequency Boost)**.** *For any $D \in \mathbb{R}^{\geqslant 0}$ and $\zeta \in \mathbb{R}^{>0}$ there is $M = M_{\mathrm{Lem\,6.6}}(D, \zeta)$ such that for any $n \geqslant M$, any $\eta \in \mathbb{R}^{>0}$, any $\gamma \in (\rho - \eta, \rho + \eta)$, any $0 < \epsilon \leqslant 1$, any family of reals $(d_k)_{k=1}^n$ satisfying $2\epsilon + \alpha \leqslant d_k \leqslant D$, and any sequence of sets $R_k \subseteq \mathcal{U}_\epsilon(d_k) \cap \mathcal{T}$ such that $R_k \cap \mathrm{fr}_\alpha^{-1}[0, \rho - \eta]$ and $R_k \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta, 1]$ are ϵ-dense in $\mathcal{U}_\epsilon(d_k)$, there is $x \in \mathcal{A}_n$ such that $|\mathrm{fr}_\alpha(x) - \gamma| < \zeta$.*

Here is a less formal statement of the lemma. In the context of Figure 14, we want to move points z_k in such a way so that the distance $\mathrm{dist}(z_0, z_n')$ will have the α-frequency ζ-close to γ, where γ is somewhere in $(\rho - \eta, \rho + \eta)$. Suppose sets R_k allow us to move z_k at our choice to the left or to the right in a way that, again at our choice, adds a real of the α-frequency at least $\rho + \eta$ or at most $\rho - \eta$. The lemma claims that under this assumption one can always find such an $x \in \mathcal{A}_n$ provided that n is sufficiently large in terms of the bound on gaps D and precision ζ. The proof is similar to the proof of the Riemann Rearrangement Theorem.

Proof. We construct M as $M = M_1 + M_2$. The integer M_1 will be picked so large that after M_1-steps the frequency does not change by more than ζ, and M_2 will ensure that we have enough intervals to move the frequency by $\eta - \zeta$. We take

$$M_1 > \frac{D+2}{\zeta} \cdot \frac{2\beta}{\alpha^2} \quad \text{and} \quad M_2 > \frac{M_1 D + 2}{\zeta} \cdot \frac{2\beta}{\alpha^2}.$$

Note that if $(y_k)_{k=1}^n$ is any sequence such that $\left|\sum_{k=1}^r (d_k - y_k)\right| < \epsilon$ for all $r \leqslant n$, and $y_k \in \mathcal{T}$, $k \leqslant n$, where $n \geqslant M_1$, then by Lemma 6.4 and the choice of M_1, for any $m \in \mathbb{N}$ satisfying $M_1 \leqslant m < n$, the contribution of y_{m+1} to the frequency of $\sum_{k=1}^{m+1} y_k$ is small:

$$\left|\mathrm{fr}_\alpha\Big(\sum_{k=1}^{m+1} y_k\Big) - \mathrm{fr}_\alpha\Big(\sum_{k=1}^{m} y_k\Big)\right| < \zeta. \tag{1}$$

We construct the required $x \in \mathcal{A}_n$ as follows. For y_1 pick any element of R_1. Suppose y_k has been constructed. Take $y_{k+1} \in R_{k+1}$ satisfying the following two conditions:

- if $\sum_{i=1}^k y_i \leqslant \sum_{i=1}^k d_i$, then $y_{k+1} \geqslant d_{k+1}$; if $\sum_{i=1}^k y_i > \sum_{i=1}^k d_i$, then $y_{k+1} \leqslant d_{k+1}$;
- if $\mathrm{fr}_\alpha(\sum_{i=1}^k y_i) \leqslant \gamma$, then $\mathrm{fr}_\alpha(y_{k+1}) \geqslant \rho + \eta$; if $\mathrm{fr}_\alpha(\sum_{i=1}^k y_i) > \gamma$, then $\mathrm{fr}_\alpha(y_{k+1}) \leqslant \rho - \eta$.

The possibility to choose such y_{k+1} is guaranteed by the assumption that

$$R_k \cap \mathrm{fr}_\alpha^{-1}[0, \rho - \eta] \text{ and } R_k \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta, 1] \text{ are } \epsilon\text{-dense in } \mathcal{U}_\epsilon(d_k).$$

It is obvious that $\sum_{k=1}^r y_k \in \mathcal{A}_r$ for all r. We claim that $\left|\mathrm{fr}_\alpha(\sum_{i=1}^n y_i) - \gamma\right| \leqslant \zeta$ for all $n \geqslant M$.

Claim 1. If $m_0 \geqslant M_1$ is such that the α-frequency jumps over γ when passing from $\sum_{i=1}^{m_0} y_i$ to $\sum_{i=1}^{m_0+1} y_i$, i.e., if

$$\mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0} y_i\Big) \leqslant \gamma \leqslant \mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0+1} y_i\Big) \quad \text{or}$$

$$\mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0} y_i\Big) \geqslant \gamma \geqslant \mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0+1} y_i\Big),$$

then $\big|\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0} y_i\big) - \gamma\big| < \zeta$ and $\big|\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0+1} y_i\big) - \gamma\big| < \zeta$. Indeed, the claim follows easily from the inequality (1), which gives

$$\Big|\mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0+1} y_i\Big) - \mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0} y_i\Big)\Big| < \zeta.$$

Claim 2. If there is $m_0 \geqslant M_1$ such that $\big|\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0} y_i\big) - \gamma\big| < \zeta$, then $\big|\mathrm{fr}_\alpha\big(\sum_{i=1}^{m} y_i\big) - \gamma\big| < \zeta$ for all $m \geqslant m_0$.

It is enough to show that the assumption $\big|\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0} y_i\big) - \gamma\big| < \zeta$ implies the same inequality with m_0+1 instead of m_0: $\big|\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0+1} y_i\big) - \gamma\big| < \zeta$; the claim will then follow by induction. To this end assume for definiteness that $\mathrm{fr}_\alpha(\sum_{i=1}^{m_0} y_i) \leqslant \gamma$ (the case $\mathrm{fr}_\alpha(\sum_{i=1}^{m_0} y_i) > \gamma$ is similar) . The construction of y_{m_0+1} shows that $\mathrm{fr}_\alpha(y_{m_0+1}) \geqslant \rho + \eta > \gamma$, and therefore

$$\mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0+1} y_i\Big) \geqslant \mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0} y_i\Big).$$

If $\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0+1} y_i\big) \leqslant \gamma$, then the conclusion follows from the assumption $\big|\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0} y_i\big) - \gamma\big| < \zeta$. If $\mathrm{fr}_\alpha\big(\sum_{i=1}^{m_0+1} y_i\big) > \gamma$, then

$$\mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0} y_i\Big) \leqslant \gamma < \mathrm{fr}_\alpha\Big(\sum_{i=1}^{m_0+1} y_i\Big),$$

and therefore the claim follows from Claim 1.

In view of Claim 1 and Claim 2, if there is an index $M_1 \leqslant m_0 \leqslant M$ such that the α-frequency jumps over γ between $\sum_{i=1}^{m_0} y_i$ and $\sum_{i=1}^{m_0+1} y_i$, then the lemma is proved. We may therefore assume that there is no such m_0 and all the α-frequencies are on the same side of γ, for instance, let us assume that $\mathrm{fr}_\alpha(\sum_{i=1}^{m} y_i) < \gamma$ for all $M_1 \leqslant m \leqslant M$. The construction of y_i ensures that $\mathrm{fr}_\alpha(y_i) \geqslant \rho + \eta$ for all $M_1 < i \leqslant M$. Set $\tilde{x} = \sum_{i=1}^{M_1} y_i$ and $\tilde{y} = \sum_{i=M_1+1}^{M} y_i$. Note that $\tilde{x} < M_1 D + 2$, $\tilde{y} \geqslant M_2\alpha$, and $\mathrm{fr}_\alpha(\tilde{y}) \geqslant \rho + \eta$. The choice of M_2 and Lemma 6.4 imply that $\big|\mathrm{fr}_\alpha(\tilde{x} + \tilde{y}) - \mathrm{fr}_\alpha(\tilde{y})\big| < \zeta$. Since we have established that

$$\mathrm{fr}_\alpha\Big(\sum_{i=1}^{M} y_i\Big) = \mathrm{fr}_\alpha(\tilde{x} + \tilde{y}) < \gamma \leqslant \rho + \eta \leqslant \mathrm{fr}_\alpha(\tilde{y}),$$

one concludes that $\big|\mathrm{fr}_\alpha(\tilde{x} + \tilde{y}) - \gamma\big| < \zeta$, and the lemma is finished by Claim 2. □

6.3. **Moving points: Constant case.** In this subsection we study the behavior of sets

$$\mathcal{A}_n\big(\epsilon, (d_k)_{k=1}^n, (R_k)_{k=1}^n\big)$$

with the constant parameters: $d_i = d_j$ and $R_i = R_j$ for all $1 \leqslant i, j \leqslant n$. We let $d = d_k$, $R = R_k$ and $\mathcal{A}_n\big(\epsilon, (d)_{k=1}^n, (R)_{k=1}^n\big)$ will be denoted simply by $\mathcal{A}_n$.

Lemma 6.7. *Sets $\mathcal{A}_n$ have the following additivity properties.*

(i) If $y_i \in R$, $1 \leqslant i \leqslant n$, are such that

$$\Big|nd - \sum_{i=1}^{n} y_i\Big| < \epsilon,$$

then $\sum_{i=1}^{n} y_i \in \mathcal{A}_n$.

(ii) *If* $x_i \in \mathcal{A}_{n_i}$, $1 \leqslant i \leqslant k$, *are such that*

$$\Big|\sum_{i=1}^{k}(x_i - n_i d)\Big| < \epsilon,$$

then $\sum_{i=1}^{k} x_i \in \mathcal{A}_{\sum_{i=1}^{k} n_i}$.

(iii) *If* $d \in \mathbb{R}$ *and* $m \leqslant n$, *then* $\mathcal{A}_m + (n-m)d \subseteq \mathcal{A}_n$.

Proof. (i) We need to show that there exists a permutation

$$\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

such that

$$\Big|kd - \sum_{i=1}^{k} y_{\pi(i)}\Big| < \epsilon \quad \text{for all } k \leqslant n.$$

Set $\pi(1) = 1$, and define $\pi(k+1)$ inductively as follows. Let

$$P = \{y_i : i \neq \pi(j) \text{ for any } 1 \leqslant j \leqslant k\}$$

to denote the set of y_i's that we haven't used yet. If $\sum_{i=1}^{k} y_{\pi(i)} \leqslant kd$, we search for the smallest $i \in P$ such that $y_i \geqslant d$ and set $\pi(k+1) = i$; if no such $i \in P$ exists, we set $\pi(k+1) = \min P$. Similarly, if $\sum_{i=1}^{k} y_{\pi(i)} > kd$, we search for the smallest $i \in P$ such that $y_i \leqslant d$ and set $\pi(k+1) = i$; if no such $i \in P$ exists, we set $\pi(k+1) = \min P$. We claim that such a permutation π satisfies

$$\Big|kd - \sum_{i=1}^{k} y_{\pi(i)}\Big| < \epsilon \quad \text{for all } k \leqslant n.$$

For $k = 1$ the inequality holds trivially. Suppose it holds for k. We check it for $k+1$, and let us assume for definiteness that $\sum_{i=1}^{k} y_{\pi(i)} \leqslant kd$. We have two cases depending on how $\pi(k+1)$ was selected. If $y_{\pi(k+1)} \geqslant d$, then

$$\Big|(k+1)d - \sum_{i=1}^{k+1} y_{\pi(i)}\Big| = \Big|d - y_{\pi(k+1)} + kd - \sum_{i=1}^{k} y_{\pi(i)}\Big| < \epsilon,$$

because

$$-\epsilon < d - y_{\pi(k+1)} \leqslant 0 \quad \text{and} \quad 0 \leqslant kd - \sum_{i=1}^{k} y_{\pi(i)} < \epsilon.$$

So, it remains to consider the case when $y_i < d$ for all $i \in P$. In this case

$$(k+1)d - \sum_{i=1}^{k+1} y_{\pi(i)} > 0.$$

Let $P' = P \setminus \{\pi(k+1)\} = P \setminus \min P$. We have

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_{\pi(i)} = \sum_{i=1}^{k+1} y_{\pi(i)} + \sum_{j \in P'} y_j$$

and $\sum_{j \in P'} y_j < |P'|d = (n-k-1)d$. If

$$(k+1)d - \sum_{i=1}^{k+1} y_{\pi(i)} \geqslant \epsilon,$$

then

$$nd - \sum_{i=1}^{n} y_i = (k+1)d - \sum_{i=1}^{k+1} y_{\pi(i)} + (n-k-1)d - \sum_{j \in P'} y_j \geqslant \epsilon,$$

contradicting the assumption $\left|nd - \sum_{i=1}^{n} y_i\right| < \epsilon$, so we have to have

$$\left|(k+1)d - \sum_{i=1}^{k+1} y_{\pi(i)}\right| < \epsilon$$

as claimed.

(ii) Since $x_i \in \mathcal{A}_{n_i}$, there are y_j^i, $1 \leqslant j \leqslant n_i$, such that

- $y_j^i \in R$;
- $\left|rd - \sum_{j=1}^{r} y_j^i\right| < \epsilon$ for all $r \leqslant n_i$;
- $\sum_{j=1}^{n_i} y_j^i = x_i$.

By (i), $\sum_{i,j} y_j^i \in \mathcal{A}_{\sum_{i=1}^{k} n_i}$, and the item follows.

(iii) Follows from item (ii), since $d \in R$ and thus $(n-m)d \in \mathcal{A}_{n-m}$ for $n > m$. □

The additivity of sets $\mathcal{A}_n$ is a luxury we have in the case of constant parameters. Given $x, y \in \mathcal{A}_m$ such that $x < md < y$ we may therefore start forming sums of x and y, e.g.,

$$x + y + y + x + y + \cdots,$$

as long as any initial segment of this sum is ϵ-close to the multiple of d. By the above lemma, all such sums will lie in $\mathcal{A}_n$ for the relevant n. The next lemma quantifies the density of $\mathcal{A}_n$ in $\mathcal{U}_\epsilon(nd)$ that can be achieved by forming such combinations. Note that if $d \in R$, then $\mathcal{A}_n + d \subseteq \mathcal{A}_{n+1}$, and therefore the degree of density cannot decrease: if $\mathcal{A}_n$ is δ-dense in $\mathcal{U}_\epsilon(nd)$, then $\mathcal{A}_{n+1}$ is δ-dense in $\mathcal{U}_\epsilon(nd + d)$.

For two reals $a, b \in \mathbb{R}$ we let $\gcd(a, b)$ denote the greatest positive real c such that both a and b are integer multiples of c. If a and b are rationally independent, we take $\gcd(a, b) = 0$.

Lemma 6.8. *Let $\epsilon > 0$, let $0 < \delta \leqslant \epsilon$, and let $x, y \in \mathcal{A}_m$, $m \geqslant 1$, be given. Set $a = x - md$ and $b = y - md$. Suppose that $d \in R$, and $a < 0 < b$. There exists $N = N_{\text{Lem 6.8}}(R, m, \epsilon, \delta, d, x, y)$ such that for all $n \geqslant N$*

- *if $\delta > \gcd(a, b)$, then the set $\mathcal{A}_n$ is δ-dense in $\mathcal{U}_\epsilon(nd)$;*
- *if $\delta \leqslant \gcd(a, b)$, then the set $\mathcal{A}_n$ is κ-dense in $\mathcal{U}_\epsilon(nd)$ for any $\kappa > \gcd(a, b)$ and moreover*

$$nd + k\gcd(a, b) \in \mathcal{A}_n \text{ for all integers } k \text{ such that } nd + k\gcd(a, b) \in \mathcal{U}_\epsilon(nd).$$

Proof. Starting with $a_0 = a$ and $b_0 = b$ we define for $k \geqslant 1$

$$a_k = a_{k-1} + l_k b_{k-1},$$
$$b_k = b_{k-1} + l'_k a_k,$$

where l_k is the largest natural number such that $a_{k-1} + l_k b_{k-1} \leqslant 0$, and l'_k is the largest natural satisfying $b_{k-1} + l'_k a_k \geqslant 0$. If $a_k = 0$, we take $l'_k = 0$. If either $a_k = 0$ or $b_k = 0$, we stop constructing the sequence. Here are a few things to note.

- $a_k \leqslant 0$ and $b_k \geqslant 0$ for all k.
- If $a_k = 0$, but $b_{k-1} \neq 0$, then $b_{k-1} = \gcd(a, b)$. Similarly, if $b_k = 0$, but $a_k \neq 0$, then $a_k = -\gcd(a, b)$.
- If $a_k \neq 0$, then $b_k \leqslant 1/2 \cdot b_{k-1}$ and $|a_k| \leqslant b_{k-1}$; in particular sequences (a_k) and (b_k) converge to 0 exponentially fast.

We take $K = K(\delta, a, b)$ to be the minimal natural such that one of a_K, b_K is 0 or $\max\{|a_K|, b_K\} < \delta$.

Claim. There is $\tilde{N} \in \mathbb{N}$ such that $a_K + \tilde{N}d$ and $b_K + \tilde{N}d \in \mathcal{A}_{\tilde{N}}$.

Proof of claim. Unraveling the formulas for a_k and b_k one checks that there are naturals $q, q', p, p' \in \mathbb{N}$ such that

$$a_K = pa + qb,$$
$$b_K = p'a + q'b. \tag{2}$$

From item (ii) of Lemma 6.7 we know that

$$a_K + (p+q)md = p(a + md) + q(b + md) = px + qy \in \mathcal{A}_{(p+q)m},$$
$$b_K + (p' + q')md = p'x + q'y \in \mathcal{A}_{(p'+q')m}.$$

Therefore by Lemma 6.7(iii) we may take $\tilde{N} = m \cdot \max\{p+q, p'+q'\}$. $\square_{\text{claim}}$

The proof now splits into three cases.

Case I: $a_K < 0$ and $b_K > 0$. In this case $\delta > \gcd(a, b)$. Let N_a and N_b be such that

$$-\epsilon < N_a a_K \leqslant -\epsilon + |a_K| \text{ and } \epsilon - b_K \leqslant N_b b_K < \epsilon,$$

and set $N = \max\{N_a, N_b\} \cdot \tilde{N}$. Again, by Lemma 6.7 and Claim for any $n \geqslant N$

- $ka_K + nd \in \mathcal{A}_n$ for all $0 \leqslant k \leqslant N_a$;
- $kb_K + nd \in \mathcal{A}_n$ for all $0 \leqslant k \leqslant N_b$.

Therefore $\mathcal{A}_n$ is δ-dense in $\mathcal{U}_\epsilon(nd)$, since $|a_K| < \delta$ and $b_K < \delta$, see Figure 15.

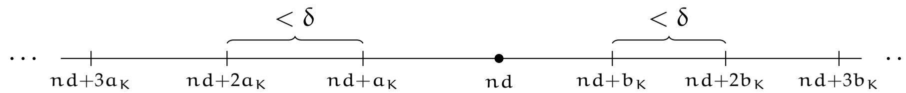


FIG. 15. Some elements of $\mathcal{A}_n$ witnessing its δ-density in $\mathcal{U}_\epsilon(nd)$.

Case II: $a_K = 0$. Note that by the choice of K, $b_{K-1} \neq 0$. We let $c = b_{K-1} = \gcd(a, b)$. Since $b_{K-1} = b_K = c$, we have $c + \tilde{N}d \in \mathcal{A}_{\tilde{N}}$, where $\tilde{N}$ is given by Claim. Thus also $c + l_K\tilde{N}d \in \mathcal{A}_{l_K\tilde{N}}$. By increasing $\tilde{N}$ if necessary, we also have $a_{K-1} + \tilde{N}d \in \mathcal{A}_{\tilde{N}}$ and $b_{K-1} + \tilde{N}d \in \mathcal{A}_{\tilde{N}}$, so by Lemma 6.7

$$-c + l_K\tilde{N}d = a_K - b_{K-1} + l_K\tilde{N}d = a_{K-1} + (l_K - 1)b_{K-1} + l_K\tilde{N}d \in \mathcal{A}_{l_K\tilde{N}}.$$

Therefore both $l_K\tilde{N}d + c$ and $l_K\tilde{N}d - c$ are elements of $\mathcal{A}_{l_K\tilde{N}}$.

Finally, let N_c be such that $\epsilon - c \leqslant N_c c < \epsilon$ and take $N = N_c l_K \tilde{N}$. Then

$$nd + kc \in \mathcal{A}_n \text{ for all } n \geqslant N_c l_K \tilde{N} \text{ and all } -N_c \leqslant k \leqslant N_c.$$

The natural $N = N_c l_K \tilde{N}$ satisfies the conclusion of the lemma.

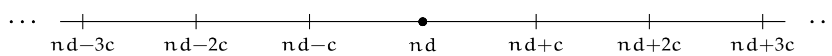


FIG. 16. Some elements of $\mathcal{A}_n$ witnessing its κ-density in $\mathcal{U}_\epsilon(nd)$ for $\kappa > \gcd(a, b) = c$.

Case III: $a_K \neq 0$, but $b_K = 0$ is treated similarly by taking $c = -a_K$. $\square$

From now on the set $R \subseteq \mathcal{U}_\epsilon(d)$ is assumed to consist of tileable reals $R \subseteq \mathcal{T}$. Note that given $d \in \mathbb{R}^{>0}$ the set of possible choices for $R \subseteq \mathcal{U}_\epsilon(d) \cap \mathcal{T}$ is finite.

Lemma 6.9. *Let $\eta > 0$. For any N there exists $M = M_{\text{Lem 6.9}}(d, \eta)$ such that for any $R \subseteq \mathcal{U}_\epsilon(d) \cap \mathcal{T}$ for which $R \cap \mathrm{fr}_\alpha^{-1}[0, \rho - \eta]$ and $R \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta, 1]$ are ϵ-dense in $\mathcal{U}_\epsilon(d)$ and for all $n \geqslant M$ sets $\mathcal{A}_n$ have at least N-many elements: $|\mathcal{A}_n| \geqslant N$.*

Proof. Set $\gamma_i = i\eta/N$ for $-N < i < N$ and let $\zeta = \eta/2N$. By Lemma 6.6 there exists $M = M_{\text{Lem 6.6}}(d, \zeta)$ so large that for any $n \geqslant M$ and any $-N < i < N$ there is $x_i \in \mathcal{A}_n$ such that $\left|\mathrm{fr}_\alpha(x_i) - \gamma_i\right| < \zeta$. Since $|\gamma_i - \gamma_j| \geqslant 2\zeta$ for $i \neq j$, elements x_i must be pairwise distinct, implying $|\mathcal{A}_n| \geqslant 2N - 1 \geqslant N$. $\square$

Lemma 6.10. *For any tileable $d > 0$, any $0 < \eta \leqslant 1$, $0 < \epsilon \leqslant 1$, and $0 < \delta \leqslant \epsilon$ there exists $M = M_{\text{Lem 6.10}}(d, \epsilon, \delta, \eta)$ such that for any $n \geqslant M$ and any tileable family $R \subseteq \mathcal{U}_\epsilon(d) \cap \mathcal{T}$ which satisfies*

- *$d \in R$;*
- *$R \cap \mathrm{fr}_\alpha^{-1}[0, \rho - \eta]$ and $R \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta, 1]$ are ϵ-dense in $\mathcal{U}_\epsilon(d)$;*

the set $\mathcal{A}_n = \mathcal{A}_n\big(\epsilon, (d)_{k=1}^n, (R)_{k=1}^n\big)$ is δ-dense in $\mathcal{U}_\epsilon(nd)$.

Proof. By Lemma 6.9 we may find M_1 such that

$$\left|\mathcal{A}_{M_1}\big(\epsilon, (d)_1^{M_1}, (R)_1^{M_1}\big)\right| \geqslant 2 + 4\epsilon/\delta$$

for all R satisfying the assumptions. We may therefore pick non-zero $y_1, y_2 \in \mathcal{A}_{M_1}$ such that $|y_1 - y_2| < \delta/2$, and $y_1 - M_1 d$, $y_2 - M_1 d$ have same signs. Let $z \in \mathcal{A}_{M_1}$ be any element such that $z - M_1 d$ has the opposite sign. Applying Lemma 6.8 to y_1 and z we can find $M_2 \geqslant M_1$ such that

- either $\mathcal{A}_n$ is δ-dense in $\mathcal{U}_\epsilon(nd)$ for any $n \geqslant M_2$, or
- $\mathcal{A}_n$ is $2c$-dense in $\mathcal{U}_\epsilon(nd)$ for any $n \geqslant M_2$, where

$$c = \gcd(y_1 - M_1 d, z - M_1 d),$$

  and moreover $nd - c, nd + c \in \mathcal{A}_n$.

Note that there are only finitely many possibilities for the choice of $y_1, y_2, z \in \mathcal{A}_{M_1}$, which let us choose M_2 so large as to work for all of them at the same time. In the first case we are done. Suppose the latter holds. Note that $y_2 + (M_2 - M_1)d \in \mathcal{A}_{M_2}$, and depending on whether $y_2 - M_1 d < 0$ or $y_2 - M_1 d > 0$, apply Lemma 6.8 to $y_2 + (M_2 - M_1)d$ and $M_2 d + c$ or to $y_2 + (M_2 - M_1)d$ and $M_2 d - c$. Again, M_3 can be picked so large as to work for all possible y_2, c, and R at the same time. We claim that the resulting M_3 works, because $c' = \gcd(y_2 - M_1 d, c) < \delta$, which guarantees hat $\mathcal{A}_n$ is δ-dense in $\mathcal{U}_\epsilon(nd)$ for all $n \geqslant M_3$. Indeed, since $y_1 - M_1 d$ is a multiple of c, both $y_1 - M_1 d$ and $y_2 - M_1 d$ are integer multiples of c', whence $0 < |y_1 - y_2| < \delta$ implies $c' < \delta$. □

6.4. **Moving points: General case.** We are now back to the situation of general sequences (d_k), (R_k) with possibly distinct d_k and R_k. Our first lemma is a close analog of Lemma 6.10 where (d_k) and (R_k) are not necessarily constant.

Lemma 6.11. *For any $D \in \mathbb{R}^{\geqslant 0}$, any $0 < \epsilon \leqslant 1$, $0 < \delta \leqslant \epsilon$, $0 < \eta \leqslant 1$, there is $M = M_{\mathrm{Lem}\,6.11}(D, \epsilon, \delta, \eta)$ such that for any $n \geqslant M$, any sequence $(d_k)_{k=1}^n$ of positive reals, $2\epsilon + \alpha \leqslant d_k \leqslant D$, any sequence $R_k \subseteq \mathcal{U}_\epsilon(d_k) \cap \mathcal{T}$ satisfying*

- *$d_k \in R_k$;*
- *$R_k \cap \mathrm{fr}_\alpha^{-1}[0, \rho - \eta]$ and $R_k \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta, 1]$ are ϵ-dense in $\mathcal{U}_\epsilon(d_k)$;*

the set $\mathcal{A}_n$ is δ-dense in $\mathcal{U}_\epsilon(\sum_{k=1}^n d_k)$.

Proof. The set $\mathcal{T} \cap (0, D]$ is finite and thus the set of possible choices of $R_k \subseteq \mathcal{U}_\epsilon(d_k) \cap \mathcal{T}$ for each k is bounded above by $2^{|\mathcal{T} \cap [0, D+1]|}$. Set

$$M = \big|\mathcal{T} \cap [0, D]\big| \cdot 2^{|\mathcal{T} \cap [0, D+1]|} \cdot \max_{\substack{d \in \mathcal{T} \\ d \leqslant D}} M_{\mathrm{Lem}\,6.10}(d, \epsilon, \delta, \eta).$$

Let $n \geqslant M$, and let d_k, $k \leqslant n$, be given. By the choice of M and the pigeon-hole principle there are indices $k_1 < k_2 < \cdots < k_N$ such that for all $i, j \leqslant N$

$$d_{k_j} = d_{k_i} =: d, \qquad R_{k_j} = R_{k_i} =: R,$$

and $N \geqslant M_{\mathrm{Lem}\,6.10}(d, \epsilon, \delta, \eta)$.

It is helpful to go back to Figure 14 for a moment. Our situation now is special in the following aspects. First of all distances $d_k \in R_k$, so once z'_{k-1} has been picked we may always set $z'_k = z'_{k-1} + d_k$, which will never violate the condition $\mathrm{dist}(z_k, z'_k) < \epsilon$. Also, we have a large collection of indices k_i, $i \leqslant N$, where all the gaps d_{k_i} and sets of admissible distances R_{k_i} are the same. The natural N is so large that Lemma 6.10 applies and the set $\mathcal{A}_N\big(\epsilon, (d)_{i=1}^N, (R)_{i=1}^N\big)$ is δ-dense in $\mathcal{U}_\epsilon(Nd)$. Recall that $x \in \mathcal{A}_N\big(\epsilon, (d)_{i=1}^N, (R)_{i=1}^N\big)$ means that there exist $y_i \in R$, $i \leqslant N$, such that $x = \sum_{i=1}^N y_i$ and $\big|\sum_{i=1}^r (d - y_i)\big| < \epsilon$ for all $r \leqslant N$. Any such x naturally corresponds to an element $\tilde{x} \in \mathcal{A}_n\big(\epsilon, (d_k)_{k=1}^n, (R_k)_{k=1}^n\big)$ defined by $\tilde{x} = \sum_{k=1}^n \tilde{y}_k$ where

$$\tilde{y}_k = \begin{cases} y_i & \text{if } k = k_i, \\ d_k & \text{otherwise.} \end{cases}$$

Since

$$\tilde{x} - \sum_{k=1}^n \tilde{y}_k = x - \sum_{i=1}^N y_i,$$

the set $\mathcal{A}_N\big(\epsilon, (d)_{i=1}^N, (R)_{i=1}^N\big)$ is δ-dense in $\mathcal{U}_\epsilon(Nd)$ if and only if $\mathcal{A}_n\big(\epsilon, (d_k)_{k=1}^n, (R_k)_{k=1}^n\big)$ is δ-dense in the set $\mathcal{U}_\epsilon\big(\sum_{k=1}^n d_k\big)$. The lemma follows. □

We are finally ready to prove the main technical result of this section. The following lemma will supply the step of induction in the proof of Theorem 9.1. Lemma 6.12 is a formalization of the "Propagation of Freedom" principle to which we alluded earlier in this section. There are two differences from Lemma 6.11. First, we no longer assume that $d_k \in R_k$. And second, the conclusion is stronger, because subsets of $\mathcal{A}_n$ of elements having prescribed α-frequencies are claimed to be δ-dense.

Lemma 6.12 (Co-sparse Induction Step)**.** *For any $D \in \mathbb{R}^{\geqslant 0}$, any $\epsilon, \delta, \eta, \nu, \nu' > 0$, where $0 < \delta \leqslant \epsilon$ and $0 \leqslant \nu' < \nu \leqslant \eta$, there exists $M = M_{\text{Lem 6.12}}(D, \epsilon, \delta, \eta, \nu, \nu') \in \mathbb{N}$ such that for any $n \geqslant M$, any sequence of reals $(d_k)_{k=1}^n$, and any sequence of tileable families $R_k \subseteq \mathcal{U}_\epsilon(d_k) \cap \mathcal{T}$, $1 \leqslant k \leqslant n$, satisfying*

(i) $2\epsilon + \alpha \leqslant d_k \leqslant D$;
(ii) $R_k \cap \mathrm{fr}_\alpha^{-1}[0, \rho - \eta]$ *and* $R_k \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta, 1]$ *are $\epsilon/6$-dense in* $\mathcal{U}_\epsilon(d_k)$;

sets $\mathcal{A}_n \cap \mathrm{fr}_\alpha^{-1}[\rho - \nu, \rho - \nu']$ and $\mathcal{A}_n \cap \mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$ are δ-dense in $\mathcal{U}_{\epsilon/2}(\sum_{k=1}^n d_k)$.

Proof. We construct $M = M_1 + M_2$ as a sum of two numbers: using the first M_1 segments we achieve the δ-density, and then use M_2 intervals to push frequencies to $[\rho - \nu, \rho - \nu']$ and $[\rho + \nu', \rho + \nu]$ intervals. Let $M_1 = M_{\text{Lem 6.11}}(D + 2, \epsilon, \delta, \eta)$, $\zeta = \frac{\nu - \nu'}{6}$, and

$$M_2' = M_{\text{Lem 6.6}}(D + 1, \zeta)$$
$$M_2'' = \left\lceil \frac{2\beta}{\alpha} \cdot \frac{M_1 D + 1}{\zeta} \right\rceil$$
$$M_2 = \max\{M_2', M_2''\}.$$

We claim that $M = M_1 + M_2$ works. Let $n \geqslant M$ be given. For a sequence d_k and R_k as in the assumptions we argue as follows. Similarly to the proof of Lemma 6.6, construct inductively $\tilde{d}_k$ for $k \leqslant M_1$ such that

(a) $|d_k - \tilde{d}_k| < \epsilon/6$;
(b) $|\sum_{k=1}^r (d_k - \tilde{d}_k)| < \epsilon/6$ for all $r \leqslant M_1$.

Set $\widetilde{R}_k = R_k \cap \mathcal{U}_{5\epsilon/6}(\tilde{d}_k)$ and note that

$$\mathcal{A}_{M_1}\big(5\epsilon/6, (\tilde{d}_k)_{k=1}^{M_1}, (\widetilde{R}_k)_{k=1}^{M_1}\big) \subseteq \mathcal{A}_{M_1}\big(\epsilon, (d_k)_{k=1}^{M_1}, (R_k)_{k=1}^{M_1}\big)$$

because for any element in $\mathcal{A}_{M_1}\big(5\epsilon/6, (\tilde{d}_k)_{k=1}^{M_1}, (\widetilde{R}_k)_{k=1}^{M_1}\big)$ each shift made by $\tilde{d}_k$ is also an admissible shift for d_k. Whence $\mathcal{A}_{M_1}\big(5\epsilon/6, (\tilde{d}_k)_{k=1}^{M_1}, (\widetilde{R}_k)_{k=1}^{M_1}\big)$ is δ-dense in $\mathcal{U}_{5\epsilon/6}\big(\sum_{k=1}^{M_1} \tilde{d}_k\big)$ by the choice of M_1. It follows that

$$\mathcal{A}_{M_1}\big(\epsilon, (d_k)_{k=1}^{M_1}, (R_k)_{k=1}^{M_1}\big) \text{ is } \delta\text{-dense in } \mathcal{U}_{4\epsilon/6}\Big(\sum_{k=1}^{M_1} d_k\Big).$$

We now improve the frequency while keeping the δ-density by constructing elements y_1, y_2 such that for $\mathcal{A}_{M_1}$ constructed above each element in $\mathcal{A}_{M_1} + y_1$ will have α-frequency in $[\rho - \nu, \rho - \nu']$ and each one in $\mathcal{A}_{M_1} + y_2$ will have frequency from $[\rho + \nu', \rho + \nu]$. Moreover, both $\mathcal{A}_{M_1} + y_1$ and $\mathcal{A}_{M_1} + y_2$ will be subsets of $\mathcal{A}_n\big(\epsilon, (d_k)_{k=1}^n, (R_k)_{k=1}^n\big)$, and the lemma will therefore be proved. Here are the details of the construction of y_1 and y_2.

Since $M_2 \geqslant M_2'$, we can apply the conclusion of Lemma 6.6 to the sequence $(d_k)_{k=M_1+1}^n$,

$$\bar{R}_k = R_k \cap \mathcal{U}_{\epsilon/6}(d_k),$$

$\gamma_1 = \rho + \frac{\nu + \nu'}{2}$, $\gamma_2 = \rho - \frac{\nu + \nu'}{2}$, and $\zeta = \frac{\nu - \nu'}{6}$. This yields elements

$$y_1, y_2 \in \mathcal{A}_n\big(\epsilon/6, (d_k)_{k=M_1+1}^n, (\bar{R}_k)_{k=M_1+1}^n\big)$$

such that

$$\big|\mathrm{fr}_\alpha(y_j) - \gamma_j\big| < \zeta \quad \text{for } j = 1, 2.$$

Observe that

$$\mathcal{A}_{M_1}\big(\epsilon, (d_k)_{k=1}^{M_1}, (R_k)_{k=1}^{M_1}\big) \cap \mathcal{U}_{5\epsilon/6}\Big(\sum_{k=1}^{M_1} d_k\Big) + y_j \subset \mathcal{A}_n\big(\epsilon, (d_k)_{k=1}^n, (R_k)_{k=1}^n\big) \quad \text{for } j = 1, 2$$

and these sets are δ-dense in $\mathcal{U}_{\epsilon/2}\big(\sum_{k=1}^{n} d_k\big)$. Finally, since $M_2 \geqslant M_2''$ from Lemma 6.4 it follows that

$$\big|\mathrm{fr}_\alpha(z) - \mathrm{fr}_\alpha(y_j)\big| < \zeta \quad \text{for any } z \in \mathcal{A}_{M_1}\big(\epsilon, (d_k)_{k=1}^{M_1}, (R_k)_{k=1}^{M_1}\big) \cap \mathcal{U}_{5\epsilon/6}\Big(\sum_{k=1}^{M_1} d_k\Big) + y_j,$$

which implies that $\mathrm{fr}_\alpha(z) \in [\rho - \nu, \rho - \nu']$ for $j = 1$ and $\mathrm{fr}_\alpha(z) \in [\rho + \nu', \rho + \nu]$ when $j = 2$. □

7. TILED REALS

Throughout this section $\bar{\eta} = (\eta_k)_{k=0}^{\infty}$ denotes a strictly decreasing sequence of positive reals converging to zero; we assume that $\eta_0 = 1$ and $\eta_1 \leqslant \min\{\rho, 1-\rho\}$. Symbol $\bar{L} = (L_k)_{k=0}^{\infty}$, with possible superscripts, will denote a strictly increasing unbounded sequence $(L_k)_{k=0}^{\infty}$ with the agreement that $L_0 = \beta$.

A **tiled** real is a number $z \in \mathcal{T} \cup \{0\}$ together with a sequence $(\sigma_i)_{i=1}^{n}$, called the **decomposition of** z, such that $\sigma_i \in \{\alpha, \beta\}$ and $\sum_{i=1}^{n} \sigma_i = z$. One should think of a tiled real as an interval which has been cut into pieces of length α and β. The set of tiled reals is denoted by $\mathfrak{T}$. By convention, $0 \in \mathfrak{T}$. Given two tiled reals $z_1, z_2 \in \mathfrak{T}$, we define their sum $z_1 + z_2 \in \mathfrak{T}$ in the natural way: if $(\sigma_i^{(1)})_{i=1}^{n_1}$ is the decomposition of z_1 and $(\sigma_i^{(2)})_{i=1}^{n_2}$ decomposes z_2, then $z_1 + z_2$ is partitioned by $(\sigma_i)_{i=1}^{n_1+n_2}$

$$\sigma_k = \begin{cases} \sigma_k^{(1)} & \text{if } k \leqslant n_1 \\ \sigma_{k-n_1}^{(2)} & \text{if } n_1 < k \leqslant n_1 + n_2. \end{cases}$$

Note that the addition of tiled reals is associative, but not commutative. If $\mathcal{C}$ is an $\{\alpha, \beta\}$-regular cross section, then any of its segments between a pair of points within an orbit naturally corresponds to a tiled real.

We shall need to control two parameters of a tiled real — its length and its α-frequency. Note that any tiled real can be viewed as a tileable one by forgetting the decomposition, and so each tiled real has an α-frequency associated to it. We introduce the following sets: given $\eta > 0$ and real $L > 0$

$$B_\eta[L] = \big\{ z \in \mathfrak{T} \;\big|\; z \leqslant L,\ |\mathrm{fr}_\alpha(z) - \rho| \leqslant \eta \big\},$$

$$SB_\eta[L] = \Big\{ z \in \mathfrak{T} \;\Big|\; z = \sum_{i=1}^{r} y_i,\ y_i \in B_\eta[L],\ r \in \mathbb{Z}^{>0} \Big\},$$

The frequency of zero, $\mathrm{fr}_\alpha(0)$, is undefined, but by our convention $0 \in B_\eta[L]$ for all L and all η. Note that $SB_\eta[L]$ is just the semigroup generated by $B_\eta[L]$. The following properties are immediate from the definitions.

Proposition 7.1. *Sets of the form* $B_\eta[L]$ *and* $SB_\eta[L]$ *satisfy the following:*

(i) $B_\eta[L] \subseteq SB_\eta[L]$.
(ii) $|\mathrm{fr}_\alpha(z) - \rho| \leqslant \eta$ *for all* $z \in SB_\eta[L] \setminus \{0\}$.
(iii) *If* $z_1, z_2 \in SB_\eta[L]$, *then also* $z_1 + z_2 \in SB_\eta[L]$.
(iv) *If* $\eta' \geqslant \eta$ *and* $L' \geqslant L$, *then* $B_\eta[L] \subseteq B_{\eta'}[L']$ *and* $SB_\eta[L] \subseteq SB_{\eta'}[L']$.

Definition 7.2. A sequence $\bar{L}$ is said to be $\bar{\eta}$**-flexible** if for any $n \in \mathbb{N}$, any $0 \leqslant \nu' < \nu \leqslant \eta_{n+1}$, the sets[9]

$$\bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho - \nu, \rho - \nu'] \text{ and } \bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$$

are asymptotically dense in $\mathbb{R}^{>0}$.

The definition of an $\bar{\eta}$-flexible sequence says that a large real can be shifted slightly to a real which for any $0 \leqslant k \leqslant n$ can be cut into pieces from $B_{\eta_k}[L_k]$ and moreover the α-frequency of the whole interval is very close to any given number in $[\rho - \eta_{k+1}, \rho + \eta_{k+1}]$. In Lemma 7.4 below, we show that any $\bar{\eta}$ admits an $\bar{\eta}$-flexible sequence $\bar{L}$. The construction of $\bar{L}$ is inductive, and the base step amounts to checking that

$$SB_{\eta_0}[L_0] \cap \mathrm{fr}_\alpha^{-1}[\rho - \nu, \rho - \nu'] \text{ and } SB_{\eta_0}[L_0] \cap \mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$$

[9] Recall that we assume that $\eta_1 \leqslant \min\{\rho, 1-\rho\}$, which ensures that both $[\rho - \nu, \rho - \nu']$ and $[\rho + \nu', \rho + \nu]$ are subintervals of $[0, 1]$.

are asymptotically dense in $\mathbb{R}^{>0}$. Since we always assume that $\eta_0 = 1$, and $L_0 = \beta$, one has $\{\alpha, \beta\} \subseteq B_{\eta_0}[L_0]$, and therefore $SB_{\eta_0}[L_0] = \mathfrak{T}$. The base case is thus covered by the following lemma.

Lemma 7.3. *Let $0 \leqslant \nu' < \nu$ be given.*

(i) If $\nu \leqslant 1 - \rho$, then the set $\mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$ is asymptotically dense in $\mathbb{R}$.
(ii) If $\nu \leqslant \rho$, then the set $\mathrm{fr}_\alpha^{-1}[\rho - \nu, \rho - \nu']$ is asymptotically dense in $\mathbb{R}$.

Proof. We give a proof of (i), item (ii) is proved similarly. Pick $\epsilon > 0$. We show that $\mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$ is ϵ-dense in $[N, \infty)$ for sufficiently large N. Since $\rho + (\nu + \nu')/2 < 1$, we may pick $x \in \mathcal{T}$ such that

$$\left|\rho + \frac{\nu + \nu'}{2} - \mathrm{fr}_\alpha(x)\right| < \zeta \quad \text{where } \zeta = \frac{\nu - \nu'}{4}.$$

Since α and β are rationally independent, the group $\langle \alpha, \beta \rangle$ is dense in $\mathbb{R}$, so we may pick $s_1, \ldots, s_n \in \langle \alpha, \beta \rangle$ such that $\{s_1, \ldots, s_n\}$ is $\epsilon/2$-dense in $[0, x]$. Let

$$s_i = p_i \alpha + q_i \beta$$

and set

$$m = \max\bigl\{|p_i|, |q_i| : i \leqslant n\bigr\}(\alpha + \beta).$$

The set $\{s_i + m : i \leqslant n\}$ therefore consists of tileable reals and is $\epsilon/2$-dense in $[m, m + x]$. Using Lemma 6.4 we may find K so big that for all $k \geqslant K$

$$\bigl|\mathrm{fr}_\alpha(kx + s_i + m) - \mathrm{fr}_\alpha(kx)\bigr| < \zeta \quad \text{for all } i \leqslant n.$$

Therefore $\mathrm{fr}_\alpha(kx + s_i + m) \in [\rho + \nu', \rho + \nu]$ for all $i \leqslant n$, and the set

$$\bigl\{kx + s_i + m : i \leqslant n, k \geqslant K\bigr\} \quad \text{is } \epsilon\text{-dense in } [Kx + m, \infty). \qquad \square$$

Lemma 7.4. *For any $\bar{\eta}$ there exists an $\bar{\eta}$-flexible sequence $\bar{L}$.*

Proof. The sequence $(L_n)_{n=0}^{\infty}$ is constructed inductively, we ensure at the n^{th} stage of the construction that

$$\bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho - \nu, \rho - \nu'] \quad \text{and} \quad \bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$$

are asymptotically dense in $\mathbb{R}$ for any choice of $0 \leqslant \nu' < \nu \leqslant \eta_{n+1}$. The base case $n = 0$ is covered by Lemma 7.3.

Suppose we have constructed $(L_k)_{k=0}^{n}$. By the inductive assumption we may choose $N \in \mathbb{R}$, such that both

$$\bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}\bigl[\rho - \eta_{n+1}, \rho - \eta_{n+2}] \text{ and } \bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta_{n+2}, \rho + \eta_{n+1}] \tag{3}$$

are 1/6-dense in $[N, \infty)$. We take

$$L_{n+1} = \max\bigl\{L_n + 1, 2(N + 2)\bigr\}.$$

and claim that

$$\bigcap_{k=0}^{n+1} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho - \nu, \rho - \nu'] \quad \text{and} \quad \bigcap_{k=0}^{n+1} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$$

are asymptotically dense in $\mathbb{R}$ for any choice of $0 \leqslant \nu' < \nu \leqslant \eta_{n+2}$. Let $0 \leqslant \nu' < \nu \leqslant \eta_{n+2}$ be given and pick $\epsilon \leqslant 1/2$. We show that

$$\bigcap_{k=0}^{n+1} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu]$$

is ϵ-dense in $[\widetilde{N}, \infty)$ for sufficiently large $\widetilde{N}$. The argument for the ϵ-density of

$$\bigcap_{k=0}^{n+1} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho - \nu, \rho - \nu']$$

is similar.

Let d be any tileable real satisfying $N+1 \leqslant d \leqslant N+2$ and $d \in \bigcap_{k=0}^{n} SB_{\eta_k}[L_k]$ which exists by (3); set

$$R = \bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathcal{U}_1(d) \cap \mathrm{fr}_\alpha^{-1}[\rho - \eta_{k+1}, \rho + \eta_{k+1}].$$

Note that by the choice of N and (3), both

$$R \cap \mathrm{fr}_\alpha^{-1}[0, \rho - \eta_{n+2}] \text{ and } R \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta_{n+2}, 1]$$

are 1/6-dense in $\mathcal{U}_1(d)$ and $R \subseteq B_{\eta_{n+1}}[L_{n+1}]$. Pick some $\tilde{\nu}, \tilde{\nu}'$ such that $0 \leqslant \nu' < \tilde{\nu}' < \tilde{\nu} < \nu \leqslant \eta_{n+2}$. By Lemma 6.12 there is

$$M = M_{\text{Lem 6.12}}(d, 1, \epsilon, \eta_{n+2}, \tilde{\nu}, \tilde{\nu}')$$

such that for any $m \geqslant M$ the set

$$\mathcal{A}_m\big(1, (d)_1^m, (R)_1^m\big) \cap \mathrm{fr}_\alpha^{-1}[\rho + \tilde{\nu}', \rho + \tilde{\nu}]$$

is ϵ-dense in $\mathcal{U}_{1/2}(md)$. Any element of $\mathcal{A}_m$ is a sum of elements in R, so by item (iii) of Proposition 7.1, this implies that

$$\bigcap_{k=0}^{n+1} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \tilde{\nu}', \rho + \tilde{\nu}] \text{ is } \epsilon\text{-dense in } \mathcal{U}_{1/2}(md). \tag{4}$$

Take $\widetilde{N} = (\widetilde{M}+1)d$ where $\widetilde{M} \geqslant M$ is so big that

$$\mathrm{fr}_\alpha(z) \in [\rho + \tilde{\nu}', \rho + \tilde{\nu}] \implies \mathrm{fr}_\alpha(z + x) \in [\rho + \nu', \rho + \nu] \tag{5}$$

holds for all tileable $z \geqslant \widetilde{M}N$ and all tileable $x \leqslant 2N + 4$ (cf. Lemma 6.4).

We show that $\widetilde{N}$ works. Let $z \in [\widetilde{N}, \infty)$ and let $m \geqslant \widetilde{M}$ be the unique natural such that

$$z \in \big[(m+1)d, (m+2)d\big).$$

By the choice of N and (3) there is some

$$y \in \bigcap_{k=0}^{n} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \eta_{n+2}, \rho + \eta_{n+1}] \cap (0, 2N+4) \subseteq B_{\eta_{n+1}}[L_{n+1}]$$

such that $|z - y - md| < 1/6$, so $z - y \in \mathcal{U}_{1/2}(md)$. Finally, since $\bigcap_{k=0}^{n+1} SB_k[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \tilde{\nu}', \rho + \tilde{\nu}]$ is ϵ-dense in $\mathcal{U}_{1/2}(md)$ by (4) there is some

$$x \in \bigcap_{k=0}^{n+1} SB_k[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \tilde{\nu}', \rho + \tilde{\nu}]$$

such that $|z - y - x| < \epsilon$. Using (5) we conclude that

$$y + x \in \bigcap_{k=0}^{n+1} SB_{\eta_k}[L_k] \cap \mathrm{fr}_\alpha^{-1}[\rho + \nu', \rho + \nu].$$

Since z was arbitrary, the lemma follows. □

Definition 7.5. We say that $z \in \mathfrak{T}$ is $(\bar{\eta}, \bar{L})$**-tiled** if $z \in SB_{\eta_n}[L_n]$ whenever $z \geqslant L_n$. Note that if $\bar{L}'$ is such that $L'_n \geqslant L_n$ for all n, then any z which is $(\bar{\eta}, \bar{L})$-tiled is also $(\bar{\eta}, \bar{L}')$-tiled.

For any given z the property of being $(\bar{\eta}, \bar{L})$-tiled depends only on a finite segment of $\bar{L}$, because the condition is vacuous for n such that $L_n > z$. On the other hand we may refer to a family of tiled reals as being $(\bar{\eta}, \bar{L})$-tiled, meaning that each member of the family is $(\bar{\eta}, \bar{L})$-tiled, and in that case all the elements of the sequence $\bar{L}$ are substantial as soon as the family has arbitrarily large tiled reals.

Lemma 7.6. *Let $\bar{\eta}$, $\bar{L}$ be given. For any $m_0 \in \mathbb{N}$, any $n_0 \in \mathbb{N}$, $n_0 \geqslant 2$, and any $K \in \mathbb{R}^{>0}$ there is $\bar{L}' = \bar{L}'_{\text{Lem 7.6}}(\bar{\eta}, \bar{L}, m_0, n_0, K)$ such that*

(i) $L'_k = L_k$ *for* $k \leqslant m_0$*;*
(ii) $L'_k \geqslant L_k$ *for* $k \in \mathbb{N}$*;*

and for any $z_1, \dots, z_{n_0}, y_1, \dots, y_{n_0-1} \in \mathfrak{T}$, *satisfying*

- $y_i \leqslant K$;
- $z_i, y_i \in SB_{\eta_k}[L_k]$ *for all* $k \leqslant m_0$;
- z_i *are* $(\bar\eta, \bar L)$*-tiled;*

the sum

$$J = z_1 + y_1 + z_2 + y_2 + \cdots + z_{n_0-1} + y_{n_0-1} + z_{n_0}$$

is $(\bar\eta, \bar L')$*-tiled and* $J \in SB_{\eta_k}[L'_k]$ *for all* $k \leqslant m_0$.

Here is a more verbose explanation of the statement. The case $n_0 = 3$ is shown in Figure 17. Numbers y_1

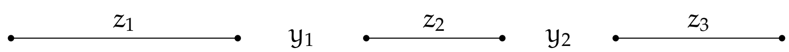


FIG. 17.

and y_2 are known to be bounded from above by K, but z_i's may be arbitrarily large. We know that z_i and y_j are all elements of $SB_{\eta_k}[L_k]$ for $k \leqslant m_0$, so their sum J is also an element of $SB_{\eta_k}[L_k]$. We assume that z_i's are $(\bar\eta, \bar L)$-flexible, i.e., if z_i is particularly long, $z_i \geqslant L_n$, then it can be partitioned into pieces of size at most L_n each having α-frequency η_n-close to ρ. The sum is not necessarily $(\bar\eta, \bar L)$-flexible, but since n_0 is fixed, for the sum to be large, at least one of the terms z_i has to be large, and the lemma claims that all tiled reals J of such form will necessarily be $(\bar\eta, \bar L')$-flexible with respect to a larger sequence $\bar L'$.

Proof. We start with the case $n_0 = 2$, and therefore $J = z_1 + y_1 + z_2$. For $i \leqslant m_0$ we put $L'_i = L_i$ and define L'_k for $k > m_0$ by induction as follows. Suppose L'_{k-1} has been defined. Let $n_1 \geqslant k$ be so large that

- $L_{n_1} \geqslant L'_{k-1}$;
- for any tileable $x \geqslant L_{n_1}$ and any tileable $\tilde z \leqslant K + L_k$ if x is η_{n_1}-close to ρ, then

$$|\mathrm{fr}_\alpha(x + \tilde z) - \rho| \leqslant \eta_k. \tag{6}$$

The possibility to choose such n_1 is based on Lemma 6.4.

Set $L'_k = 4L_{n_1} + K$.

We now check that the sequence $\bar L'$ constructed this way satisfies the conclusions of the lemma for $n_0 = 2$. Suppose we are given z_1, y_1, and z_2. Since $z_1, y_1, z_2 \in SB_{\eta_k}[L_k]$, $k \leqslant m_0$, it is immediate to see that $J \in SB_{\eta_k}[L'_k]$ for all $k \leqslant m_0$. We show that for $k > m_0$ one has $J \in SB_{\eta_k}[L'_k]$ whenever $J \geqslant L'_k$. If $J \geqslant L'_k$, then $z_1 \geqslant 2L_{n_1}$ or $z_2 \geqslant 2L_{n_1}$. Suppose for definiteness that $z_1 \geqslant 2L_{n_1}$. Since z_1 is $(\bar\eta, \bar L)$-tiled this implies that $z_1 = \sum_{i=1}^{r_1} x_i$ with $x_i \in B_{\eta_{n_1}}[L_{n_1}]$. Note that $x_i \in B_{\eta_k}[L'_k]$ for all i, because $L_{n_1} \leqslant L'_k$ and $k \leqslant n_1$. We now have two cases.

Case 1: $z_2 < L_k$. In this case let j be the largest index $j \leqslant r_1$ such that

$$x_j + x_{j+1} + \cdots + x_{r_1} \geqslant L_{n_1}.$$

and note that since $x_i \leqslant L_{n_1}$ we have to have

$$L_{n_1} \leqslant x_j + x_{j+1} + \cdots + x_{r_1} \leqslant 2L_{n_1}.$$

Since $2L_{n_1} + K + L_k \leqslant L'_k$, using $y_1 + z_2 \leqslant K + L_k$,

$$x_j + \cdots + x_{r_1} \in SB_{\eta_{n_1}}[L_{n_1}],$$

and equation (6) we get

$$x_j + x_{j+1} + \cdots + x_{r_1} + y_1 + z_2 \in B_{\eta_k}[L'_k]$$

and therefore $J \in SB_{\eta_k}[L'_k]$ is witnessed by the decomposition

$$J = x_1 + x_2 + \cdots + x_{j-1} + (x_j + x_{j+1} + \cdots + x_{r_1} + y_1 + z_1)$$

in which every summand is an element of $B_{\eta_k}[L'_k]$.

Case 2: $z_2 \geqslant L_k$. There is a decomposition $z_2 = \sum_{i=1}^{r_2} x'_i$ with $x'_i \in B_{\eta_k}[L_k]$ as z_2 is $(\bar\eta, \bar L)$-tiled. We therefore have $x'_i \in B_{\eta_k}[L'_k]$ and if, as in previous case, $j \leqslant r_1$ is the largest index such that

$$x_j + x_{j+1} + \cdots + x_{r_1} \geqslant L_{n_1},$$

then $x_j + x_{j+1} + \cdots + x_{r_1} + y_1 \in B_{\eta_k}[L'_k]$ by (6). Thus

$$J = x_1 + x_2 + \cdots + x_{j-1} + (x_j + x_{j+1} + \cdots + x_{r_1} + y_1) + x'_1 + \cdots + x'_{r_2}$$

with each summand being an element of $B_{\eta_k}[L'_k]$. This proves the lemma for $n_0 = 2$.

For $n_0 > 2$ the lemma follows easily by induction. Suppose the lemma has been proved for $n_0 - 1$ and the sequence $\bar{L}'$ has been constructed. Apply this lemma to $\bar{L}'$ and $n_0 = 2$ to get $\bar{L}'' = \bar{L}'_{\text{Lem 7.6}}(\bar{\eta}, \bar{L}', m_0, 2, K)$. We claim that $\bar{L}''$ works for n_0. Indeed, suppose z_i, y_i are given. By inductive assumption

$$J' = z_1 + y_1 + z_2 + y_2 + \cdots + y_{n_0-2} + z_{n_0-1}$$

is $(\bar{\eta}, \bar{L}')$-tiled. Note that z_{n_0} by assumption is $(\bar{\eta}, \bar{L})$-tiled and is therefore also $(\bar{\eta}, \bar{L}')$-tiled. By the choice of $\bar{L}''$, $J = J' + y_{n_0-1} + z_{n_0}$ must be $(\bar{\eta}, \bar{L}'')$-tiled and the lemma follows. □

Note that $z_i = 0$ is allowed in this lemma. The case $z_1 = 0$ is used in the proof of the Lemma 7.12.

Remark 7.7. A simple observation is that if $\bar{L}'$ satisfies the conclusion of Lemma 7.6, then any larger sequence (which also starts with L_k, $k \leqslant m_0$) will do so as well. This implies the following immediate strengthening: given $\bar{\eta}$, $\bar{L}$, m_0, n_0, and K as above there exists $\bar{L}'$, $L'_k = L_k$ for $k \leqslant m_0$, such that all elements of the form

$$z_1 + y_1 + z_2 + y_2 + \cdots + y_{\tilde{n}-1} + z_{\tilde{n}},$$

for **all** $\tilde{n} \leqslant n_0$ are $(\bar{\eta}, \bar{L}')$-tiled.

Definition 7.8. Let x and y be tileable reals. We say that $\mathrm{fr}_\alpha(x)$ and $\mathrm{fr}_\alpha(y)$ are **on the same side** of ρ if either $\big(\mathrm{fr}_\alpha(x) \leqslant \rho$ and $\mathrm{fr}_\alpha(y) \leqslant \rho\big)$ or $\big(\mathrm{fr}_\alpha(x) \geqslant \rho$ and $\mathrm{fr}_\alpha(y) \geqslant \rho\big)$. Similarly, we refer to $\mathrm{fr}_\alpha(x)$ and $\mathrm{fr}_\alpha(y)$ as being **on the different sides** of ρ if either $\big(\mathrm{fr}_\alpha(x) \leqslant \rho$ and $\mathrm{fr}_\alpha(y) \geqslant \rho\big)$ or $\big(\mathrm{fr}_\alpha(x) \geqslant \rho$ and $\mathrm{fr}_\alpha(y) \leqslant \rho\big)$. Note that these notions are not direct negations of each other, since we use non-strict inequalities.

We say that y **flips** x if $\mathrm{fr}_\alpha(x)$ and $\mathrm{fr}_\alpha(x+y)$ are on the different sides of ρ. Finally, a tileable $z \in \mathcal{T}$ is said to be N-**near** ρ, $N \in \mathbb{N}$, if either $N\alpha$ or $N\beta$ flips z.

Unraveling the definitions, $z = p\alpha + q\beta$ is N-near ρ if one of the two possibilities holds:

- $\mathrm{fr}_\alpha(z) \leqslant \rho$ and $\mathrm{fr}_\alpha(z + N\alpha) = \frac{p+N}{p+q+N} \geqslant \rho$;
- $\mathrm{fr}_\alpha(z) \geqslant \rho$ and $\mathrm{fr}_\alpha(z + N\beta) = \frac{p}{p+q+N} \leqslant \rho$.

In plain words it means that adding N tiles of the right type brings the α-frequency to the other side of ρ. Note that the two items above are essentially mutually exclusive — only reals $z \in \mathcal{T}$ with α-frequency $\mathrm{fr}_\alpha(z) = \rho$ satisfy both of them. All such z are 0-near ρ.

It is convenient to note that $z \in \mathcal{T}$ is N-near ρ if and only if *both* $\mathrm{fr}_\alpha(z + N\alpha) \geqslant \rho$ and $\mathrm{fr}_\alpha(z + N\beta) \leqslant \rho$. Indeed, if $\mathrm{fr}_\alpha(z) \geqslant \rho$, then the first inequality is always true, while the second one shows that z is N-near ρ. Similarly, if $\mathrm{fr}_\alpha(z) \leqslant \rho$, then $\mathrm{fr}_\alpha(z + N\beta) \leqslant \rho$ is vacuous, and $\mathrm{fr}_\alpha(z + N\alpha) \geqslant \rho$ implies that $N\alpha$ flips z.

Here are the relevant properties of these notions that will be helpful in the proof of Lemma 7.12.

Proposition 7.9. *Let $x, y \in \mathcal{T}$ be tileable reals, let $N, N' \in \mathbb{N}$ be natural numbers, and set*

$$\tilde{\rho} = \max\Big\{\Big\lceil \frac{\rho}{1-\rho} \Big\rceil, \Big\lceil \frac{1-\rho}{\rho} \Big\rceil\Big\}.$$

Note that $\tilde{\rho} \geqslant 1$.

(i) If x is N-near ρ and $N' \geqslant N$, then x is N'-near ρ.
(ii) If x is N-near ρ and y is N'-near ρ, then $x + y$ is $(N + N')$-near ρ.
(iii) If y is N-near ρ, $\mathrm{fr}_\alpha(y)$ and $\mathrm{fr}_\alpha(x+y)$ are on the same side of ρ, while $\mathrm{fr}_\alpha(x)$ and $\mathrm{fr}_\alpha(x+y)$ are on the different sides of ρ, then $x + y$ is also N-near ρ.
(iv) If y flips x and y is N-near ρ, then $x + y$ is also N-near ρ.
(v) Any tileable x is $\big(\tilde{\rho}\lfloor x/\alpha \rfloor\big)$-near ρ.
(vi) If x and $x + y$ are both N-near ρ, then y is $\big(N + N\tilde{\rho}\lceil \beta/\alpha \rceil\big)$-near ρ.
(vii) If x is N-near ρ and $x > 2\beta^2 N/\alpha\epsilon$, then $\big|\mathrm{fr}_\alpha(x) - \rho\big| < \epsilon$.

Proof. Item (i) is straightforward.

(ii) Since x is N-near ρ and y is N'-near ρ, we have

$$\mathrm{fr}_\alpha(x+N\alpha) \geqslant \rho,\ \mathrm{fr}_\alpha(x+N\beta) \leqslant \rho \quad \text{and} \quad \mathrm{fr}_\alpha(y+N'\alpha) \geqslant \rho,\ \mathrm{fr}_\alpha(y+N'\beta) \leqslant \rho.$$

Thus

$$\mathrm{fr}_\alpha(x+y+N\alpha+N'\alpha) \geqslant \rho \quad \text{and} \quad \mathrm{fr}_\alpha(x+y+N\beta+N'\beta) \leqslant \rho.$$

Depending on whether $\mathrm{fr}_\alpha(x+y) \leqslant \rho$ or $\mathrm{fr}_\alpha(x+y) \geqslant \rho$, either $N\alpha + N'\alpha$ or $N\beta + N'\beta$ flips $x+y$.

(iii) If $\mathrm{fr}_\alpha(x+y) = \rho$, then there is nothing to prove as such $x+y$ is 0-near ρ. If $\mathrm{fr}_\alpha(x+y) \neq \rho$, then we have two possibilities: either $\mathrm{fr}_\alpha(y) \leqslant \rho$, $\mathrm{fr}_\alpha(x+y) < \rho$, and $\mathrm{fr}_\alpha(x) \geqslant \rho$ or another case with all inequalities reversed. We deal with the first case, leaving the other one to the reader. Since y is N-near ρ, $\mathrm{fr}_\alpha(y+z) \geqslant \rho$, where $z = N\alpha$. Therefore also $\mathrm{fr}_\alpha(x+y+z) \geqslant \rho$, because $\mathrm{fr}_\alpha(x) \geqslant \rho$. Together with the assumption $\mathrm{fr}_\alpha(x+y) \leqslant \rho$ this witnesses the conclusion of the item.

(iv) If y flips x, then $\mathrm{fr}_\alpha(y)$ and $\mathrm{fr}_\alpha(x+y)$ are on the same side of ρ, while $\mathrm{fr}_\alpha(x)$ and $\mathrm{fr}_\alpha(x+y)$ are on the different sides of ρ. The item now follows from (iii).

(v) Let $x = p\alpha + q\beta$ and let $N = \tilde{\rho}\lfloor x/\alpha \rfloor$. It is enough to show that

$$\frac{p}{p+q+N} \leqslant \rho \leqslant \frac{p+N}{p+q+N}.$$

To this end note that $\max\{p, q\} \leqslant \lfloor x/\alpha \rfloor$ and consider the following inequalities.

$$\frac{p}{p+q+N} \leqslant \frac{p}{p+N} = \frac{1}{1+N/p} \leqslant \frac{1}{1+\frac{\tilde{\rho}\lfloor x/\alpha\rfloor}{\lfloor x/\alpha \rfloor}} = \frac{1}{1+\tilde{\rho}} \leqslant \rho.$$

$$\frac{p+N}{p+q+N} = \frac{1}{1+\frac{q}{p+N}} \geqslant \frac{1}{1+\frac{q}{N}} \geqslant \frac{1}{1+\frac{\lfloor x/\alpha\rfloor}{\tilde{\rho}\lfloor x/\alpha\rfloor}} = \frac{1}{1+\tilde{\rho}^{-1}} \geqslant \rho.$$

(vi) Suppose first that $\mathrm{fr}_\alpha(x)$ and $\mathrm{fr}_\alpha(y)$ are on the same side of ρ. For instance, let us assume that $\mathrm{fr}_\alpha(x) \leqslant \rho$ and $\mathrm{fr}_\alpha(y) \leqslant \rho$. In this case $\mathrm{fr}_\alpha(x+y) \leqslant \rho$ and we claim that y is, in fact, N-near ρ (which is enough in view of item (i)). Indeed, since $x+y$ is N-near ρ by the assumption, one has $\mathrm{fr}_\alpha(x+y+z) \geqslant \rho$, where $z = N\alpha$. But this means that $\mathrm{fr}_\alpha(y+z)$ has to be at least ρ, for if $\mathrm{fr}_\alpha(y+z) < \rho$, then $\mathrm{fr}_\alpha(x) \leqslant \rho$ would imply $\mathrm{fr}_\alpha(x+y+z) < \rho$. Therefore $\mathrm{fr}_\alpha(y) \leqslant \rho$ and $\mathrm{fr}_\alpha(y+z) \geqslant \rho$, i.e., y is N-near ρ.

Now to the remaining case when the α-frequencies of x and y are on the different sides of ρ. Let $z = N\alpha$ or $z = N\beta$ be such that $\mathrm{fr}_\alpha(x+z)$ is on the same side of ρ as is $\mathrm{fr}_\alpha(y)$. Such z exists, since x is N-near ρ by the assumption. Items (v) and (i) imply that z is $\big(\tilde{\rho}N\lceil\beta/\alpha\rceil\big)$-near ρ, and item (ii) shows that $x+y+z$ is $\big(N + N\tilde{\rho}\lceil\beta/\alpha\rceil\big)$-near ρ. But now $\mathrm{fr}_\alpha(x+z)$ and $\mathrm{fr}_\alpha(y)$ are on the same side of ρ, and both $x+y+z$ and $x+z$ are $\big(N + N\tilde{\rho}\lceil\beta/\alpha\rceil\big)$-near ρ. By the previously considered case so is y.

(vii) Since x is N-near ρ, one may take $z = N\alpha$ or $z = N\beta$ such that

$$\big|\mathrm{fr}_\alpha(x) - \rho\big| \leqslant \big|\mathrm{fr}_\alpha(x+z) - \mathrm{fr}_\alpha(x)\big|.$$

The lower bound on x and $\alpha < \beta$ show that

$$\frac{z}{x} < \frac{N\beta}{2N\beta^2/\alpha\epsilon} = \frac{\alpha\epsilon}{2\beta}.$$

The item now follows from Lemma 6.4. □

Besides the concept of N-nearness, we shall also need the notion of N-farness.

Definition 7.10. We say that a tileable $z = p\alpha + q\beta$ is N-**far** from ρ if

- $\mathrm{fr}_\alpha(z) \leqslant \rho$ implies $q \geqslant N$ and $\mathrm{fr}_\alpha(z - N\beta) < \rho$;
- $\mathrm{fr}_\alpha(z) \geqslant \rho$ implies $p \geqslant N$ and $\mathrm{fr}_\alpha(z - N\alpha) > \rho$.

Note that this time the two items in the definition above are truly mutually exclusive; a tileable y that is N-far from ρ cannot have the α-frequency equal to ρ. The definition of N-farness is similar to the direct negation of being N-near ρ, and it is designed primarily so that the following condition holds: If y is N-far from ρ and x is N-near ρ, then

$$\mathrm{fr}_\alpha(y) < \rho \implies \mathrm{fr}_\alpha(x+y) < \rho \quad \text{and} \quad \mathrm{fr}_\alpha(y) > \rho \implies \mathrm{fr}_\alpha(x+y) > \rho. \tag{7}$$

The following lemma shows that any sufficiently large tileable x with the α-frequency bounded away from ρ (and also from the endpoints 0 and 1) is automatically N-far from ρ.

Lemma 7.11. *Let $\nu > 0$ be so small that $\nu < \rho - \nu$ and $\rho + \nu < 1 - \nu$.*

(i) For any N there exists $K = K(\nu, N)$ such that for any tileable $x = p\alpha + q\beta$ satisfying $x > K$ and $\mathrm{fr}_\alpha(x) \in [\nu, 1-\nu]$ one has $\min\{p, q\} \geqslant N$.

(ii) For any N there exists $K = K(\nu, N)$ such that for any tileable x

$$x > K \textit{ and } \mathrm{fr}_\alpha(x) \in [\nu, \rho - \nu] \cup [\rho + \nu, 1 - \nu] \implies x \textit{ is } N\textit{-far from } \rho.$$

Proof. (i) One can take $K = N(\alpha + \beta)/\nu$. Indeed, if $x > K$, $x = p\alpha + q\beta$, then either p or q has to be larger than N/ν. Suppose first that $q > N/\nu$, but $p < N$. In this case

$$\mathrm{fr}_\alpha(x) = \frac{p}{p+q} < \frac{N}{q} < \nu.$$

If on the other hand $p > N/\nu$, but $q < N$, then

$$\mathrm{fr}_\alpha(x) = \frac{p}{p+q} = 1 - \frac{q}{p+q} > 1 - \frac{N}{p} > 1 - \nu.$$

Thus $\mathrm{fr}_\alpha(x) \in [\nu, 1-\nu]$ implies $p \geqslant N$ and $q \geqslant N$ as claimed.

(ii) Applying item (i), we can pick K so large that $\min\{p, q\} \geqslant N$ for all tileable $x = p\alpha + q\beta$ with the α-frequency in $[\nu, 1-\nu]$. Any such x can therefore be written as $x = x' + y'$, where $x' \in \mathfrak{T}$ and $y' = N\alpha$, and also as $x = x'' + y''$, where $y'' = N\beta$. Using Lemma 6.4 we may increase K so that for any $x \geqslant K$ one has

$$\left|\mathrm{fr}_\alpha(x' + y') - \mathrm{fr}_\alpha(x')\right| < \nu \text{ and } \left|\mathrm{fr}_\alpha(x'' + y'') - \mathrm{fr}_\alpha(x'')\right| < \nu.$$

Since by the assumption either $\mathrm{fr}_\alpha(x) \leqslant \rho - \nu$ or $\mathrm{fr}_\alpha(x) \geqslant \rho + \nu$, we conclude that either all three frequencies $\mathrm{fr}_\alpha(x)$, $\mathrm{fr}_\alpha(x')$, and $\mathrm{fr}_\alpha(x'')$ are $< \rho$ or they are all $> \rho$, whence x is N-far from ρ. □

The next lemma encapsulates the step of induction in Theorem 8.1.

Lemma 7.12 (Sparse induction step)**.** *Let $\bar{L}$ be an $\bar{\eta}$-flexible sequence. Given $0 < \epsilon \leqslant 1$, $m_0 \in \mathbb{N}$, and $N \in \mathbb{N}$ there exist a real number $K = K_{\mathrm{Lem\,7.12}}(\bar{\eta}, \bar{L}, \epsilon, m_0, N)$, a sequence $\bar{L}' = \bar{L}'_{\mathrm{Lem\,7.12}}(\bar{\eta}, \bar{L}, \epsilon, m_0, N)$, and a natural number $N' = N'_{\mathrm{Lem\,7.12}}(\bar{\eta}, \bar{L}, \epsilon, m_0, N)$ such that*

- *$L'_k = L_k$ for all $k \leqslant m_0$;*
- *$L'_k \geqslant L_k$ for all k;*

for any $n \geqslant 2$, any family $z_1, \ldots, z_n \in \mathfrak{T}$, $y_1, \ldots, y_{n-1} \in \mathbb{R}^{>0}$ satisfying

- *z_i are $(\bar{\eta}, \bar{L})$-tiled;*
- *$z_i \in \mathrm{SB}_{\eta_k}[L_k]$ for $k \leqslant m_0$;*
- *z_i are N-near ρ;*
- *$K/2 \leqslant y_i \leqslant K$;*

there are tiled $\tilde{y}_1, \ldots, \tilde{y}_{n-1} \in \mathfrak{T}$ such that

(i) $\left|\sum_{i=1}^{r}(\tilde{y}_i - y_i)\right| < \epsilon$ for all $r \leqslant n-1$;
(ii) $\tilde{y}_i \in \bigcap_{j=0}^{m_0} \mathrm{SB}_{\eta_j}[L_j]$;
(iii) $\tilde{y}_i$ are $(\bar{\eta}, \bar{L}')$-tiled

and setting

$$\tilde{J} = z_1 + \tilde{y}_1 + z_2 + \tilde{y}_2 + \cdots + z_{n-1} + \tilde{y}_{n-1} + z_n$$

one also has

(iv) $\tilde{J}$ is $(\bar{\eta}, \bar{L}')$-tiled;
(v) $\tilde{J} \in \mathrm{SB}_{\eta_k}[L'_k]$ for $k \leqslant m_0 + 1$;
(vi) $\tilde{J}$ is N'-near ρ.

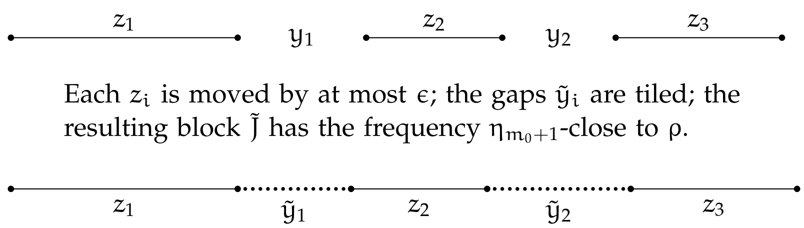


FIG. 18. Constructing $\tilde{J}$.

The set up of this lemma differs from the one of Lemma 7.6 in the following aspects. The reals y_i are not necessarily tileable. In the context of Figure 18, we perturb each y_i into $\tilde{y}_i$ in a way that results in shifting each z_i by no more than ϵ. Similarly to Lemma 7.6 the sums $\tilde{J}$ are claimed to be $(\bar{\eta}, \bar{L}')$-tiled, but the crucial difference is that this time we do not fix the number of summands n, but the assumption on z_i's is stronger: they are additionally assumed to be N-near ρ for a fixed natural N. Moreover, each z_i may only be η_{m_0}-close to ρ, while the resulting $\tilde{J}$ is at least η_{m_0+1}-close to ρ, so the approximation of the α-frequency improves. Also, note that $\tilde{J}$ satisfies the assumptions similar to those imposed on z_i which will allow us to continue the process inductively.

Proof. We begin by claiming that one may take K so large that both sets of

$$y^-, y^+ \in \bigcap_{j=0}^{m_0} SB_{\eta_j}[L_j]$$

satisfying the following items are ϵ-dense in $[K/2-1, \infty)$:

a) $fr_\alpha(y^-) < \rho$, $fr_\alpha(y^+) > \rho$;
b) $|fr_\alpha(y^-) - \rho| < \eta_{m_0+1}/2$ and $|fr_\alpha(y^+) - \rho| < \eta_{m_0+1}/2$.
c) y^- and y^+ are N-far from ρ.

Indeed, by the definition of an $\bar{\eta}$-flexible sequence, one can find $K \geqslant 4$ so large that sets of those y^-, y^+ that satisfy

$$fr_\alpha(y^-) \in (\rho - \eta_{m_0+1}/2, \rho - \eta_{m_0+1}/3), \quad fr_\alpha(y^+) \in (\rho + \eta_{m_0+1}/3, \rho + \eta_{m_0+1}/2)$$

are ϵ-dense in $[K/2-1, \infty)$. This guarantees fulfillment of items (a) and (b). In view of Lemma 7.11(ii), item (c) automatically holds for K large enough. By enlarging K even more, we may assume that

$$|fr_\alpha(y^- + x) - \rho| < \eta_{m_0+1} \quad \text{and} \quad |fr_\alpha(y^+ + x) - \rho| < \eta_{m_0+1} \tag{8}$$

holds for all tileable $x \leqslant 2L_{m_0+1}$ and all $y^-, y^+ \geqslant K/2-1$ as above. This concludes the requirements on K.

We now describe the construction of $\tilde{y}_i$. Since z_1 and z_2 are N-near ρ and since $y_1 \geqslant K/2$, we may find $\tilde{y}_1 \in \bigcap_{j=0}^{m_0} SB_{\eta_j}[L_j]$ such that

- $\tilde{y}_1$ is N-far from ρ;
- if $fr_\alpha(z_1+z_2) \leqslant \rho$ we want $fr_\alpha(\tilde{y}_1) > \rho$; if $fr_\alpha(z_1+z_2) > \rho$ we require $fr_\alpha(\tilde{y}_1) < \rho$;
- $|y_1 - \tilde{y}_1| < \epsilon$;
- $|fr_\alpha(\tilde{y}_1) - \rho| < \eta_{m_0+1}/2$.

The tiled reals $\tilde{y}_k$ are defined recursively: if elements $\tilde{y}_1, \ldots, \tilde{y}_{k-1}$ have been chosen, we may find $\tilde{y}_k \in \bigcap_{j=0}^{m_0} SB_{\eta_j}[L_j]$ such that

- $|\tilde{y}_k - y_k| < \epsilon$;
- $|fr_\alpha(\tilde{y}_k) - \rho| < \eta_{m_0+1}/2$;
- $\tilde{y}_k \geqslant y_k$ if $\sum_{j=1}^{k-1} \tilde{y}_j \leqslant \sum_{j=1}^{k-1} y_j$ and $\tilde{y}_k < y_k$ if $\sum_{j=1}^{k-1} \tilde{y}_j > \sum_{j=1}^{k-1} y_j$;
- if the frequency

$$fr_\alpha\Big(\sum_{j=1}^{k-1}(z_j + \tilde{y}_j) + z_k + z_{k+1}\Big) \leqslant \rho,$$

then we pick $fr_\alpha(\tilde{y}_k) > \rho$; otherwise we ensure $fr_\alpha(\tilde{y}_k) < \rho$.

The number N' is set to be $N' = N + \tilde{\rho}\lfloor (K+1)/\alpha \rfloor$, where $\tilde{\rho}$ as in Proposition 7.9.

Claim 1. For all $k < n$ the tiled reals $z_1 + \tilde{y}_1 + z_2 + \cdots + \tilde{y}_k + z_{k+1}$ are N'-near ρ.

The important part of this claim is that the nearness parameter of these sums does not grow with k. Nonetheless, the proof of the claim is by induction on k, with the base case $k = 0$ following from the assumption that z_1 is N-near ρ, $N' > N$, and Proposition 7.9(i).

Suppose $w = z_1 + \tilde{y}_1 + \cdots + \tilde{y}_{k-1} + z_k$ is known to be N'-near ρ. We aim at showing that $w + \tilde{y}_k + z_{k+1}$ is also N'-near ρ. If z_{k+1} flips w, then $w + z_{k+1}$ is N-near ρ by the assumption on z_{k+1} and 7.9(iv), and therefore $w + \tilde{y}_k + z_{k+1}$ is N'-near ρ by 7.9(v) and 7.9(ii) (recall that $\tilde{y}_k \leqslant K + 1$). Similarly, if $\tilde{y}_k$ flips $w + z_{k+1}$, then $w + \tilde{y}_k + z_{k+1}$ is N'-near ρ by 7.9(v) and 7.9(iv).

We may therefore assume that neither z_{k+1} flips w, nor $\tilde{y}_k$ flips $w + z_{k+1}$, in which case all three frequencies $\mathrm{fr}_\alpha(w)$, $\mathrm{fr}_\alpha(w + z_{k+1})$, $\mathrm{fr}_\alpha(w + z_{k+1} + \tilde{y}_k)$ are on the same side of ρ. In fact, since in the definition of the flip we used non-strict inequalities, none of these three frequencies can be equal to ρ. Let us assume for notational convenience that they are all $< \rho$; the case of them being above ρ is, of course, similar. So, we assume that $\mathrm{fr}_\alpha(w + z_{k+1}) < \rho$, and therefore the last requirement on $\tilde{y}_k$ ensures that $\mathrm{fr}_\alpha(\tilde{y}_k) > \rho$. Moreover, $\tilde{y}_k$ is N-far from ρ, and therefore $\mathrm{fr}_\alpha(\tilde{y}_k + z_{k+1}) \geqslant \rho$ by (7). Finally, w is N'-near ρ by the inductive assumption, hence $\mathrm{fr}_\alpha(w) < \rho$, $\mathrm{fr}_\alpha(w + \tilde{y}_k + z_{k+1}) < \rho$, and $\mathrm{fr}_\alpha(\tilde{y}_k + z_{k+1}) \geqslant \rho$ imply that $w + \tilde{y}_k + z_{k+1}$ is N'-near ρ by Proposition 7.9(iii). This finishes the proof of Claim 1.

We still need to construct the sequence $\bar{L}'$, but note that items (i) and (vi) do not depend on $\bar{L}'$ and are satisfied by the construction of $\tilde{y}_i$ and Claim 1 taken for $k = n - 1$. Moreover, item (iii) will be trivially satisfied once we ensure that $L'_{m_0+1} > K + 1 \geqslant \tilde{y}_i$. Also, it is easy to see that (v) is automatic once $L'_{m_0+1} > 2L_{m_0+1} + K + 1$. Indeed, $\tilde{J} \in SB_{\eta_j}[L'_j]$ for $j \leqslant m_0$, because $z_i, \tilde{y}_i \in SB_{\eta_j}[L'_j]$. By the assumptions on the choice of $\tilde{y}_i$, we have $|\mathrm{fr}_\alpha(\tilde{y}_i) - \rho| < \eta_{m_0+1}$. If $z_i \geqslant L_{m_0+1}$, then $z_i \in SB_{\eta_{m_0+1}}[L_{m_0+1}]$, because z_i is assumed to be $(\bar{\eta}, \bar{L})$-tiled. If, on the other hand, $z_i < L_{m_0+1}$, then $|\mathrm{fr}_\alpha(z_i + \tilde{y}_i) - \rho| < \eta_{m_0+1}$ by (8). So, in either case $z_i + \tilde{y}_i \in SB_{\eta_{m_0+1}}[L'_{m_0+1}]$ for all i, and also $z_{n-1} + \tilde{y}_{n-1} + z_n \in SB_{\eta_{m_0+1}}[L'_{m_0+1}]$ for a similar reason. This shows that $\tilde{J} \in SB_{\eta_{m_0+1}}[L'_{m_0+1}]$ as (v) claims.

It therefore remains to construct $\bar{L}'$ that ensures satisfaction of (iv). For $k \leqslant m_0$ we set $L'_k = L_k$ and define L'_k for $k > m_0$ inductively. Let N'' be equal to $N' + N'\tilde{\rho}\lceil \beta/\alpha \rceil$.

Claim 2. For any $1 \leqslant j_1 < j_2 \leqslant n_0$ the sum $\sum_{i=j_1}^{j_2}(\tilde{y}_i + z_{i+1})$ is N''-near ρ.

Indeed, by Claim 1 both sums

$$\sum_{i=1}^{j_1-1}\big(z_i + \tilde{y}_i\big) + z_{j_1} \quad \text{and} \quad \sum_{i=1}^{j_2}\big(z_i + \tilde{y}_i\big) + z_{j_2+1}$$

are N'-near ρ. Therefore 7.9(vi) implies that the sum $\sum_{i=j_1}^{j_2}\big(\tilde{y}_i + z_{i+1}\big)$ is N''-near ρ.

Claim 3. For any $k > m_0$ there exists $n(k)$ such that

$$\Big|\mathrm{fr}_\alpha\Big(\sum_{i=j_1}^{j_2-1}\tilde{y}_i + z_{i+1}\Big) - \rho\Big| \leqslant \eta_k \quad \text{whenever } j_2 - j_1 \geqslant n(k).$$

Since $\tilde{y}_i \geqslant K/2 - 1 \geqslant 1$, one has $\sum_{i=j_1}^{j_2-1}(\tilde{y}_i + z_{i+1}) \geqslant j_2 - j_1$. The claim now follows from Claim 2 and 7.9(vii).

Fixing $n(k)$ as in Claim 3, we are now ready to define L'_k, $k \geqslant m_0 + 1$, as follows. Apply Lemma 7.6 together with Remark 7.7 for the given $\bar{\eta}, \bar{L}, m_0$ and $2n(k) + 1, K + 2$. Let $\bar{L}''$ be the output of this application and set

$$L'_{m_0+1} = \max\big\{2L_{m_0+1} + K + 2, L''_{m_0+1}\big\} \text{ and } L'_k = \max\{L'_{k-1} + 1, L''_k\} \text{ if } k > m_0 + 1.$$

We verify that all elements of the form $\tilde{J}$ are $(\bar{\eta}, \bar{L}')$-tiled. Take $k > m_0$ and consider

$$\tilde{J} = z_1 + \tilde{y}_1 + z_2 + \tilde{y}_2 + \cdots + z_{n-1} + \tilde{y}_{n-1} + z_n.$$

Assuming that $\tilde{J} \geqslant L_k$, we aim at showing that $\tilde{J} \in SB_{\eta_k}[L'_k]$. If $n \leqslant 2n(k)+1$, then this follows immediately from the choice of $\bar{L}''$ given by Lemma 7.6. So, we may assume that $n > 2n(k)+1$. In this case, we may find

$$1 = j_0 < j_1 < j_2 < \cdots < j_{s-1} < j_s = n$$

such that $n(k) \leqslant j_{i+1} - j_i \leqslant 2n(k)$ for all $0 \leqslant i < s$. Consider the elements

$$\tilde{J}_0 = z_1 + \sum_{l=j_0}^{j_1-1} (\tilde{y}_l + z_{l+1}) \quad \text{and} \quad \tilde{J}_i = \sum_{l=j_i}^{j_{i+1}-1} (\tilde{y}_l + z_{l+1}) \text{ for } 0 < i \leqslant s-1.$$

In this notation we have $\tilde{J} = \tilde{J}_0 + \tilde{J}_1 + \cdots + \tilde{J}_{s-2} + \tilde{J}_{s-1}$. Since $n(k)$ was chosen according to Claim 3, the

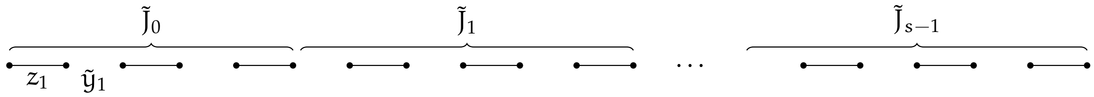


FIG. 19. Partitioning $\tilde{J}$ into pieces $\tilde{J}_i$.

α-frequency of each element $\tilde{J}_i$, $i \geqslant 1$, is η_k-close to ρ. The frequency of the first summand $\tilde{J}_0$ is also η_k-close to ρ. Indeed, by Claim 1 it is N'-near ρ, and $N' \leqslant N''$, hence $\tilde{J}_0$ is also N''-near ρ, and therefore $\mathrm{fr}_\alpha(\tilde{J}_0)$ is η_k-close to ρ by 7.9(vii) and the choice of $n(k)$.

Finally, we claim that all $\tilde{J}_i$ are elements of $SB_{\eta_k}[L'_k]$ implying that so is $\tilde{J}$. If $\tilde{J}_i < L'_k$, then $\tilde{J}_i \in B_{\eta_k}[L'_k]$ and we are done. Otherwise, Lemma 7.6 (together with Remark 7.7) applies[10] and by the choice of L'_k we get $\tilde{J}_i \in SB_{\eta_k}[L'_k]$. □

8. REGULAR CROSS SECTIONS OF SPARSE FLOWS

We now prove the main theorem under the additional assumption that the flow is sparse.

Theorem 8.1 (Regular cross sections of sparse flows)**.** *Let $\mathfrak{F}$ be a free sparse Borel flow on a standard Borel space Ω. There is a Borel $\{\alpha, \beta\}$-regular cross section $\mathcal{C}$ such that moreover for any $\eta > 0$ there exists $N(\eta)$ such that for all $x \in \mathcal{C}$ and $n \geqslant N(\eta)$*

$$\left| \rho - \frac{1}{n} \sum_{k=0}^{n-1} \chi_{\mathcal{C}_\alpha}\left(\phi^k_{\mathcal{C}}(x)\right) \right| < \eta. \tag{9}$$

Proof. Let $\epsilon_n = 2^{-n} \cdot \frac{\min\{\alpha,1\}}{3}$, $n \geqslant 1$, which will serve as bounds on the size of jumps at step n, and let η_n be any strictly decreasing positive sequence converging to zero such that $\eta_0 = 1$ and $\eta_1 < \min\{\rho, 1-\rho\}$. Note that

$$\sum_{n=1}^{\infty} \epsilon_n \leqslant \alpha/3,$$

so if we start with two points which are at least α apart, and if each one is moved by at most $\sum \epsilon_n$, no two points will be "glued together."

Based on Lemma 7.12, we construct reals K_n, naturals N_n, and sequences $\bar{L}^{(n)}$, $n \in \mathbb{N}$, as follows. To start, pick any $\bar{\eta}$-flexible sequence $\bar{L}^{(0)}$, which exists by Lemma 7.4, pick $K_0 \geqslant 4 \cdot \max\{1, \beta\}$ such that $SB_{\eta_0}[L^{(0)}_0]$ is ϵ_1-dense in $[K_0 - 2, \infty)$ and put $N_0 = 1$. Define

$$\begin{aligned} K_{n+1} &= \max\bigl\{K_{\mathrm{Lem\,7.12}}(\bar{\eta}, \bar{L}^{(n)}, \epsilon_{n+1}, n, N_n),\ K_n + 4\bigr\}, \\ \bar{L}^{(n+1)} &= \bar{L}'_{\mathrm{Lem\,7.12}}(\bar{\eta}, \bar{L}^{(n)}, \epsilon_{n+1}, n, N_n), \\ N_{n+1} &= N'_{\mathrm{Lem\,7.12}}(\bar{\eta}, \bar{L}^{(n)}, \epsilon_{n+1}, n, N_n). \end{aligned}$$

Note that by construction $L^{(n)}_k = L^{(n+1)}_k$ for all $k \leqslant n$. We construct Borel cross sections $\mathcal{C}_n$, and Borel function $h_{n+1} : \mathcal{C}_n \to (-\epsilon_n, \epsilon_n)$, which will represent shifts of points in $\mathcal{C}_n$.

We start with an application of Lemma 5.2 which gives a sparse cross section $\mathcal{C}_0$ such that

[10]For an application of Lemma 7.6 to $\tilde{J}_i$, $i \geqslant 1$, we add an extra $\tilde{z}_{j_i} = 0$ at the beginning.

(i) $\vec{\mathrm{gap}}_{\mathcal{C}_0}(x) > K_0$ for all $x \in \mathcal{C}_0$.
(ii) Each $E_{\mathcal{C}_0}^{\leqslant K_{n+1}}$-class consists of at least two $E_{\mathcal{C}_0}^{\leqslant K_n}$-classes for all $n \in \mathbb{N}$.
(iii) Distance between adjacent $E_{\mathcal{C}_0}^{\leqslant K_n}$-classes within a given $E_{\mathcal{C}_0}^{\leqslant K_{n+1}}$-class is 1-close to $(K_{n+1}+K_n)/2$. Since $K_{n+1} \geqslant K_n + 4$, and since $\sum \epsilon_n \leqslant 1/3$, even if each point of $\mathcal{C}_0$ is perturbed by at most $\sum \epsilon_n$, no two $E_{\mathcal{C}_0}^{\leqslant K_n}$-classes will be "glued", nor any $E_{\mathcal{C}_0}^{\leqslant K_n}$-class will "split" into two or more classes. This guarantees that during our construction the structure of $E^{\leqslant K_n}$-classes will remain intact.

Step 1: Constructing $\mathcal{C}_1$. Consider an $E_{\mathcal{C}_0}^{\leqslant K_1}$-class. It consists of a number of points $x_1 < x_2 < \cdots < x_n$ such that

$$K_0 < \mathrm{dist}(x_i, x_{i+1}) \leqslant K_1 \quad \text{for all } i < n.$$

As was explained in Subsection 4.2, we shall move each x_i by at most ϵ_1. This is done via an application of Lemma 7.12. In the context of this lemma, $y_i = \mathrm{dist}(x_i, x_{i+1})$ correspond to sizes of the gaps, and $z_i = 0$

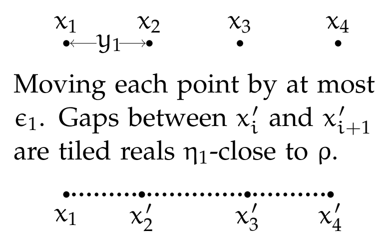


FIG. 20. Constructing $\mathcal{C}_1$. A single $E_{\mathcal{C}_0}^{\leqslant K_1}$-class is shown.

correspond to sizes of $E_{\mathcal{C}_0}^{\leqslant K_0}$-classes, which are single points, thus have zero length and are N_0-near ρ. By property (iii) of the cross section $\mathcal{C}_0$,

$$\frac{K_1}{2} \leqslant \frac{K_0 + K_1}{2} - 1 \leqslant y_i \leqslant \frac{K_0 + K_1}{2} + 1 \leqslant K_1 \quad \text{for all } i < n,$$

and also assumptions on z_i in Lemma 7.12 are trivially satisfied, so this lemma, and we move as it prescribes each x_i to x_i' and tile the gaps $\mathrm{dist}(x_i', x_i')$. The process is depicted in Figure 20. Since the argument of Lemma 7.12 describes an effective algorithm of performing such a tiling, this procedure can be run in a Borel way over all $E_{\mathcal{C}_0}^{\leqslant K_1}$-classes resulting in a cross section $\mathcal{C}_1$.

To summarize, $\mathcal{C}_1$ has points of two kinds. It contains points from $\mathcal{C}_0$ shifted by at most ϵ_1; the shift is given by a Borel function $h_1 : \mathcal{C}_0 \to (-\epsilon_1, \epsilon_1)$ such that

$$\mathcal{C}_0 + h_1 := \{x + h_1(x) : x \in \mathcal{C}_0\} \subseteq \mathcal{C}_1.$$

All other points in $\mathcal{C}_1$ were added during "tiling the gaps" procedure inside $E_{\mathcal{C}_1}^{\leqslant K_1}$-classes. Note that $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes are in one-to-one correspondence with $E_{\mathcal{C}_0}^{\leqslant K_1}$-classes.

Since any $E_{\mathcal{C}_1}^{\alpha,\beta}$-class $[x]_{E_{\mathcal{C}_1}^{\alpha,\beta}}$ can be viewed as a tileable real of length $\mathrm{dist}\big(\min[x]_{E_{\mathcal{C}_1}^{\alpha,\beta}}, \max[x]_{E_{\mathcal{C}_1}^{\alpha,\beta}}\big)$, the conclusion of Lemma 7.12 guarantees the following:

- Any such class is necessarily $(\bar{\eta}, \bar{L}^{(1)})$-tiled (by 7.12(iv)).
- $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes are N_1-near ρ (by 7.12(vi)).
- Every $E_{\mathcal{C}_1}^{\alpha,\beta}$-class is in $SB_{\eta_i}[L_i^{(1)}]$ for $i = 0, 1$ (by 7.12(v)); and is, in particular, η_1-close to ρ.

First of all, note that these items form the set of assumptions on elements z_i in Lemma 7.12. During the next step of the construction, when $\mathcal{C}_2$ will be defined, $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes will play the role of elements z_i in that lemma. Also, for the last item we need to fix a witness, i.e., we may fix a Borel sub cross section $\mathcal{D}_1^{(1)} \subseteq \mathcal{C}_1$ such that

- $\vec{\mathrm{gap}}_{\mathcal{D}_1^{(1)}}(x) \in B_{\eta_1}[L_1^{(1)}]$ for all $x \in \mathcal{D}_1^{(1)}$ such that $x \, E_{\mathcal{C}_1}^{\alpha,\beta} \, \phi_{\mathcal{D}_1^{(1)}}(x)$;
- $\min[x]_{E_{\mathcal{C}_1}^{\alpha,\beta}} \in \mathcal{D}_1^{(1)}$ and $\max[x]_{E_{\mathcal{C}_1}^{\alpha,\beta}} \in \mathcal{D}_1^{(1)}$ for all $x \in \mathcal{C}_1$.

Geometrically $\mathcal{D}_1^{(1)}$ gives a partition of each $E_{\mathcal{C}_1}^{\alpha,\beta}$-class into pieces of length at most $L_1^{(1)}$ each having α-frequency η_1-close to ρ. This finishes the first step of the construction.

Step 2: Constructing $\mathcal{C}_2$. Consider the relation $E_{\mathcal{C}_1}^{\leqslant K_2}$. Each $E_{\mathcal{C}_1}^{\leqslant K_2}$-class consists of at least two $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes with gaps y_i between them being 2-close[11] to $(K_2 + K_1)/2$ (see Figure 21). Note that $K_2 \geqslant K_1 + 4$ implies

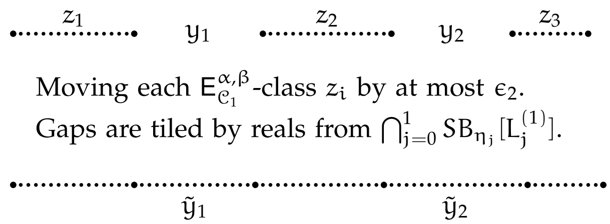


FIG. 21. Constructing $\mathcal{C}_2$.

$$K_2/2 \leqslant \frac{K_1 + K_2}{2} - 2 \leqslant y_i \leqslant \frac{K_1 + K_2}{2} + 2 \leqslant K_2.$$

We set z_i's to be the tiled reals that correspond to $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes. By the choice of K_2, $\bar{L}^{(2)}$, and N_2 given by Lemma 7.12, one may move each $E_{\mathcal{C}_1}^{\alpha,\beta}$-class by at most ϵ_2 and tile the gaps by tiled reals from $\bigcap_{j=0}^{1} SB_{\eta_j}[L_j^{(1)}]$ as shown in Figure 21. This defines the cross sections $\mathcal{C}_2$ which consists of shifted points from $\mathcal{C}_1$ and newly added points in between $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes. Let $h_2 : \mathcal{C}_1 \to (-\epsilon_2, \epsilon_2)$ be the shift function:

$$\mathcal{C}_1 + h_2 := \{x + h_2(x) : x \in \mathcal{C}_1\} \subseteq \mathcal{C}_2.$$

Note that h_2 is constant on $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes.

Again, the conclusion of Lemma 7.12 ensures that each $E_{\mathcal{C}_2}^{\alpha,\beta}$-class satisfies the necessary conditions for another round of application of the same lemma. Moreover, it ensures that each $E_{\mathcal{C}_2}^{\alpha,\beta}$-class is an element of $SB_{\eta_j}[L_j^{(2)}]$ for $j \leqslant 2$. We now define sub cross sections $\mathcal{D}_j^{(2)} \subseteq \mathcal{C}_2$, $j = 1, 2$ as follows. For $j = 1$ we start with $\mathcal{D}_1^{(1)}$ which partitions $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes. The cross section $\mathcal{D}_1^{(2)}$ will consist of points of two types. First it will contain the copy of $\mathcal{D}_1^{(1)}$ inside $\mathcal{C}_2$, i.e., it will have points

$$\mathcal{D}_1^{(1)} + h_2 := \left\{x + h_2(x) : x \in \mathcal{D}_1^{(1)}\right\}.$$

Recall that in the conclusion of Lemma 7.12 tiled reals $\tilde{y}_i$ are taken from $SB_{\eta_j}[L_j']$ for $j \leqslant m_0$, i.e., in our case the gaps between the $E_{\mathcal{C}_1}^{\alpha,\beta}$-classes inside $\mathcal{C}_2$ are elements of $SB_{\eta_1}[L_1^{(1)}]$, hence each such gap can be partitioned into pieces $B_{\eta_1}[L_1^{(1)}]$. We include all such partition points into $\mathcal{D}_1^{(2)}$. This results in $\mathcal{D}_1^{(2)}$ partitioning each $E_{\mathcal{C}_2}^{\alpha,\beta}$-classes into pieces of size at most $L_1^{(1)}$ each having α-frequency η_1-close to ρ. Note that (modulo the h_2-shift) $\mathcal{D}_1^{(2)}$ extends $\mathcal{D}_1^{(1)}$.

The remaining sub cross section $\mathcal{D}_2^{(2)}$ is defined based on 7.12(v). More precisely, each $E_{\mathcal{C}_2}^{\alpha,\beta}$-class is guaranteed to be an element of $SB_{\eta_2}[L_2^{(2)}]$, and we let $\mathcal{D}_2^{(2)}$ to represent a partition of each such class into pieces of size at most $L_2^{(2)}$ each having α-frequency η_2-close to ρ. This concludes the second step of the construction.

Step $k+1$**:** Constructing $\mathcal{C}_{k+1}$ and $\mathcal{D}_j^{(k+1)}$, $j \leqslant k+1$ from $\mathcal{C}_k$ and $\mathcal{D}_j^{(k)}$, $j \leqslant k$. The construction continues in a similar fashion as in step 2, and produces a cross section $\mathcal{C}_{k+1}$ and a Borel shift function $h_{k+1} : \mathcal{C}_k \to (-\epsilon_{k+1}, \epsilon_{k+1})$ such that

$$\mathcal{C}_k + h_{k+1} := \{x + h_{k+1}(x) : x \in \mathcal{C}_k\} \subseteq \mathcal{C}_{k+1}.$$

The shift function is constant on $E_{\mathcal{C}_k}^{\alpha,\beta}$-classes. These data is produced via the construction of Lemma 7.12 for the parameters $\epsilon = \epsilon_{k+1}$, $m_0 = k$, $N = N_k$, $\bar{L} = \bar{L}^{(k)}$. Specifically $m_0 = k$ ensures $L_j^{(k+1)} = L_j^{(k)}$

[11]It was 1-close in $\mathcal{C}_0$, but we have moved points during the construction of $\mathcal{C}_1$ by ϵ_1.

for $j \leqslant k$, and the gaps between $E^{\alpha,\beta}_{\mathcal{C}_k}$-classes inside $E^{\alpha,\beta}_{\mathcal{C}_{k+1}}$-classes represent tiled reals from $\bigcap_{j=0}^{m_0} SB_{\eta_j}[L_j^{(k)}]$. This allows us to "extend" each $\mathcal{D}_j^{(k)}$ to $\mathcal{D}_j^{(k+1)}$, $j \leqslant k$. Finally, $\mathcal{D}_{k+1}^{(k+1)}$ is constructed based on the conclusion 7.12(v) that each $E^{\alpha,\beta}_{\mathcal{C}_{k+1}}$-class is an element of $SB_{\eta_{m_0+1}}[L_{k+1}^{(k+1)}]$. Sub cross sections $\mathcal{D}_j^{(k+1)}$ will satisfy:

- $\vec{\text{gap}}_{\mathcal{D}_j^{(k+1)}}(x) \in B_{\eta_j}[L_j^{(k+1)}]$ for all $x \in \mathcal{D}_j^{(k+1)}$ such that $x\, E^{\alpha,\beta}_{\mathcal{C}_{k+1}}\, \phi_{\mathcal{D}_j^{(k+1)}}(x)$;
- $\min[x]_{E^{\alpha,\beta}_{\mathcal{C}_{k+1}}} \in \mathcal{D}_j^{(k+1)}$ and $\max[x]_{E^{\alpha,\beta}_{\mathcal{C}_{k+1}}} \in \mathcal{D}_j^{(k+1)}$ for all $x \in \mathcal{C}_{k+1}$ and $j \leqslant k+1$.

Once cross sections $\mathcal{C}_n$ have been constructed, the desired section $\mathcal{C}$ is defined to be the "limit" of $\mathcal{C}_n$. More formally, let $f_{n,n+1} : \mathcal{C}_n \to \mathcal{C}_{n+1}$ be given by $f_{n,n+1}(x) = x + h_{n+1}(x)$, set

$$f_{m,n} = f_{n-1,n} \circ f_{n-2,n-1} \circ \cdots \circ f_{m+1,m+2} \circ f_{m,m+1}$$

to be the embedding $\mathcal{C}_m \to \mathcal{C}_n$ for $m \leqslant n$ with the natural agreement that $f_{m,m}$ is the identity map. Define

$$H_n : \mathcal{C}_n \to \Big(-\sum_{k=n+1}^{\infty} \epsilon_k, \sum_{k=n+1}^{\infty} \epsilon_k\Big)$$

to be given by

$$H_n(x) = \sum_{k=n}^{\infty} h_{k+1}\big(f_{n,k}(x)\big).$$

The function H_n is just the "total shift" of each point in $\mathcal{C}_n$. Note that

$$\mathcal{C}_m + H_m \subseteq \mathcal{C}_n + H_n \quad \text{for all } m \leqslant n.$$

The limit cross section $\mathcal{C}$ is defined to be the (increasing) union

$$\mathcal{C} = \bigcup_k \big(\mathcal{C}_k + H_k\big) := \bigcup_k \big\{\, x + H_k(x) : x \in \mathcal{C}_k \,\big\}.$$

It is immediate from the construction that $\mathcal{C}$ is an $\{\alpha, \beta\}$-regular cross section.

The moreover part of the theorem follows from the properties of cross sections $\mathcal{D}_j^{(k)}$. First, we set for $j \geqslant 1$

$$\mathcal{D}_j = \bigcup_{k \geqslant j} \big(\mathcal{D}_j^{(k)} + H_k\big).$$

For each j the cross section $\mathcal{D}_j$ partitions $\mathcal{C}$ into pieces of size at most $L_j^{(j)}$ each piece having α-frequency η_j-close to ρ. Since $\eta_j \to 0$ as $j \to \infty$, Lemma 6.5 implies the moreover part of the theorem. □

9. Battle for every last orbit

In this section we finally prove that a free Borel flow can always be $\{\alpha, \beta\}$-tiled.

Theorem 9.1 (Main Theorem)**.** *Let $\mathfrak{F}$ be a free Borel flow on a standard Borel space Ω. There is an $\{\alpha, \beta\}$-regular cross section $\mathcal{C}$ such that moreover for any $\eta > 0$ there exists $N(\eta)$ such that for all $x \in \mathcal{C}$ and $n \geqslant N(\eta)$*

$$\left| \rho - \frac{1}{n} \sum_{k=0}^{n-1} \chi_{\mathcal{C}_\alpha}\big(\phi^k_{\mathcal{C}}(x)\big) \right| < \eta. \tag{10}$$

Proof. Let $\mathfrak{F}$ be a free Borel flow on a standard Borel space Ω. Not unlike the proof of Theorem 8.1 we fix sequence

$$(\epsilon_n)_{n=1}^{\infty} = \Big(2^{-n} \cdot \frac{\min\{\alpha, 1\}}{3}\Big)_{n=1}^{\infty}$$

and $(\eta_n)_{n=0}^{\infty}$ is any strictly decreasing positive sequence converging to zero such that $\eta_0 = 1$ and $\eta_1 < \min\{\rho, 1-\rho\}$. We also let $\nu'_n = \eta_{n+1} + \frac{\eta_n - \eta_{n+1}}{3}$ and $\nu_n = \eta_{n+1} + 2\frac{\eta_n - \eta_{n+1}}{3}$. In our argument we shall need to take an interval strictly inside $[\eta_{n+1}, \eta_n]$ and we are going to use $[\nu'_n, \nu_n]$ for this purpose since

$$\eta_{n+1} < \nu'_n < \nu_n < \eta_n.$$

Similarly to the proof of Theorem 8.1, ϵ_n controls the maximum shift of points, the total shift will therefore be bounded by $\sum_{n=1}^{\infty} \epsilon_n \leqslant 1/3$. We construct a sequence of Borel cross sections $\mathcal{C}_n$ and the desired cross section $\mathcal{C}$ will be defined as the "limit" of this sequence.

Pick K_0 so large that

- For any $x \geqslant K_0$ sets $\mathrm{fr}_\alpha^{-1}[\rho-\eta_1, \rho-\eta_2]$ and $\mathrm{fr}_\alpha^{-1}[\rho+\eta_2, \rho+\eta_1]$ are $\epsilon_1/6$-dense in $\mathcal{U}_{\epsilon_1}(x)$. The possibility to choose such K_0 is guaranteed by Lemma 7.3.
- To avoid some possible collapses, we also make sure that K_0 is large compared to $1, \alpha$, and β. For instance, $K_0 > 2\max\{1, \beta\}$ will work.

We let $\mathcal{C}_0$ to be a cross section such that $\vec{\mathrm{gap}}_{\mathcal{C}_0}(x) \in [K_0+1, K_0+2]$ for all $x \in \mathcal{C}_0$, which exists by Corollary 2.3. Each point will be shifted by at most $\sum \epsilon_n$, therefore the distance between $\mathcal{C}_0$-neighbors will always be somewhere between K_0 and K_0+3.

We now construct $\mathcal{C}_1$. For this we take a large natural number N_1, precise bounds on which to be provided later. In short, $\mathcal{C}_1$ is constructed by selecting pairs of points in $\mathcal{C}_0$ with at least N_1-many points between any two pairs, moving the right point within each pair by at most ϵ_1 so as to make the gap tileable, and adding points to tile the gaps, see Figure 22.

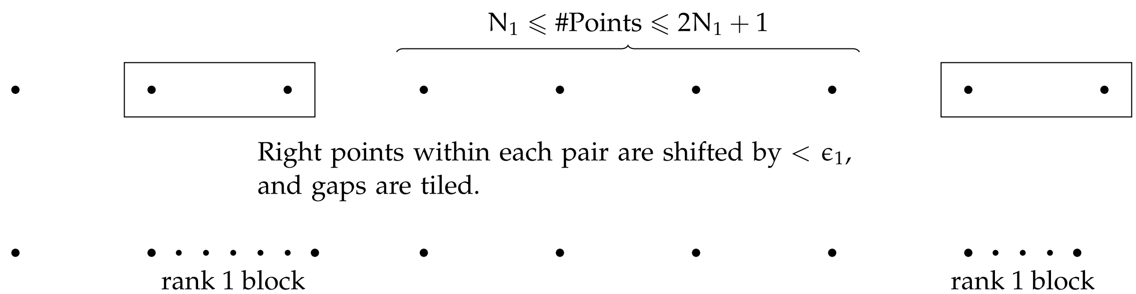


FIG. 22. Constructing $\mathcal{C}_1$ from $\mathcal{C}_0$.

In symbols, we select a sub cross section $\mathcal{P} \subseteq \mathcal{C}_0$ such that

- $\mathcal{P}$ consists of pairs: for any $x \in \mathcal{P}$ exactly one of the two things happens — either $\phi_{\mathcal{C}_0}(x) \in \mathcal{P}$ or $\phi_{\mathcal{C}_0}^{-1}(x) \in \mathcal{P}$.
- Between any two pairs there are at least N_1, at most $2N_1+1$ points from $\mathcal{C}_0$: if $x \in \mathcal{P}$ is the right point of a pair, then $\phi_{\mathcal{P}}(x) = \phi_{\mathcal{C}_0}^{n(x)}(x)$ with $N_1 \leqslant n(x) \leqslant 2N_1+1$.

Given such $\mathcal{P}$, we shift the right point of each pair to make the distance to the left one tileable. That is for each right point $x \in \mathcal{P}$ we find $h_1(x) \in (-\epsilon_1, \epsilon_1)$ such that

$$\mathrm{dist}\big(\phi_{\mathcal{C}_0}^{-1}(x), x+h_1(x)\big) \in \mathrm{fr}_\alpha^{-1}[\rho-\eta_1, \rho+\eta_1].$$

Such an h_1 can be found by the choice of K_0. It is convenient to extend h_1 to a function on $\mathcal{C}_0$ by declaring $h_1(x) = 0$ whenever x is not a right point of a pair in $\mathcal{P}$. And we can make sure that the function $h_1 : \mathcal{C}_0 \to (-\epsilon_1, \epsilon_1)$ is Borel.

Consider the relation $\mathsf{E}_{\mathcal{C}_1}^{\alpha,\beta}$ on $\mathcal{C}_1$. Equivalence classes of $\mathsf{E}_{\mathcal{C}_1}^{\alpha,\beta}$ are of two types: some of them consist of a single point, which is necessarily an element of $\mathcal{C}_0$; others consist of two points from $\mathcal{C}_0$ (one of which may be shifted) together with newly added points in the midst. We shall refer to the latter as *blocks of rank* 1. Isolated points from $\mathcal{C}_1$ are in this sense *blocks of rank* 0, see Figure 22. Note that each block of rank 1 in $\mathcal{C}_1$ corresponds to a tiled real of α-frequency η_1-close to ρ.

This finishes the definition of $\mathcal{C}_1$, but we still owe the reader a bound on N_1. For this we let $D_1 = K_0+3$, it will serve as an upper bound on the distance between points in $\mathcal{C}_1$. Let $M_1 = M_{\mathrm{Lem}\,6.12}(D_1, \epsilon_2/6, \eta_2, \nu_2, \nu_2')$. *Our first requirement is* $N_1 \geqslant M_1$. There will be one more largeness assumption, but let us elaborate on this condition first.

Take a segment between rank 1 blocks in $\mathcal{C}_1$ as shown in Figure 23. Let $d_1, \ldots, d_n$ denote the gaps inside this segment, where $N_1 \leqslant n \leqslant 2N_1+1$, and note that $d_k \geqslant K_0$. Let

$$R_k = \mathcal{T} \cap \mathcal{U}_{\epsilon_1}(d_k) \cap \mathrm{fr}_\alpha^{-1}[\rho-\eta_1, \rho+\eta_1]$$

denote the set of admissible shifts between the kth and $k-1$th points when the former is allowed to be shifted by at most ϵ_1. Since $n \geqslant M_1$, Lemma 6.12, applied relative to (d_k) and (R_k), implies that both

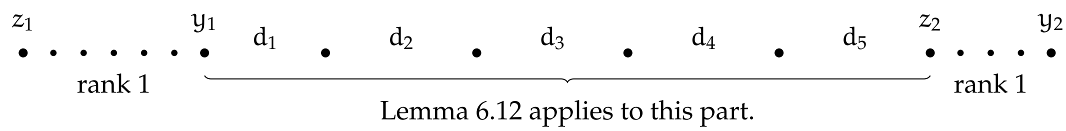


FIG. 23. Propagation of freedom principle.

$$\mathcal{A}_n\big(\epsilon_1,(d_k),(R_k)\big) \cap \mathrm{fr}_\alpha^{-1}[\rho-\nu_2,\rho-\nu_2'] \quad \text{and} \quad \mathcal{A}_n\big(\epsilon_1,(d_k),(R_k)\big) \cap \mathrm{fr}_\alpha^{-1}[\rho+\nu_2',\rho+\nu_2]$$

are $\epsilon_2/6$-dense in $\mathcal{U}_{\epsilon_1/2}(\sum_k d_k) = \mathcal{U}_{\epsilon_2}(\sum_k d_k)$. In the notation of Figure 23, it means that one can move each rank 0 point by at most ϵ_1 and the rank 1 block $[z_2, y_2]$ by at most ϵ_2 so as to make all the gaps tileable. Moreover, one has many possible ways of doing this with additional control on the α-frequencies. Set

$$R = \mathcal{A}_n\big(\epsilon_1,(d_k),(R_k)\big) \cap \mathrm{fr}_\alpha^{-1}[\rho-\eta_2,\rho+\eta_2] \cap \mathcal{U}_{\epsilon_2}\big(\sum_k d_k\big).$$

This set will correspond to admissible ways of shifting rank 1 blocks during the next construction step.

It is perhaps helpful to list some properties of R.

- $R \subseteq \mathcal{U}_{\epsilon_2}\big(\sum_{k=1}^n d_k\big)$.
- Both $R \cap \mathrm{fr}_\alpha^{-1}[\rho-\nu_2,\rho-\nu_2']$ and $R \cap \mathrm{fr}_\alpha^{-1}[\rho+\nu_2',\rho+\nu_2]$ are $\epsilon_2/6$-dense in $\mathcal{U}_{\epsilon_2}\big(\sum_{k=1}^n d_k\big)$.
- R agrees with R_k, $k=1,\dots,n$, in the sense that each $a \in R$ can be decomposed as $a = \sum_{k=1}^n b_k$ with $b_k \in R_k$.

It is the last item that is particularly specific to this argument compared to the one in Theorem 8.1. When dealing with sparse flows, we did not have any rank 0 blocks between rank 1 blocks, therefore to get such an R we only needed to know that the gap from y_1 to z_2 is large enough. But now, we must consider only shifts that agree with blocks of lower rank in the midst.

Note also that the above list of properties for R are precisely what Lemma 6.12 is designed for. Here is a tiny problem though. The way Lemma 6.12 is stated, it does not apply to blocks of points, so we shall need to pick representatives from each block. These will be the left points of rank 1 blocks, e.g., points z_1 and z_2 in Figure 23. Admissible tilings will therefore be given by $\mathrm{dist}(z_1,y_1)+R$ rather than just R. More formally, we define $\mathcal{D}_1 \subseteq \mathcal{C}_1$ to consist of the left endpoints of rank 1 blocks in $\mathcal{C}_1$. Note that the gaps in $\mathcal{D}_1$ are bounded,

$$\mathrm{dist}(z_1,z_2) \leqslant \mathrm{dist}(z_1,y_1) + (2N_1+2)D_1 \leqslant (2N_1+3)D_1.$$

We set $D_2 = (2N_1+3)D_1$. For an adjacent pair $z_1, z_2 \in \mathcal{D}_1$ we have the set of **admissible shifts**

$$R(z_1,z_2) = \mathrm{dist}(z_1,y_1) + R.$$

Lemma 6.12 will be applied to segments of $\mathcal{D}_1$.

While adding $\mathrm{dist}(z_1,y_1)$ does not ruin the density part, and $\mathrm{dist}(z_1,y_1)+R$ is still $\epsilon_2/6$-dense in the set $\mathcal{U}_{\epsilon_2}\big(\mathrm{dist}(z_1,z_2)\big)$, it may slightly alter the α-frequencies of elements from R. This is where we use ν_2 and ν_2'. Recall that they were selected to satisfy $\eta_3 < \nu_2' < \nu_2 < \eta_2$. So, if N_1 is large enough (cf. Lemma 6.4), then the α-frequency of $\mathrm{dist}(z_1,y_1)+a$ differs from the α-frequency of a by at most $\min\{\eta_2-\nu_2, \nu_2'-\eta_3\}$ for any $a \in R$. In fact, since rank 2 interval will include both $[z_1,y_1]$ and $[z_2,y_2]$ segments, we want the combined influence of $\mathrm{dist}(z_1,y_1)$ and $\mathrm{dist}(z_2,y_2)$ on the α-frequency of any $a \in R$ to be small. This brings us to the second largeness assumption on N_1. *The natural N_1 is so large that for any $a \in R$ and any tileable $d \leqslant 2D_1$ one has*

$$\big|\mathrm{fr}_\alpha(a+d) - \mathrm{fr}_\alpha(a)\big| < \min\{\eta_2-\nu_2, \nu_2'-\eta_3\}.$$

Under these assumptions the set $R(z_1,z_2)$ satisfies:

- $R(z_1,z_2) \subseteq \mathcal{U}_{\epsilon_2}\big(\mathrm{dist}(z_1,z_2)\big)$.
- The set $R(z_1,z_2) \cap \mathrm{fr}_\alpha^{-1}[\rho-\eta_2,\rho-\eta_3]$ as well as the set $R(z_1,z_2) \cap \mathrm{fr}_\alpha^{-1}[\rho+\eta_3,\rho+\eta_2]$ are both $\epsilon_2/6$-dense in $\mathcal{U}_{\epsilon_2}\big(\mathrm{dist}(z_1,z_2)\big)$.

- $R(z_1, z_2)$ agrees with the sets R_k in the sense that each $a \in R(z_1, z_2)$ is of the form
$$a = \mathrm{dist}(z_1, y_1) + \sum_{k=1}^{n} b_k$$
for some $b_k \in R_k$.

It is the sets $R(z_1, z_2)$ that will play the role of R_k in the application of Lemma 6.12 during the next stage, when we construct $\mathcal{C}_2$. The process depicted in Figure 22 is run in Borel way over all orbits of the flow.

We are now officially done with the second step of our construction, which can now be continued in a very similar fashion. A cross section $\mathcal{C}_2$ is constructed by selecting adjacent[12] pairs of rank 1 blocks, shifting the right block in each pair by at most ϵ_2 according to some element of $R(z_1, z_2)$ and tiling the gaps. Each $\mathsf{E}^{\alpha,\beta}_{\mathcal{C}_2}$-class obtained this way is called a rank 2 block. The pairs are selected in such a way that between any two of them there are at least N_2, at most $2N_2 + 1$ rank 1 blocks, for large enough $N_2 \in \mathbb{N}$. The process is depicted in Figure 24. The function $h_2 : \mathcal{C}_1 \to (-\epsilon_1, \epsilon_1)$ represents shifts of points. Note that if $x \in \mathcal{C}_1$ is a

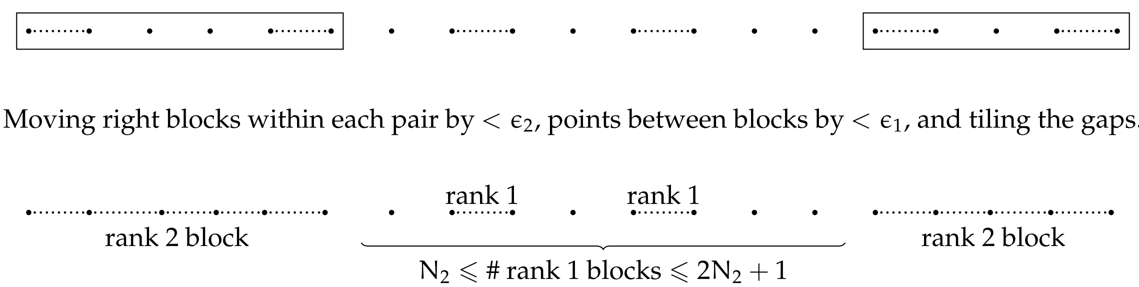


FIG. 24. Constructing $\mathcal{C}_2$ out of $\mathcal{C}_1$.

member of a rank 1 block, then $h_2(x) \in (-\epsilon_2, \epsilon_2)$. The number N_2 is assumed to be large in the same sense as N_1:

- First of all $N_2 \geqslant M_2 = M_{\mathrm{Lem\,6.12}}(D_2, \epsilon_3/6, \eta_3, \nu_3, \nu_3')$. This by the same principle implies that given a pair of neighboring rank 2 blocks the right one can be shifted by at most ϵ_3, rank 1 blocks in between by no more than ϵ_2, and points of rank 0 by ϵ_1, and all this can be done in such a way that gaps become tileable and the α-frequency of the whole segment becomes η_2-close to ρ. To each pair of adjacent rank 2 blocks we therefore may associate a set R of admissible shifts.
- Secondly, N_2 is so large that the contribution of α-frequency of a rank 2 block to the α-frequency of any element in R is so small that the frequency of their sum is still in $[\rho - \eta_3, \rho + \eta_3]$.

At the end of the day, we construct cross sections $\mathcal{C}_n$, sub sections $\mathcal{D}_n \subseteq \mathcal{C}_n$, and shift maps $h_{n+1} : \mathcal{C}_n \to (-\epsilon_1, \epsilon_1)$ such that

(i) $x + h_{n+1}(x) \in \mathcal{C}_{n+1}$ for all $x \in \mathcal{C}_n$.
(ii) h_{n+1} is constant on $\mathsf{E}^{\alpha,\beta}_{\mathcal{C}_n}$-classes.
(iii) If $x \in \mathcal{C}_n$ belongs to a rank k block, then $h_{n+1}(x) \in (-\epsilon_{k+1}, \epsilon_{k+1})$.
(iv) If $h_{n+1}(x) \neq 0$, then $x + h_{n+1}(x)$ belongs to a rank $n+1$ block in $\mathcal{C}_{n+1}$.
(v) Each orbit in $\mathcal{C}_n$ has blocks of rank n, and $\mathcal{D}_n$ consists of left endpoints of rank n blocks.
(vi) Cross sections $\mathcal{D}_n$ have bounded gaps: $\vec{\mathrm{gap}}_{\mathcal{D}_n}(x) \leqslant D_n$ for all $x \in \mathcal{D}_n$.
(vii) To every pair of adjacent points in $\mathcal{D}_n$ we associate a set of admissible shifts which consists of tileable reals η_n-close to ρ.

Let $f_{n,n+1} : \mathcal{C}_n \to \mathcal{C}_{n+1}$ be given by $f_{n,n+1}(x) = x + h_{n+1}(x)$, set
$$f_{m,n} = f_{n-1,n} \circ f_{n-2,n-1} \circ \cdots \circ f_{m,m+1}$$

[12] Adjacent among rank 1 intervals, between such a pair there is, of course, a bunch of rank 0 points.

to be the embedding $\mathcal{C}_m \to \mathcal{C}_n$ for $m \leqslant n$ with the natural agreement that $f_{m,m}$ is the identity map. With these notations we may define

$$H_n : \mathcal{C}_n \to \Big(-\sum_{k=1}^{\infty} \epsilon_k, \sum_{k=1}^{\infty} \epsilon_k\Big) \quad \text{to be} \quad H_n(x) = \sum_{k=n}^{\infty} h_{k+1}\big(f_{n,k}(x)\big).$$

Despite the fact that each h_{n+1} can for some points of $\mathcal{C}_n$ be as large as ϵ_1, the sum converges. This follows from items (iii) and (iv) above.

The limit cross section $\mathcal{C}$ is defined to be the union

$$\mathcal{C} = \bigcup_n \{\, x + H_n(x) : x \in \mathcal{C}_n \,\}.$$

Note that this union is increasing by (ii). It is easy to see that the $\{\alpha, \beta\}$-chain relation $E^{\alpha,\beta}_{\mathcal{C}}$ on $\mathcal{C}$ is the union of (shifted) $E^{\alpha,\beta}_{\mathcal{C}_n}$ relations: $x\, E^{\alpha,\beta}_{\mathcal{C}}\, y$ if and only if there is n such that $\big(x - H_n(x)\big)\, E^{\alpha,\beta}_{\mathcal{C}_n}\, \big(y - H_n(y)\big)$.

For $\mathcal{C}$ to be an $\{\alpha, \beta\}$-regular cross section, the relation $E^{\alpha,\beta}_{\mathcal{C}}$ must coincide with $E_{\mathcal{C}}$, which is not necessarily the case. But $\mathcal{C}$ has arbitrarily large regular blocks, which as we noted earlier in Subsection 4.1 is enough. We remind the reader, that all orbits fall into three categories.

- On some orbits $\mathcal{C}$ is $\{\alpha, \beta\}$-regular, these are precisely the orbits on which $E^{\alpha,\beta}_{\mathcal{C}}$ consists of a single equivalence class.
- On other orbits $E^{\alpha,\beta}_{\mathcal{C}}$ may have at least two classes, one of which is infinite.
- Finally, all $E^{\alpha,\beta}_{\mathcal{C}}$-classes are finite one some orbits.

The decomposition of the space into orbits of these types is Borel, and we may deal with each of them separately. The restriction of the flow on the second type of orbits is smooth, since we may select finite endpoints of infinite classes for a Borel transversal. Orbits of the third type are sparse. The sparse cross sections is given by the endpoints of $E^{\alpha,\beta}_{\mathcal{C}}$-classes. We may therefore employ Theorem 8.1 and build the required cross section $\mathcal{C}$ on this part of the space. It is in this sense that our argument here is complementary to the one of Theorem 8.1 — where the current method fails, the one from 8.1 succeeds.

We are finally left with the orbits on which $\mathcal{C}$ is indeed an $\{\alpha, \beta\}$-regular cross section. There is no loss in generality to assume that all the orbit are of this sort. We need to verify (10). During the construction of $\mathcal{C}_n$, we also defined sub sections $\mathcal{D}_n \subseteq \mathcal{C}_n$, which consisted of left endpoints of blocks of rank n, as well as reals D_n such that all the gaps in $\mathcal{D}_n$ are bounded by D_n. Let $\widetilde{\mathcal{D}}_n$ be the sub section of $\mathcal{C}$ that corresponds to $\mathcal{D}_n$,

$$\widetilde{\mathcal{D}}_n = \{\, x + H_n(x) : x \in \mathcal{D}_n \,\}.$$

Since gaps in $\widetilde{\mathcal{D}}_n$ are still bounded, in view of Lemma 6.5 it is enough to show that all the gaps in $\widetilde{\mathcal{D}}_n$ have α-frequencies η_n-close to ρ:

$$\big|\mathrm{fr}_\alpha\big(\vec{\mathrm{gap}}_{\widetilde{\mathcal{D}}_n}(x)\big) - \rho\big| < \eta_n.$$

Indeed, by construction for any adjacent $z_1, z_2 \in \mathcal{D}_n$, we selected a family $R(z_1, z_2)$ of admissible shifts such that any $a \in R(z_1, z_2)$ has α-frequency η_n-close to ρ. Since for any $m \geqslant n$, shifts at stage m agree with admissible shifts at stage n, it means that whenever z_1, z_2 are shifted, the $\mathrm{dist}(z_1, z_2)$ becomes tileable with α-frequency η_n-close to ρ. Finally, since we are working on the part of the space, where all orbits of $\mathcal{C}$ are $\{\alpha, \beta\}$-regular, it follows that all adjacent z_1, z_2 are tiled at some stage of the construction, and the theorem follows. □

10. Lebesgue orbit equivalence

In this closing section we would like to offer an application of our Main Theorem to the classification of Borel flows up to Lebesgue orbit equivalence. But first we need to do some preliminary work.

10.1. **Compressible equivalence relations.** Let E be a countable Borel equivalence relation on a standard Borel space X. A **partial full group** of E is the set of all partial Borel injections which act within E-classes:

$$[\![\mathsf{E}]\!] = \left\{ f : A \to B \;\middle|\; A, B \subseteq X \text{ are Borel, } f \text{ is a Borel bijection and } x \,\mathsf{E}\, f(x) \text{ for all } x, y \in A \right\}.$$

For Borel $A, B \subseteq X$ we let $A \sim B$ to denote existence of a bijection $f : A \to B$, $f \in [\![\mathsf{E}]\!]$. Recall that E is said to be **compressible** if either of the following two equivalent conditions is satisfied:

- There exist a Borel set $A \subseteq X$ such that $X \sim A$ and $X \setminus A$ intersects each E-class.
- There are Borel $A_i \subseteq X$, $i \in \mathbb{N}$, such that $X \sim A_i$ for all i and $A_i \cap A_j = \varnothing$ for $i \neq j$.

A celebrated theorem of Nadkarni [Nad90] gives a powerful characterization of compressible relations.

Theorem (Nadkarni [Nad90], Becker–Kechris [BK96, Chapter 4])**.** *An aperiodic*[13] *countable Borel equivalence relation* E *is compressible if and only if it does not admit a finite invariant measure.*

Definition 10.1. A subset $A \subseteq X$ is said to be E-**syndetic** (or just **syndetic** when E is understood) if there exist $n \in \mathbb{N}$ and Borel sets $A_i \subseteq X$, $1 \leqslant i \leqslant n$, such that $A \sim A_i$ and

$$X = \bigcup_{i=1}^{n} A_i.$$

Here is a typical example of a syndetic set. Suppose E is the orbit equivalence relation of an aperiodic Borel automorphism $T : X \to X$, $\mathsf{E} = \mathsf{E}_X^T$. Suppose $A \subseteq X$ has bounded gaps in the sense that there is $N \in \mathbb{N}$ such that for each $x \in A$ there are $1 \leqslant i, j \leqslant N$ satisfying $T^i(x) \in A$ and $T^{-j}(x) \in A$. If A intersects each orbit of T then A is necessarily E-syndetic as we may take $A_i = T^i(A)$ for $0 \leqslant i \leqslant N$.

The concept of syndeticity allows for a convenient reformulation of compressibility.

Proposition 10.2. *Let* E *be an aperiodic countable Borel equivalence relation on a standard Borel space* X*. It is compressible if and only if* $A \sim B$ *for any two Borel syndetic sets* $A, B \subseteq X$*.*

Proof. We begin by proving necessity. Suppose E is compressible. Since the set X itself is obviously syndetic and $\sim$ is a symmetric and transitive relation, it is enough to show that $A \sim X$ holds for any Borel syndetic set $A \subseteq X$. Pick a syndetic set $A \subseteq X$ and let E_A be the restriction of E onto A:

$$\mathsf{E}_A = \mathsf{E} \cap (A \times A).$$

Our first goal is to show that E_A is compressible. Let $f_i \in [\![\mathsf{E}]\!]$, $1 \leqslant i \leqslant n$, $\mathrm{dom}(f_i) = A$, be such that $X = \bigcup_{i=1}^{n} f_i(A)$. Set $\tilde{A}_1 = A$ and define $\tilde{A}_i$ for $i \leqslant n$ inductively by setting

$$\tilde{A}_{i+1} = f_{i+1}^{-1}\Big(X \setminus \bigcup_{k=1}^{i} f_k(A)\Big).$$

Sets $\tilde{A}_i \subseteq A$ correspond to the partition of X:

$$X = \bigsqcup_{i=1}^{n} f_i(\tilde{A}_i).$$

If μ is a finite E_A-invariant measure on A, then $\sum_{i=1}^{n} (f_i)_* \mu_i$ is a finite E-invariant measure on X, where $\mu_i = \mu|_{\tilde{A}_i}$. By Becker–Kechris–Nadkarni Theorem the compressibility of E implies the compressibility of E_A.

We may therefore pick Borel injections $\tau_i \in [\![\mathsf{E}_A]\!]$, $i \in \mathbb{N}$, such that $\tau_i(A) \cap \tau_j(A) = \varnothing$ for $i \neq j$. Let $h : X \to A$ be given by

$$h(x) = \tau_i\big(f_i^{-1}(x)\big) \quad \text{for } x \in f_i(\tilde{A}_i).$$

Note that the map $h : X \to A$ is injective and witnesses $X \sim h(X)$. On the other hand the identity map $\mathrm{id} : A \to X$ is an injection in the other direction. The usual Schröder–Bernstein argument produces a bijection $\theta : A \to X$, $\theta \in [\![\mathsf{E}]\!]$, thus proving $A \sim X$ as required.

It remains to show sufficiency. Since X is necessarily syndetic, it is enough to construct a syndetic set $B \subseteq X$ such that $X \setminus B$ intersects each E-class; any $f \in [\![\mathsf{E}]\!]$ satisfying $f(X) = B$ will then witness compressibility

[13] An equivalence relation is aperiodic if each equivalence class is infinite.

of E. By the proof of Feldman–Moore Theorem [FM77] we may pick a countable family $h_i \in [\![\mathsf{E}]\!]$, $i \in \mathbb{N}$, and Borel $B_i \subseteq X$ such that $\mathrm{dom}(h_i) = B_i$, $B_i \cap h_i(B_i) = \varnothing$, and

$$x \mathsf{E} y \iff x = y \text{ or } \Big(\exists n\ x \in B_n \text{ and } h_n(x) = y\Big).$$

Define inductively sets $\tilde{B}_n$ by setting $\tilde{B}_0 = B_0$ and

$$\tilde{B}_{n+1} = \Big\{x \in B_{n+1} : x, h_{n+1}(x) \notin \bigcup_{i \leqslant n} \big(\tilde{B}_i \cup h_i(\tilde{B}_i)\big)\Big\}.$$

Let $B = \bigcup_i \tilde{B}_i$ and set $\theta : B \to X$ to be given by $\theta(x) = h_n(x)$, where n is such that $x \in \tilde{B}_n$ (note that such n is necessarily unique). The map $\theta \in [\![\mathsf{E}]\!]$, sets B and $\theta(B)$ are disjoint and moreover, for each $x \in X$ the set $[x]_{\mathsf{E}} \setminus \big(B \cup \theta(B)\big)$ has cardinality at most 1, for if

$$y_1, y_2 \in [x]_{\mathsf{E}} \setminus \big(B \cup \theta(B)\big) \text{ are distinct,}$$

then $y_1 \in \tilde{B}_n$ for the smallest n such that $h_n(y_1) = y_2$, contradicting the definition of B. The restriction of E onto the set of $x \in X$ where

$$\big|[x]_{\mathsf{E}} \setminus \big(B \cup \theta(B)\big)\big| = 1$$

is smooth and we may modify the set B and the map θ on this part to ensure that $X = B \sqcup \theta(B)$. The set B is thus E-syndetic and $X \setminus B$ intersects every E-class as desired. ☐

10.2. **Matching equidense sets.** In the previous subsection we worked in the context of countable Borel equivalence relations. We now turn to a more specific situation of orbit equivalence relations that arise from the actions of $\mathbb{Z}$. In this case each orbit naturally inherits the linear order from $\mathbb{Z}$.

Let $\mathsf{T} : X \to X$ be an aperiodic automorphism. We say that a subset $A \subseteq X$ has **uniform frequency** $\rho \in [0, 1]$ if for every $\eta > 0$ there exists N such that for all $n \geqslant N$ and all $x \in X$

$$\Big|\rho - \frac{1}{n}\sum_{k=0}^{n-1} \chi_A\big(\mathsf{T}^k(x)\big)\Big| < \eta.$$

Note that when $\rho > 0$, any set $A \subseteq X$ of uniform frequency ρ is necessarily $\mathsf{E}_X^{\mathsf{T}}$-syndetic. Note also that the Main Theorem constructs an $\{\alpha, \beta\}$-regular cross section for which sets of α- and β-points have uniform frequency ρ and $1 - \rho$ respectively.

Theorem 10.3. *Let $\mathsf{T} : X \to X$ be an aperiodic Borel automorphism. If $A, B \subseteq X$ are sets of the same uniform frequency $\rho \in (0, 1]$, then $A \sim B$.*

Proof. Define inductively sets A_k and B_k, $k \in \mathbb{N}$, by the formula

$$A_k = \big\{x \in A \setminus \bigcup_{i<k} A_i : \mathsf{T}^k(x) \in B \setminus \bigcup_{i<k} B_i\big\},$$
$$B_k = \mathsf{T}^k(A_k).$$

Note that sets A_k are pairwise disjoint and $A_k \subseteq A$ for all k. Similarly, for the sets B_k. Set $A_\infty = \bigcup_k A_k$ and $B_\infty = \bigcup_k B_k$. Define $\tilde{\theta} : A_\infty \to B_\infty$ by $\tilde{\theta}(x) = \mathsf{T}^k(x)$ whenever $x \in A_k$.

We claim that $\mu(A \setminus A_\infty) = 0$ for any T-invariant probability measure μ on X. Indeed, let μ be such a measure which we furthermore assume ergodic. By Birkhoff's Ergodic Theorem $\mu(A) = \rho = \mu(B)$. Since $\mu(A_\infty) = \mu(B_\infty)$, we get $\mu(A \setminus A_\infty) = \mu(B \setminus B_\infty)$. Suppose $\mu(A \setminus A_\infty) > 0$. By ergodicity $\mu\big(\bigcup_k \mathsf{T}^k(A \setminus A_\infty)\big) = 1$, and therefore

$$\Big(\bigcup_k \mathsf{T}^k(A \setminus A_\infty)\Big) \cap (B \setminus B_\infty) \neq \varnothing.$$

If $x \in A \setminus A_\infty$ is such that $\mathsf{T}^k(x) \in B \setminus B_\infty$ for some $k \in \mathbb{N}$, then $x \in A_k$ contradicting $x \notin A_\infty$. We conclude that $\mu(A \setminus A_\infty) = 0$. By ergodic decomposition the same is true for all (not necessarily ergodic) invariant probability measures μ on X.

Set $Z = [A \setminus A_\infty]_{E_X^T} \cup [B \setminus B_\infty]_{E_X^T}$. By above $\mu(Z) = 0$ for any T-invariant measure, and so the restriction of T onto Z is compressible. Proposition 10.2 applies and produces a map $\bar\theta \in [\![T|_Z]\!]$ such that $\bar\theta(A \cap Z) = B \cap Z$. The map θ defined by

$$\theta(x) = \begin{cases} \tilde\theta(x) & \text{when } x \in A \setminus Z, \\ \bar\theta(x) & \text{when } x \in A \cap Z, \end{cases}$$

witnesses $A \sim B$. □

10.3. **Lebesgue orbit equivalence.** Recall that an **orbit equivalence** between two group actions $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ is a Borel bijection $\phi : X_1 \to X_2$ which sends orbits onto orbits: $\phi\big(\mathrm{Orb}_{\Gamma_1}(x)\big) = \mathrm{Orb}_{\Gamma_2}\big(\phi(x)\big)$ for all $x \in X_1$. In the language of equivalence relations, two actions are orbit equivalent if and only if the orbit equivalence relations they generate are isomorphic. Actions of $\mathbb{Z}$ give rise to **hyperfinite** equivalence relations, i.e., equivalence relations which can be written as an increasing union of finite equivalence relations. Hyperfinite relations have been classified by Dougherty, Jackson, and Kechris in [DJK94]. The important part of their classification is as follows.

Theorem (Dougherty–Jackson–Kechris). *Non-smooth aperiodic hyperfinite equivalence relations are isomorphic if and only if they admit the same number of invariant probability ergodic measures.*

When one considers actions of non-discrete groups, e.g., Borel flows, the number of invariant ergodic probability measures is no longer an invariant of orbit equivalence. When actions under consideration are free, one can overcome this obstacle by considering the notion of Lebesgue orbit equivalence. For simplicity of the presentation we restrict ourselves to the case of Borel flows. For rigorous proofs of the following statements and for a more general treatment see [Slu15].

Let $\mathfrak{F}$ be a free Borel flow on a standard Borel space X. Recall from Subsection 2.1 that any orbit of the action can be endowed with a copy of the Lebesgue measure: for $A \subseteq X$ let $\lambda_x(A) = \lambda(\{r \in \mathbb{R} : x + r \in A\})$. It is immediate to see that $\lambda_x = \lambda_y$ for all $y \in \mathrm{Orb}(x)$, i.e., the measure λ_x depends only on the orbit of x, but not on the x itself.

Suppose $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are free Borel flows on X_1 and X_2 respectively, and let $\phi : X_1 \to X_2$ be an orbit equivalence between these flows. We say that ϕ is a **Lebesgue orbit equivalence** (LOE for short) if ϕ is a Lebesgue measure preserving map when restricted onto any orbit: $\phi_*\lambda_x = \lambda_{\phi(x)}$. Any LOE map preserves the number of invariant ergodic probability measures.

In [Slu15] the analog of DJK classification has been proved for free Borel actions of $\mathbb{R}^n$. Here we would like to show how the Main Theorem gives a simple proof of this classification for the particular case of flows. Recall the notation from Subsection 2.5, where $\mathcal{E}(\mathfrak{F})$ denotes the set of invariant ergodic probability measures for the flow $\mathfrak{F}$.

Theorem 10.4. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be free non-smooth Borel flows. These flows are LOE if and only if*

$$\big|\mathcal{E}(\mathfrak{F}_1)\big| = \big|\mathcal{E}(\mathfrak{F}_2)\big|.$$

Proof. As we mentioned earlier, $\Longrightarrow$ is an easy direction and the reader is referred to Section 4 of [Slu15]. Suppose $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have the same number of invariant measures. By the Main Theorem each flow can be represented as a flow under a two-valued function, and moreover one may assume that gaps of each type occur with uniform frequency 1/2 within every orbit. More formally, we may pick Borel $\{\alpha, \beta\}$-regular cross sections $\mathcal{C}^1 \subseteq X_1$ and $\mathcal{C}^2 \subseteq X_2$, α and β are some positive rationally independent reals, say 1 and $\sqrt{2}$, such that both $\mathcal{C}^i_\alpha$ and $\mathcal{C}^i_\beta$ occur with uniform frequency 1/2 within every orbit, $i = 1, 2$, where

$$\mathcal{C}^i_\alpha = \big\{x \in \mathcal{C}^i : \vec{\mathrm{gap}}_{\mathcal{C}^i}(x) = \alpha\big\}.$$

By Theorem 10.3, there are Borel bijections $\theta^i : \mathcal{C}^i_\alpha \to \mathcal{C}^i_\beta$ which preserve the orbit equivalence relation: $\theta^i \in [\![E_{\mathfrak{F}_i}]\!]$.

Note that the cross sections $\mathcal{C}^i_\alpha$ have to have bounded gaps. Indeed, $\mathcal{C}^i_\alpha$ has uniform frequency 1/2 in $\mathcal{C}^i$, and therefore there is N such that for all $n \geqslant N$ and all $x \in \mathcal{C}^i$ one has

$$\Big|\frac{1}{2} - \frac{1}{n}\sum_{k=0}^{n-1} \chi_{\mathcal{C}^i_\alpha}\big(\phi^k_{\mathcal{C}^i}(x)\big)\Big| < 1/4.$$

In particular, for any $x \in \mathcal{C}^i$ there is $0 \leqslant k < N$ such that $\phi^k_{\mathcal{C}^i}(x) \in \mathcal{C}^i_\alpha$; in other words, any interval inside $\mathcal{C}^i$ of length at least N contains a point from $\mathcal{C}^i_\alpha$.

In view of Theorem 2.6 the induced automorphisms $\phi_{\mathcal{C}^1_\alpha}$ and $\phi_{\mathcal{C}^2_\alpha}$ have the same number of invariant ergodic probability measure. Therefore, by the DJK classification there exists a Borel orbit equivalence $\psi : \mathcal{C}^1_\alpha \to \mathcal{C}^2_\alpha$ between $\mathsf{E}_{\mathcal{C}^1_\alpha}$ and $\mathsf{E}_{\mathcal{C}^2_\alpha}$. Maps θ^i allow us to extend ψ to an orbit equivalence $\mathcal{C}^1 \to \mathcal{C}^2$ by setting

$$\psi\big(\theta^1(x)\big) = \theta^2\big(\psi(x)\big) \quad \text{for all } x \in \mathcal{C}^1_\alpha.$$

The map $\psi : \mathcal{C}^1 \to \mathcal{C}^2$ sends α-points to α-points and also β-points to β-points. This allows us to extend it linearly to a LOE $\psi : X_1 \to X_2$ by setting

$$\psi(x + r) = \psi(x) + r$$

where $r \in [0, \alpha)$ if $x \in \mathcal{C}^1_\alpha$ and $r \in [0, \beta)$ whenever $x \in \mathcal{C}^1_\beta$. The map ψ when restricted onto any orbit is a piecewise translation with countably many pieces and is therefore obviously Lebesgue measure preserving. □

The construction of LOE presented in [Slu15] is based on a similar idea on a sparse piece, but uses a different construction on a compressible part, resulting in a LOE which is a piecewise translation with countably many pieces, but sizes of pieces may be arbitrarily small on some orbits. The argument presented above has the advantage of constructing LOE with all translation pieces being of size α or β.

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