NON-GENERICITY PHENOMENA IN ORDERED FRAÏSSÉ CLASSES

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ABSTRACT. We show that every two-dimensional class of topological similarity, and hence every diagonal conjugacy class of pairs, is meager in the group of order preserving bijections of the rationals and in the group of automorphisms of the ordered rational Urysohn space.

1. INTRODUCTION

The size of conjugacy classes (in the topological sense: dense, meager, comeager, etc.) in the groups of automorphisms of Fraïssé limits has recently become an active area of research. This is partially due to the newly revealed connections between combinatorial properties of Fraïssé classes and algebraic, topological, and dynamical properties of the groups of automorphisms of their limits. One of the most astonishing links was established by Kechris, Pestov and Todorcevic in [KPT05] and displays a close relationship between Ramsey theory (a purely combinatorial area) and extreme amenability (a classical dynamical notion).

Another reason for the interest in the size of conjugacy classes of Polish groups in general, and groups of automorphisms of Fraïssé limits in particular, comes from the special importance of some concrete groups, e.g., the group of automorphisms of the countable atomless Boolean algebra (which is isomorphic via Stone's theorem to the group of homeomorphisms of the Cantor space), the group of isometries of the rational Urysohn space, and the group of order preserving bijections of the rationals. The reader may consult [KR07] for details, examples, and a deep structural theory for the groups with large conjugacy classes.

In [Tru07] J. K. Truss looked at different possible notions of genericity of conjugacy classes and discussed advantages of each. In general, conjugacy classes are objects that are difficult to understand, and the conjugacy relation may sometimes be very complicated (complete analytic). Motivated by the work of Truss, in this paper we look at a coarser equivalence relation than conjugacy, namely at classes of topological similarity. They are much easier to work with and can be used to prove *non*-genericity in some cases.

Let us recall the definition from [Ros09].

Definition 1.1. If G is a topological group, an n-tuple $(g_1, \dots, g_n) \in G^n$ is said to be *topologically similar* to an n-tuple $(f_1, \dots, f_n) \in G^n$ if the map F sending $g_i \mapsto f_i$ extends (necessarily uniquely) to an isomorphism, that is continuous and has continuous inverse, between the groups generated by these tuples

$$F : \langle g_1, \dots, g_n \rangle \to \langle f_1, \dots, f_n \rangle.$$

We denote this relation by E^n_{TS}.

There is another natural relation on the n-tuples in G, namely the relation of diagonal conjugation, i.e. $(g_1, \dots, g_n)$ is conjugate to $(f_1, \dots, f_n)$ if there is some $\alpha \in G$ such that

$$(\alpha g_1 \alpha^{-1}, \dots, \alpha g_n \alpha^{-1}) = (f_1, \dots, f_n).$$

More generally, if G is a topological subgroup of H, then one can restrict the conjugacy relation in H to G, i.e. say that an n-tuple $(g_1, \dots, g_n) \in G^n$ conjugates in H to an n-tuple $(f_1, \dots, f_n) \in G^n$ if there is some $\beta \in H$ such that

$$(\beta g_1 \beta^{-1}, \dots, \beta g_n \beta^{-1}) = (f_1, \dots, f_n).$$

It is easy to see that this is an equivalence relation on n-tuples of G and we denote it by E^n_H (this relation also depends on G and on the embedding of G into H, but this information is usually clear from the context). The following proposition is obvious.

Proposition 1.2. *Let G be a topological group and $n \in \mathbb{N}$, then*

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(i) the relation of topological similarity is an equivalence relation;
(ii) if H is a topological group such that $G \leqslant H$ is a topological subgroup of it, then E^n_H is finer (not necessarily strictly) than E^n_{TS}. In particular, E^n_G is finer than E^n_{TS}.

Equivalence classes of topological similarity on n-tuples are called *n-dimensional similarity classes*, in particular a two-dimensional similarity class is a set of pairs.

Since conjugacy classes refine classes of topological similarity, if one wants to prove meagerness of the former, it suffices to prove meagerness of the latter (it suffices, but may be impossible, meagerness of conjugacy classes does not imply meagerness of classes of topological similarity). This sometimes turns out to be an easier task. For example, Rosendal in [Ros09] developed this idea to find a simple proof of A. del Junco's result, that each conjugacy class of measure preserving automorphisms of the standard Lebesgue space is meager.

Hodkinson showed (see [Tru07] for the details) that in the group of order preserving automorphisms of the rationals all conjugacy classes of pairs are meager (though it is known that there is a comeager one-dimensional conjugacy class in $\mathrm{Aut}(\mathbb{Q})$). In this paper we strengthen this result and show that all two-dimensional classes of topological similarity are meager in this group.

The paper is organized as follows. In the second section we give a brief introduction to the theory of Fraïssé classes, which is the right context for the technique developed in this paper. The third section is devoted to the strengthening of the Hodkinson's result on classes of diagonal conjugation in the rationals, and in the fourth and fifth sections an analogous theorem is proved for the group of order preserving isometries of the ordered rational Urysohn space.

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2. Brief introduction to Fraïssé classes

In this section we give a short introduction to the theory of Fraïssé classes. In the next two sections we deal with two examples of them, so it is useful to keep in mind this more general setting. The classical text on Fraïssé classes is a beautiful book by Hodges [Hod93].

Let L be a relational first order language. We use the standard notation: solid arrows correspond to "for all" quantifiers, and dashed arrows represent maps that "exist".

Definition 2.1. Let $\mathbf{A}$ and $\mathbf{B}$ be two L-structures. A map $f : \mathbf{A} \to \mathbf{B}$ is called a *strong homomorphism* if for any $\mathcal{R} \in L$ of arity m and for any $x_1, \ldots, x_m \in \mathbf{A}$

$$\mathcal{R}^{\mathbf{A}} x_1 \ldots x_m \iff \mathcal{R}^{\mathbf{B}} f(x_1) \ldots f(x_m).$$

The map is a *strong embedding* if it is an injective strong homomorphism (if we adopt a convention that the equality sign "$=$" is always in the set of relations then any strong homomorphism is necessarily injective).

Definition 2.2. Let $\mathcal{K}$ be a class of finite L-structures. For L-structures $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A} \leqslant \mathbf{B}$ we mean "$\mathbf{A}$ strongly embeds into $\mathbf{B}$". $\mathcal{K}$ is called a Fraïssé class if the following properties hold:

(HP) If $\mathbf{A} \leqslant \mathbf{B}$ and $\mathbf{B} \in \mathcal{K}$ then $\mathbf{A} \in \mathcal{K}$;
(JEP) For $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B} \in \mathcal{K}$ there is some $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \leqslant \mathbf{C}$ and $\mathbf{B} \leqslant \mathbf{C}$;
(AP) For $\mathbf{A} \in \mathcal{K}$, $\mathbf{B} \in \mathcal{K}$, $\mathbf{C} \in \mathcal{K}$ and embeddings $i : \mathbf{A} \to \mathbf{B}$, $j : \mathbf{A} \to \mathbf{C}$ there are $\mathbf{D} \in \mathcal{K}$ and embeddings $k : \mathbf{B} \to \mathbf{D}$, $l : \mathbf{C} \to \mathbf{D}$ such that $k \circ i = l \circ j$, i.e., the following diagram commutes

$$\begin{tikzcd} & \mathbf{B} \arrow[dr, dashed, "k"] & \\ \mathbf{A} \arrow[ur, "i"] \arrow[dr, "j"'] & & \mathbf{D} \\ & \mathbf{C} \arrow[ur, dashed, "l"'] & \end{tikzcd}$$

(Inf) $\mathcal{K}$ contains structures of arbitrarily high finite cardinality and has up to isomorphism only countably many structures.

Basic examples of Fraïssé classes are: finite sets, finite linear orders, finite graphs and finite metric spaces with rational distances (to satisfy (Inf) condition). With a Fraïssé class $\mathcal{K}$ one can associate its Fraïssé limit (which is unique up to an isomorphism).

Definition 2.3. The countably infinite structure $\mathbb{K}$ is called a *Fraïssé limit* of the class $\mathcal{K}$ if the following holds:

(i) Finite substructures of $\mathbb{K}$ up to isomorphism are exactly the elements of $\mathcal{K}$;
(ii) $\mathbb{K}$ is ultrahomogeneous (that is any isomorphism between finite substructures of $\mathbb{K}$ extends to a full automorphism of $\mathbb{K}$).

Fraïssé limits of the above Fraïssé classes are: $\mathbb{N}$ –- countably infinite set, $\mathbb{Q}$ –- dense linear ordering without endpoints, $\mathbb{G}$ –- random graph, $\mathbb{QU}$ –- rational Urysohn space.

In the next section we deal with the simplest linearly ordered Fraïssé class: with the rationals $\mathbb{Q}$. The fourth and fifth sections are devoted to the case of the linearly ordered rational Urysohn space.

3. Topological Similarity Classes in the Groups $\mathrm{Aut}(\mathbb{Q})$ and $\mathrm{Homeo}^+([0,1])$

Let $\mathbb{Q}$ denote the rational numbers viewed as a linearly ordered set. By an open interval $I = (a, b) \subset \mathbb{Q}$ we mean the set of rational numbers $\{c : a < c < b\} \subset \mathbb{Q}$. A closed interval $[a, b]$ also includes endpoints a and b. If I is a bounded interval (open or closed) $L(I)$ will denote its left endpoint and $R(I)$ will be its right endpoint. If $\mathbf{A} \subset \mathbb{Q}$ is a finite subset, $\min(\mathbf{A})$ and $\max(\mathbf{A})$ will denote its minimal and maximal elements respectively.

Let G denote the group $\mathrm{Aut}(\mathbb{Q})$ of order preserving bijections of the rationals.

Definition 3.1. A partial isomorphism of $\mathbb{Q}$ is an order preserving bijection p between *finite* subsets $\mathbf{A}$ and $\mathbf{B}$ of $\mathbb{Q}$.

It is a basic property of the rationals (and, as mentioned earlier, of a Fraïssé limit in general) that each partial isomorphism can be extended (certainly, not uniquely) to a full automorphism.

Letters p and q (with possible sub- or superscripts) will denote partial isomorphisms; let $\mathrm{dom}(p)$ be the domain of p, and $\mathrm{ran}(p)$ be its range. If $I \subseteq \mathbb{Q}$ then $p|_I$ denotes the restriction of p on $I \cap \mathrm{dom}(p)$; $\mathrm{F}(p)$ will be the set of fixed points in the domain of p, i.e.,

$$\mathrm{F}(p) = \{c \in \mathrm{dom}(p) : p(c) = c\}.$$

First we recall that G is a Polish group (i.e., a separable completely metrizable topological group) in the topology given by the basic open sets

$$U(p) = \{g \in G : g \text{ extends } p\},$$

where p is a partial isomorphism of $\mathbb{Q}$. Note that if p and q are two partial isomorphisms and q extends p then $U(q) \subseteq U(p)$; we will use this observation frequently. We denote the identity element of G by 1.

We use the words generic and comeager as synonyms. For example, a property is generic in the group G if the set of elements with this property is comeager in G.

Let $F(s, t)$ denote the free group on two generators: s and t; elements of $F(s, t)$ are reduced words on the alphabet $\{s, t, s^{-1}, t^{-1}\}$. Every element $w \in F(s, t)$ has certain length associated to it, namely the length of the reduced word w. This length is denoted by $|w|$. If $u, v \in F(s, t)$ are words, we say that the word $uv \in F(s, t)$ is reduced if $|uv| = |u| + |v|$, that is there is no cancellation between u and v.

If $w \in F(s, t)$ is a reduced word, $w = t^{n_k} s^{m_k} \cdots t^{n_1} s^{m_1}$, and p, q are partial isomorphisms, then we can define a partial isomorphism $w(p, q)$ by $w(p, q)(c) = q^{n_k} p^{m_k} \cdots q^{n_1} p^{m_1}(c)$, whenever the right-hand side is defined. The *orbit* of c under $w(p, q)$ is by definition

$$\mathrm{Orb}_{w(p,q)}(c) = \cup_{l=1}^{k} \{p^{i \,\mathrm{sign}(m_l)} q^{n_{l-1}} \cdots p^{m_1}(c),\ q^{j \,\mathrm{sign}(n_l)} p^{m_l} \cdots p^{m_1}(c) : i = 0, \ldots, |m_l|,\ j = 0, \ldots, |n_l|\}.$$

We say that a word w starts from the word v if w can be written as $w = vu$ for some word u, where vu is reduced. Similarly, we say that w ends in v if there is a word u such that $w = uv$, where uv is reduced. On the one hand this is consistent with the intuitive understanding of these notions for, say, left-to-right languages. On the other hand, we consider left actions, and then *the end of the word acts first*, i.e., if $w = st$ then $w(p, q)(c) = p(q(c))$. This may be a bit confusing, we apologize for that and emphasize this possible confusion.

Definition 3.2. Let p be a partial isomorphism of $\mathbb{Q}$. An interval $(a,b) \subset \mathbb{Q}$ is called *p-increasing* if $a, b \in \mathrm{dom}(p)$, $p(a) = a$, $p(b) = b$ and $p(c) > c$ for any $c \in \mathrm{dom}(p) \cap (a,b)$. The definition of *$p$-decreasing* interval is analogous. Note that if $[a,b] \cap \mathrm{dom}(p) = \{a,b\}$ and $p(a) = a$, $p(b) = b$ then the interval (a,b) is both p-increasing and p-decreasing. An interval is *p-monotone* if it is either p-increasing or p-decreasing.

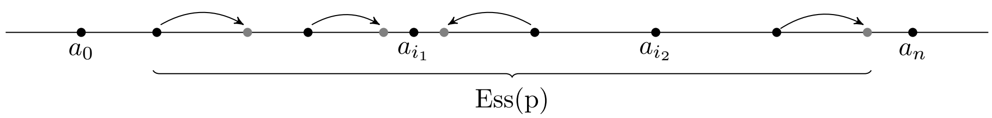
Definition 3.3. Let p be a partial isomorphism. Let $\mathrm{dom}(p) = \{a_0, \ldots, a_n\}$ and assume that $a_0 < \ldots < a_n$. We say that p is *informative* if $p(a_0) = a_0$, $p(a_n) = a_n$ and there is a list $\{i_0, \ldots, i_r\}$ of indices such that

(i) $i_0 = 0$, $i_r = n$;
(ii) $a_{i_k} = p(a_{i_k})$ for $0 \leqslant k \leqslant r$;
(iii) for any $0 \leqslant k < r$ the interval $(a_{i_k}, a_{i_{k+1}})$ is p-monotone.

If p is an informative partial isomorphism and $\mathrm{dom}(p) = \{a_0, \ldots, a_n\}$ as above then we set

$$\mathrm{Ess(p)} = \big(\mathrm{dom}(p) \cup \mathrm{ran}(p)\big) \setminus \{a_0, a_n\}$$

and refer to it as to *the set of essential points of p*.



Definition 3.4. A pair (p,q) of partial isomorphisms is called *piecewise elementary* if the following holds

(i) p and q are informative;
(ii) $\min(\mathrm{dom}(p)) = \min(\mathrm{dom}(q))$,
(iii) $\max(\mathrm{dom}(p)) = \max(\mathrm{dom}(q))$.

If additionally $\mathrm{F}(p) \cap \mathrm{F}(q)$ has cardinality at most 2 (i.e., consists of the above minimum and maximum) then the pair (p,q) is called *elementary*.

Let (p,q) be a piecewise elementary pair, and $\mathrm{F}(p) \cap \mathrm{F}(q) = \{a_0, \ldots, a_n\}$ with $a_i < a_j$ for $i < j$. Set $I_j = [a_j, a_{j+1}]$, then $(p|_{I_j}, q|_{I_j})$ is elementary for any $0 \leqslant j < n$. Thus every piecewise elementary pair (p,q) can be decomposed into finitely many elementary pairs.

The following obvious lemma partially explains the importance of piecewise elementary pairs.

Lemma 3.5. *For any non-empty open $V \subseteq G \times G$ there is a piecewise elementary pair (p,q) such that $U(p) \times U(q) \subseteq V$.*

Definition 3.6. Let (p,q) be an elementary pair. We say that a triple (p', q', w) *liberates p in (p,q)*, where p' and q' are partial isomorphisms that extend p and q respectively, and $w \in F(s,t)$ is a reduced word, if the following holds

(i) p' and q' are informative;
(ii) $\min(\mathrm{dom}(p')) = \min(\mathrm{dom}(p))$, $\min(\mathrm{dom}(q')) = \min(\mathrm{dom}(q))$;
(iii) $\max(\mathrm{dom}(p')) = \max(\mathrm{dom}(p))$, $\max(\mathrm{dom}(p')) = \max(\mathrm{dom}(p))$;
(iv) the word w starts from a non-zero power of t, $w = t^n v$ for $n \neq 0$;
(v) $w(p', q')(c)$ is defined for any $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ and
$$w(p',q')(\min(\mathrm{Ess(p)} \cup \mathrm{Ess(q)})) > \max(\mathrm{Ess(p')}),$$
(vi) there is an open interval J such that $R(J) = \max(\mathrm{dom}(q))$, q' is monotone on J and $w(p',q')(c) \in J$ for any $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$; moreover, if $n > 0$ in the item (iv), then J is q'-increasing, and it is q'-decreasing otherwise.

Similarly, we say that a triple (p', q', w) *liberates q in (p,q)* if the above holds with roles of p and q, s and t interchanged.

For a piecewise elementary pair (p,q), we say that a triple (p', q', w) liberates p [liberates q] in (p,q) if

(i) $\min(\mathrm{dom}(p')) = \min(\mathrm{dom}(p))$, $\min(\mathrm{dom}(q')) = \min(\mathrm{dom}(q))$;
(ii) $\max(\mathrm{dom}(p')) = \max(\mathrm{dom}(p))$, $\max(\mathrm{dom}(p')) = \max(\mathrm{dom}(p))$;
(iii) for any interval I, such that $(p|_I, q|_I)$ is elementary, the triple $(p'|_I, q'|_I, w)$ liberates $p|_I$ [liberates $q|_I$] in $(p|_I, q|_I)$.

Lemma 3.7. *For any elementary pair (p, q) there is a triple (p', q', w) that liberates p [liberates q] in (p, q).*

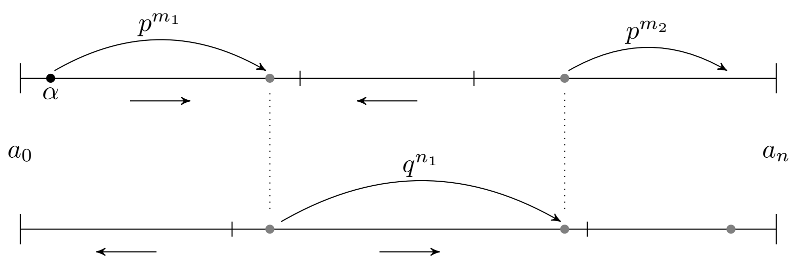
Proof. We show the existence of a triple that liberates p, and the second clause then follows by symmetry.

Extending p and q if necessary, we may assume that

(i) $\mathrm{Ess(p)} \neq \emptyset$, $\mathrm{Ess(q)} \neq \emptyset$;
(ii) $I_1, \ldots, I_k$ are all the (open) intervals of monotonicity for p and $J_1, \ldots, J_l$ are all the (open) intervals of monotonicity for q; we list them in increasing order, i.e., $I_i < I_{i+1}$, $J_j < J_{j+1}$;
(iii) $I_1 \cap \mathrm{dom}(p) \neq \emptyset$ and $J_1 \cap \mathrm{dom}(q) \neq \emptyset$;
(iv) $I_k \cap \mathrm{dom}(p) = \emptyset$ and $J_l \cap \mathrm{dom}(q) = \emptyset$;
(v) $L(I_k) > L(J_l)$.

Let $\alpha = \min(\mathrm{Ess(p)} \cup \mathrm{Ess(q)})$. Then $\alpha \in I_1 \cap J_1$ by (iii) (and in particular α is not a fixed point of p or q). We first find an informative extension p_1 of p that has the same intervals of monotonicity as p and $m_1 \in \mathbb{Z}$ (the sign of m_1 depends on whether p is increasing or decreasing) such that $p_1^{m_1}(\alpha)$ is defined and is "close enough" to the right endpoint of I_1. "Close enough" exactly means the following. Since by assumptions $R(I_1)$ is not fixed by q (because (p, q) is elementary), there is some j_1 such that $R(I_1) \in J_{j_1}$ and we want $p_1^{m_1}(\alpha) \in J_{j_1}$. At the second step we find an informative extension q_1 of q (also with the same intervals of monotonicity) and $n_1 \in \mathbb{Z}$ (similarly the sign of n_1 depends on whether q is increasing or decreasing) such that $q_1^{n_1} p_1^{m_1}(\alpha)$ is defined and is "close enough" in the above sense to the right endpoint of J_{j_1}. We proceed in this way and stop as soon as the image of α reaches J_l, i.e., we obtain extensions $\bar{p}$, $\bar{q}$ of p and q and a word $u = s^{m_{N+1}} v$, where $v = t^{n_N} s^{m_N} \cdots t^{n_1} s^{m_1}$ such that $u(\bar{p}, \bar{q})(\alpha)$ is defined, lies in J_l and $v(\bar{p}, \bar{q})(\alpha) \notin J_l$. Note that since we added to the domain of q only points of the orbit of α under u, this implies $\mathrm{dom}(\bar{q}) \cap J_l = \emptyset$. Also by induction $\bar{p}$ and $\bar{q}$ are informative with the same decomposition into intervals of monotonicity as for p and q.

The following figure illustrates the construction (horizontal arrows indicate monotonicity of partial isomorphisms, bars stand for fixed points, the black dot is the minimal element α, and gray dots are its images under w):



We now take extensions $\bar{p}'$, $\bar{q}'$ of $\bar{p}$ and $\bar{q}$ such that

(i) $u(\bar{p}', \bar{q}')(c)$ is defined for every $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$;
(ii) $\bar{p}'$ and $\bar{q}'$ are informative with the same decomposition into intervals of monotonicity as for $\bar{p}$ and $\bar{q}$;
(iii) the minimum and maximum of the domains of $\bar{p}'$ and $\bar{q}'$ are equal to the minimum and maximum of the domains of $\bar{p}$ and $\bar{q}$;
(iv) $\bar{q}'$ is monotone on J_l (this is possible since $J_l \cap \mathrm{dom}(\bar{q}) = \emptyset$).

Set $p' = \bar{p}'$. Finally extending $\bar{q}'$ to q' we can find $M \in \mathbb{Z} \setminus \{0\}$ such that

$$q'^M u(p', q')(\alpha) > \max(\mathrm{Ess(p')}).$$

And so let $w = t^M u$, then (p', q', w) liberates p in (p, q). □

Remark 3.8. In the lemma above we started our construction by applying a power of p, but we likewise could start it by applying a power of q.

Remark 3.9. We view rationals as a dense linear ordering without endpoints. But note that if we have the usual metric on $\mathbb{Q}$ then the above construction gives us p', q', and w such that $w(p', q')(\alpha)$ is as close in this metric to the endpoint $\max(\mathrm{dom}(p))$ as one wants. We will use this observation later.

Lemma 3.10. *For any elementary pair (p, q) and any word u there are a word v, and partial isomorphisms p' and q' such that the triple (p', q', vu) liberates p [liberates q] in (p, q) and $|vu| = |v| + |u|$ (i.e., no cancellation between v and u happens).*

Proof. First we take extensions p_1 and q_1 of p and q respectively such that $u(p_1,q_1)(c)$ is defined for any $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$, (p_1,q_1) is elementary and

$$\min(\mathrm{dom}(p_1)) = \min(\mathrm{dom}(p)) = \min(\mathrm{dom}(q)) = \min(\mathrm{dom}(q_1)),$$
$$\max(\mathrm{dom}(p_1)) = \max(\mathrm{dom}(p)) = \max(\mathrm{dom}(q)) = \max(\mathrm{dom}(q_1)).$$

By Lemma 3.7 one can find a word v and extensions p', q' of p_1, q_1 such that (p',q',v) liberates p_1 in (p_1,q_1). By Remark 3.8 we may also assume that there is no cancellation in vu. We claim that (p',q',vu) liberates p in (p,q). Items (i-iv) from the definition of liberation are obvious.

For item (v) note that by construction $u(p_1,q_1)(c)$ for all $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ is defined. Since p' and q' extend p_1 and q_1 we get that $u(p',q')(c)$ is defined for all $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ and since (p',q',v) liberates p_1 in (p_1,q_1) we have that for all $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$ the expression $v(p',q')(u(p_1,q_1)(c))$ is defined (just because $u(p_1,q_1)(c) \in \mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)}$). This shows that $vu(p',q')(c)$ is defined for $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$. Also we have

$$v(p',q')\big(\min(\mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)})\big) > \max(\mathrm{Ess(p')}).$$

Finally $u(p_1,q_1)(\mathrm{Ess(p)} \cup \mathrm{Ess(q)}) \subseteq \mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)}$ implies

$$vu(p',q')\big(\min(\mathrm{Ess(p)} \cup \mathrm{Ess(q)})\big) > \max(\mathrm{Ess(p')}).$$

Item (vi) follows immediately from the fact that (p',q',v) liberates p_1 in (p_1,q_1) and from the observation that

$$u(p_1,q_1)\big(\,\mathrm{Ess(p)} \cup \mathrm{Ess(q)}\,\big) \subseteq \mathrm{Ess(p_1)} \cup \mathrm{Ess(q_1)}\,. \qquad \square$$

Lemma 3.11. *Let (p,q) be a piecewise elementary pair and assume a triple (p',q',w) liberates p [liberates q] in (p,q). Let $u = t^n v$ [$u = s^m v$] be a reduced word such that uw is irreducible. Then there is a triple (p'',q'',uw) that liberates p [liberates q] in (p,q). Moreover, one can take p'' to be an extension of p' and q'' to be an extension of q'.*

Proof. By the definition of liberation for piecewise elementary pairs it is enough to prove the statement for elementary triples only. So assume (p,q) is elementary. Since w liberates p in (p,q) it has to start with a non-zero power l of t, i.e, $w = t^l *$. We prove the statement by induction on $|u|$. If u is empty the statement is trivial. Now consider the inductive step. Either $u = *t^k$ and the sign of k matches the sign of l (because uw has to be reduced by assumptions) or $u = *s^k$ with $k \neq 0$. In the former case extend q' to q_1' in such a way that $(t^k w)(p',q_1')(c)$ is defined for $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$, then $(p',q_1',t^k w)$ will be a p-liberating tuple by the item (vi) of the definition of liberation. In the second case we can find p_1' such that $(p_1',q',s^k w)$ liberates q' in (p',q') by taking p_1' such that $(s^k w)(p_1',q')(c) > \max(\mathrm{Ess(p')} \cup \mathrm{Ess(q')})$ for any $c \in \mathrm{Ess(p')} \cup \mathrm{Ess(q')}$. This proves the induction step and the lemma. $\square$

Lemma 3.12. *Let (p,q) be a piecewise elementary pair and $u \in F(s,t)$. Then there is a triple (p',q',w) that liberates p [liberates q] in (p,q) and such that $w = vu$ is reduced.*

Proof. We prove the statement by induction on the number of elementary components of (p,q). Lemma 3.10 covers the base of induction. Assume we have proved the lemma for r-many elementary components and inductively constructed a triple $(\bar{p}_r,\bar{q}_r,w_r)$ that liberates p_r in (p_r,q_r), where p_r and q_r are restrictions of p and q to the first r-many elementary components. Consider the restrictions $\tilde{p}_{r+1}$, $\tilde{q}_{r+1}$ of p and q to the $r+1$ elementary component. By the base of induction (i.e., Lemma 3.10) we can find extensions $\tilde{p}'_{r+1}$, $\tilde{q}'_{r+1}$ of $\tilde{p}_{r+1}$ and $\tilde{q}_{r+1}$ and a word v_{r+1} such that $(\tilde{p}'_{r+1},\tilde{q}'_{r+1},v_{r+1}w_r)$ liberates $\tilde{p}_{r+1}$ in $(\tilde{p}_{r+1},\tilde{q}_{r+1})$ and $v_{r+1}w_r$ is irreducible. By Lemma 3.11 we can also extend p_r and q_r to p_r', q_r' in such a way that $(p_r',q_r',v_{r+1}w_r)$ liberates p_r in (p_r,q_r). Now set $\bar{p}_{r+1}$ to coincide with p_r' on the first r-many elementary components and with $\tilde{p}'_{r+1}$ on the $r+1$ component. Define $\bar{q}_{r+1}$ similarly. Then $(\bar{p}_{r+1},\bar{q}_{r+1},w_{r+1})$ liberates p_{r+1} in (p_{r+1},q_{r+1}). This proves the induction step and the lemma. $\square$

Lemma 3.13. *For any pair (p,q) of partial isomorphisms and any word $u \in F(s,t)$ there are extensions p' and q' of p and q respectively and a reduced word $w = vu$ such that $w(p',q')(c) = c$ for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$.*

Proof. By Lemma 3.5 it is enough to prove the statement for a piecewise elementary pair (p,q). By Lemma 3.12 we can find extensions $\bar{p}$, $\bar{q}$ and a word v such that $(\bar{p},\bar{q},vu)$ liberates p in (p,q). By the definition of liberation we can now extend $\bar{p}$ to p' by declaring

$$p'(c) = c, \text{ for any } c \in vu(\bar{p},\bar{q})(\mathrm{Ess(p)} \cup \mathrm{Ess(q)}).$$

Now set $q' = \bar{q}$ and $w = u^{-1}v^{-1}svu$. Then $w(p', q')(c) = c$ for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$. □

Lemma 3.14. *Fix a sequence $\{u_k\}$ of reduced words. For a generic $(f, g) \in G \times G$ there is a sequence of reduced words $w_k = v_k u_k$ such that $w_k(f, g) \to 1$.*

Proof. Take an enumeration $\{c_i\} = \mathbb{Q}$ of the rationals. Let

$$B_n^k = \{(f, g) \in G \times G : \exists w = vu_k \text{ reduced and } w(f, g)(c_i) = c_i \text{ for } 0 \leqslant i \leqslant n\}.$$

We claim that each B_n^k is dense and open. Indeed, assume for a certain n one has $(f, g) \in B_n^k$. This is witnessed by a word w. Set

$$D = \cup_{i=0}^n \mathrm{Orb}_{w(f,g)}(c_i)$$

and let $p = f|_D$, $q = g|_D$. Then $(f, g) \in U(p) \times U(q) \subseteq B_n^k$ and so B_n^k is open. Density follows from Lemma 3.13.

Now by the Baire theorem $\cap_{n,k} B_n^k$ is a dense G_δ. The lemma follows. □

Theorem 3.15. *Each two-dimensional topological similarity class in G is meager.*

Proof. Assume towards a contradiction that there is a pair $(f_1, g_1) \in G \times G$ that has a non-meager class of topological similarity. Then by Lemma 3.14 there must be a sequence $w_n = v_n t^n s^n$ of reduced words such that (f_1, g_1) converges to the identity along this sequence (we apply Lemma 3.14 with the sequence $u_k = t^k s^k$).

Take and fix $a \in \mathbb{Q}$. Set

$$F_a = \{(f, g) \in G \times G : f(a) = a = g(a)\}.$$

Let

$$C_n = \{(x, y) \in G \times G : \exists m > n \; w_m(x, y)(a) \neq a\}.$$

Then C_n is open and dense in $(G \times G) \setminus F_a$. To see density take a basic open set $U(p) \times U(q) \subseteq (G \times G) \setminus F_a$ and assume $p(a) \neq a$ (the case when $p(a) = a$, but $q(a) \neq a$ is similar). For some $k > n$ $p^k(a)$ is not in the domain of p. Thus the set

$$\{b \in \mathbb{Q} : \exists f \in U(p) \; f^{k+1}(a) = b\}$$

is infinite, and so (by induction) there are infinitely many values that $w_{k+1}(f, g)(a)$ may attain for a pair $(f, g) \in U(p) \times U(q)$. Hence $w_{k+1}(f, g)(a) \neq a$ for some (f, g). And so C_n is dense in $G \times G \setminus F_a$. An application of the Baire theorem shows that $\cap C_n$ is a dense G_δ and so for a generic $(f, g) \in (G \times G) \setminus F_a$ one has $w_n(f, g)(a) \not\to a$ in the discrete topology. Since $\cup_a (G \times G) \setminus F_a = (G \times G) \setminus \{1 \times 1\}$ we get a contradiction with the assumption that $w_n(f_1, g_1) \to 1$ and that the class of topological similarity of (f_1, g_1) is non-meager. □

Homeomorphisms of the unit interval. We now turn to the group of homeomorphisms of the unit interval. This is a Polish group in the natural topology, given by the basic open sets:

$$U(f; a_1, \ldots, a_n; \varepsilon) = \{g \in \mathrm{Homeo}([0, 1]) : |g(a_i) - f(a_i)| < \varepsilon\}.$$

We may write this neighborhood as $U(p; \varepsilon)$, where $p = f|_{\{a_1, \ldots, a_n\}}$ is a partial isomorphism. Since $\mathbb{Q}$ is dense in $[0, 1]$, we may assume that p is a partial isomorphism of the rationals: this will give us a base of open sets.

This group $\mathrm{Homeo}([0, 1])$ has a normal subgroup of index 2, namely the subgroup $\mathrm{Homeo}^+([0, 1])$ of order preserving homeomorphisms. If $H = \mathrm{Homeo}^+([0, 1])$, then $\mathrm{Aut}(\mathbb{Q}) = G$ naturally embeds into H (this embedding is a continuous injective homomorphism, its inverse, though, is not continuous), and the image of G under this embedding is dense in H.

Theorem 3.16. *Every two-dimensional class of topological similarity in H is meager.*

Proof. We imitate the proof of Theorem 3.15. If $\{x_m\}$ is an enumeration of the rationals $\mathbb{Q} \cap [0, 1]$, then $\{x_m\}$ is dense in $[0, 1]$. Set

$$A_{m,n} = \{f \in H : |f(x_m) - x_m| > 1/n \text{ and } |f^{-1}(x_m) - x_m| > 1/n\},$$
$$B_{m,n} = \{(f, g) \in H \times H : f \in A_{m,n} \text{ or } g \in A_{m,n}\}.$$

Note that $B_{m,n}$ is open for every m and n. Then $\cup_{m,n} B_{m,n} = H \times H \setminus \{(1, 1)\}$ and so it is enough to prove that each two-dimensional class of topological similarity is meager in each of $B_{m,n}$.

Let u_k be a sequence of words such that for every piecewise elementary pair (p, q) (here p and q are partial isomorphisms of the rationals, as before) there are infinitely many k such that for some p'_k, q'_k, (p'_k, q'_k, u_k) liberates p in (p, q). Then by Lemma 3.14 for a generic pair $(f, g) \in G \times G$ there is a sequence of reduced words $w_k = v_k u_k$ such that $w_k(f, g) \to 1$. This implies that for a generic pair $(f, g) \in H \times H$ there is a sequence w_k as above (because the topology in H is coarser than in G). If there is a non-meager two-dimensional class of topological similarity then there is a sequence of reduced words $\{w_k\} = \{v_k u_k\}$ (for some $\{v_k\}$) such that the set of pairs $(f_1, g_1) \in H \times H$ that converges to the identity along w_k is non-meager.

Fix now m, n and a sequence of reduced words $w_k = v_k u_k$. Set

$$C_k = \{(f, g) \in H \times H : \exists K > k\ |w_K(f, g)(x_m) - x_m| > 1/2n\}.$$

Each C_k is open, and we claim that it is also dense in $B_{m,n}$. Let $V \subseteq B_{m,n}$ be an open set. Without loss of generality we may assume that $V = U(p; \varepsilon_1) \times U(q; \varepsilon_2)$, where p and q are partial isomorphisms of the rationals. Let

$$\delta = \min\{|x_m - c| : c \in \mathrm{F}(p) \cap \mathrm{F}(q)\} > 1/n.$$

Then there is $K > k$ and p', q' such that (p', q', u_K) liberates p in (p, q). Now repeat the proof of Lemma 3.11 and use Remark 3.9 to get p'', q'' that extend p' and q' and such that $|w_K(p'', q'')(x_m) - x_m| \geqslant 1/2\delta$. Hence each C_k is dense in $B_{m,n}$. Now by the Baire theorem the intersection $\cap_k C_k$ is a dense G_δ in $B_{m,n}$ and thus for any specific sequence w_k the set of elements $(f_1, g_1) \in H \times H$ that converges to the identity along this sequence is meager in $B_{m,n}$. Finally we showed that each two-dimensional topological similarity class is meager in $B_{m,n}$ for any m, n and so is in $H \times H$. □

4. Extensions of Partial Isometries

In this section we prove several results, that will be used later, when dealing with the ordered Urysohn space. But we believe that some of the theorems below are of independent interest for understanding the group of isometries of the Urysohn space.

We recall that the Urysohn space $\mathbb{U}$ is a complete separable metric space, that is uniquely characterized by the following properties:

- Every finite metric space can be isometrically embedded into $\mathbb{U}$;
- $\mathbb{U}$ is ultrahomogeneous, that is each partial isometry between finite subsets of $\mathbb{U}$ extends to a full isometry of $\mathbb{U}$.

There is a rational counterpart $\mathbb{QU}$ of the Urysohn space. It is called rational Urysohn space. This is a countable metric space with rational distances, characterized by similar properties:

- Every finite metric space with rational distances can be isometrically embedded into $\mathbb{QU}$;
- $\mathbb{QU}$ is ultrahomogeneous.

The groups of isometries $\mathrm{Iso}(\mathbb{U})$ and $\mathrm{Iso}(\mathbb{QU})$ of these spaces are Polish groups when endowed with the topology of pointwise convergence (for this $\mathbb{QU}$ is viewed as a discrete topological space).

Definition 4.1. Let $(\mathbf{A}, d)$ be a finite metric space with at least two elements. The *density* of $\mathbf{A}$ is denoted by $\mathcal{D}(\mathbf{A})$ and is the minimal distance between two distinct points in $\mathbf{A}$:

$$\mathcal{D}(\mathbf{A}) = \min\{d(x, y) : x, y \in \mathbf{A}, x \neq y\}.$$

Definition 4.2. An ordered metric space is a triple $(\mathbf{A}, d, <)$, where d is a metric on $\mathbf{A}$ and $<$ is a linear ordering on $\mathbf{A}$.

Definition 4.3. A *partial isometry* or *partial isomorphism* of a metric space $\mathbf{C}$ is an isometry $p : \mathbf{A} \to \mathbf{B}$ between *finite* subspaces $\mathbf{A}, \mathbf{B} \subseteq \mathbf{C}$. A partial isomorphism of an ordered metric space is a partial isometry of the metric space that also preserves the ordering on its domain.

Definition 4.4. Let p be a partial isometry of a metric space. Then we let $\mathrm{dom}(p)$ denote the domain of p and $\mathrm{ran}(p)$ denote its range. A point $x \in \mathrm{dom}(p)$ is called *periodic* if there is a natural number $n > 0$ such that

$$x, p(x), \ldots, p^n(x) \in \mathrm{dom}(p) \quad \text{and} \quad p^n(x) = x.$$

The set of periodic points is denoted by $\mathrm{Z}(p)$. A point $x \in \mathrm{dom}(p)$ is called *fixed* if $p(x) = x$ and the set of fixed points is denoted by $\mathrm{F}(p)$.

In this section we deal mostly with the classical Urysohn space, but some of the results will be later applied to the ordered rational Urysohn space. The following proposition will let us do that.

Proposition 4.5. *Let $\mathbf{A}$ be a finite ordered metric space, and let p be a partial isomorphism of $\mathbf{A}$. Let $\mathbf{B}$ be a finite metric space (with no ordering) and let q be a partial isometry of $\mathbf{B}$ with $\mathrm{Z}(q) = \mathrm{F}(q)$. Suppose that $\mathbf{A} \subseteq \mathbf{B}$ as metric spaces and q extends p. If*

$$\forall x \in \mathrm{dom}(q)\ q(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p)$$

then there is a linear ordering on $\mathbf{B}$ that extends an ordering on $\mathbf{A}$ and such that q becomes a partial isomorphism of an ordered metric space $\mathbf{B}$.

Proof. We prove the statement by induction on $|\mathbf{B} \setminus \mathbf{A}|$. If $\mathbf{A} = \mathbf{B}$ the statement is obvious. For the inductive step we consider two cases.

Case 1. There is some $x \in \mathbf{A}$ such that $x \in \mathrm{dom}(q)$ but $x \notin \mathrm{dom}(p)$. Then by the assumption, $q(x) \in \mathbf{B} \setminus \mathbf{A}$. Now extend the linear ordering on $\mathbf{A}$ to a partial ordering on $\mathbf{A} \cup \{q(x)\}$ by declaring for $y \in \mathbf{A}$

$$q(x) < y \iff \exists z \in \mathrm{dom}(p)\ (p(z) \leqslant y)\&(x < z),$$
$$y < q(x) \iff \exists z \in \mathrm{dom}(p)\ (y \leqslant p(z))\&(z < x).$$

It is straightforward to check that this relation is indeed a partial ordering on $\mathbf{A} \cup \{q(x)\}$. Extend this partial ordering to a linear ordering on $\mathbf{A} \cup \{q(x)\}$ in any way. Then q is a partial isomorphism of $\mathbf{A} \cup \{q(x)\}$ and we apply the induction.

Case 2. Assume the opposite to the first case happens. Then $q|_{\mathbf{A}} = p$. Take any $x \in \mathrm{dom}(q) \setminus \mathbf{A}$ (if there is no such x then $\mathrm{dom}(p) = \mathrm{dom}(q)$ and the statement is obvious). Assume first that x is not a fixed point of q. Then define a linear ordering on $\mathbf{A} \cup \{x, q(x)\}$ by declaring

$$\forall y \in \mathbf{A}\ (y < x)\&(y < q(x))\&(x < q(x)).$$

Then q is a partial isomorphism of $\mathbf{A} \cup \{x, q(x)\}$ and we can apply the induction hypothesis. If x was a fixed point then we declare

$$\forall y \in \mathbf{A}\ (y < x),$$

and, again, induction does the rest. □

Definition 4.6. Let $\mathbf{A} = (\mathbf{A}, d_{\mathbf{A}})$, $\mathbf{B} = (\mathbf{B}, d_{\mathbf{B}})$, and $\mathbf{C} = (\mathbf{C}, d_{\mathbf{C}})$ be finite metric spaces and $i : \mathbf{A} \to \mathbf{B}$, $j : \mathbf{A} \to \mathbf{C}$ be isometries. We define the *free amalgam* $\mathbf{D} = \mathbf{B} *_{\mathbf{A}} \mathbf{C}$ of metric spaces as follows: substituting $\mathbf{B}$ and $\mathbf{C}$ by isomorphic copies we may assume that $\mathbf{B} \cap \mathbf{C} = \mathbf{A}$. Set $\mathbf{D} = \mathbf{B} \cup \mathbf{C}$ and define the metric $d_{\mathbf{D}}$ by:

$$d_{\mathbf{D}}(x,y) = \begin{cases} d_{\mathbf{B}}(x,y) & \text{if } x,\, y \in \mathbf{B}, \\ d_{\mathbf{C}}(x,y) & \text{if } x,\, y \in \mathbf{C}, \\ \min\limits_{z \in \mathbf{A}}\{d_{\mathbf{B}}(x,z) + d_{\mathbf{C}}(z,y)\} & \text{if } x \in \mathbf{B} \text{ and } y \in \mathbf{C}. \end{cases}$$

Note that the first and the second clauses agree for x, $y \in \mathbf{A}$.

If $\mathbf{A}$ is empty then we set $R = diam(\mathbf{B}) + diam(\mathbf{C})$, $\mathbf{D} = \mathbf{B} \sqcup \mathbf{C}$ and

$$d_{\mathbf{D}}(x,y) = \begin{cases} d_{\mathbf{B}}(x,y) & \text{if } x,\, y \in \mathbf{B}, \\ d_{\mathbf{C}}(x,y) & \text{if } x,\, y \in \mathbf{C}, \\ R & \text{otherwise.} \end{cases}$$

The core of our arguments will be the following seminal result due to Sławomir Solecki established in 2005, see [Sol05]. The second item is slightly modified compared to the original statement, but the modification follows from the proof in [Sol05] without any additional work.

Theorem 4.7 (Solecki)**.** *Let a finite metric space $\mathbf{A}$ and a partial isometry p of $\mathbf{A}$ be given. There exist a finite metric space $\mathbf{B}$ with $\mathbf{A} \subseteq \mathbf{B}$ as metric spaces, an isometry $\bar{p}$ of $\mathbf{B}$ extending p, and a natural number M such that*

(i) $\bar{p}^{2M} = 1_B$;

(ii) if $a \in \mathbf{A}$ is aperiodic then $\bar{p}^j(a) \neq a$ for $0 < j < 2M$, and moreover for any j such that $0 < j < 2M$ $\bar{p}^j(a) \in \mathbf{A}$ iff $\bar{p}^{j-1}(a) \in \mathrm{dom}(p)$;

(iii) $\mathbf{A} \cup \bar{p}^M(\mathbf{A})$ is the free amalgam of $\mathbf{A}$ and $\bar{p}^M(\mathbf{A})$ over $(\mathrm{Z}(p), id_{\mathrm{Z}(p)}, \bar{p}^M|_{\mathrm{Z}(p)})$.

Moreover, the distances in $\mathbf{B}$ may be taken from the additive semigroup generated by the distances in $\mathbf{A}$.

Definition 4.8. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be metric spaces and let $\mathbf{C}$ be embedded into $\mathbf{A}$ and $\mathbf{B}$. We say that $\mathbf{B}$ *extends* $\mathbf{A}$ *over* $\mathbf{C}$ if there exists an embedding $i : \mathbf{A} \to \mathbf{B}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{A} \arrow[dr, dashed, "i"] & \\
\mathbf{C} \arrow[u, hook] \arrow[r, hook] & \mathbf{B}
\end{tikzcd}$$

We say that $\mathbf{A}$ and $\mathbf{B}$ are *disjoint over* $\mathbf{C}$ if neither $\mathbf{B}$ extends $\mathbf{A}$ over $\mathbf{C}$ nor $\mathbf{A}$ extends $\mathbf{B}$ over $\mathbf{C}$.

Lemma 4.9. *Let $\mathbf{A}$ be a finite metric space, let p be a partial isometry of $\mathbf{A}$, and $x \in \mathrm{dom}(p)$ be a non-periodic point $x \notin \mathrm{Z}(p)$ such that and $x \notin \mathrm{ran}(p)$ (i.e., $p^{-1}(x)$ is undefined). Then there are metric spaces $\mathbf{A}_1$ and $\mathbf{A}_2$ that both extend $\mathbf{A}$: $\mathbf{A} \subset \mathbf{A}_1$ and $\mathbf{A} \subset \mathbf{A}_2$, and partial isometries p_1 of $\mathbf{A}_1$ and p_2 of $\mathbf{A}_2$ that both extend p and such that $x \notin \mathrm{ran}(p_1) \cup \mathrm{ran}(p_2)$ and $\mathrm{Orb}_{\mathrm{p}_1}(x)$ and $\mathrm{Orb}_{\mathrm{p}_2}(x)$ are disjoint over $\mathrm{Orb}_{\mathrm{p}}(x)$.*

Moreover, one can assume that

$$\begin{aligned}
&\mathrm{Z}(p_1) = \mathrm{Z}(p) = \mathrm{Z}(p_2),\\
&\forall x \in \mathrm{dom}(p_1)\ p_1(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p),\\
&\forall x \in \mathrm{dom}(p_2)\ p_2(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p).
\end{aligned}$$

Proof. Apply Theorem 4.7 to get a full isometry $\bar{p}$ of a finite metric space $\mathbf{B}$ that extends p and a natural number M. Set

$$\bar{\mathbf{A}} = \mathbf{A} \cup \bar{p}(\mathbf{A}) \cup \ldots \cup \bar{p}^{2M-1}(\mathbf{A}) \cup \{y\},$$

where y is a new point, i.e., a point not in $\mathbf{B}$. Let $\delta = \mathcal{D}(\mathbf{A})$ denote the density of $\mathbf{A}$ and fix an $\varepsilon > 0$ such that $\varepsilon \leqslant 2\delta$. We turn $\bar{\mathbf{A}}$ into a metric space by defining the distance between $a, b \in \bar{\mathbf{A}}$, $a \neq b$ as follows.

$$\begin{aligned}
&d_{\bar{\mathbf{A}}}(a,b) = d_{\mathbf{B}}(a,b) \quad \text{when } a, b \neq y;\\
&d_{\bar{\mathbf{A}}}(a,y) = d_{\mathbf{B}}(a,x) \quad \text{when } a \neq x, y;\\
&d_{\bar{\mathbf{A}}}(x,y) = \varepsilon.
\end{aligned}$$

We claim that $(\bar{\mathbf{A}}, d_{\bar{\mathbf{A}}})$ is a metric space. We have to check the triangle inequality (other conditions are obviously fulfilled). For this note that both $\bar{\mathbf{A}} \setminus \{y\}$ and $\bar{\mathbf{A}} \setminus \{x\}$ are isometrically embeddable into $\mathbf{B}$, where the triangle inequality is known to be satisfied. So one needs to prove two claims.

Claim 1. For any $z \in \bar{\mathbf{A}}$

$$d_{\bar{\mathbf{A}}}(x,y) \leqslant d_{\bar{\mathbf{A}}}(x,z) + d_{\bar{\mathbf{A}}}(z,y).$$

If $z \in \{x, y\}$ then the statement is obvious. If $z \notin \{x,y\}$ then $d_{\bar{\mathbf{A}}}(x,z) + d_{\bar{\mathbf{A}}}(z,y) \geqslant 2\delta$ and $d_{\bar{\mathbf{A}}(x,y)} = \varepsilon \leqslant 2\delta$ and Claim 1 follows.

Claim 2. For any $z \in \bar{\mathbf{A}}$

$$\begin{aligned}
d_{\bar{\mathbf{A}}}(x,z) \leqslant d_{\bar{\mathbf{A}}}(x,y) + d_{\bar{\mathbf{A}}}(y,z),\\
d_{\bar{\mathbf{A}}}(z,y) \leqslant d_{\bar{\mathbf{A}}}(z,x) + d_{\bar{\mathbf{A}}}(x,y).
\end{aligned}$$

Note that for $z \notin \{x,y\}$ one has $d_{\bar{\mathbf{A}}}(y,z) = d_{\bar{\mathbf{A}}}(x,z)$. From this both inequalities follow immediately.

So $\bar{\mathbf{A}}$ is a metric space. We denote it by $\bar{\mathbf{A}}(\varepsilon)$ to signify the dependence on epsilon. Define a partial isometry $\hat{p}$ on $\bar{\mathbf{A}}(\varepsilon)$ by

$$\hat{p}(z) = \bar{p}(z),$$

whenever $z \in \bar{\mathbf{A}}$ and $\bar{p}(z) \in \bar{\mathbf{A}}$; and $\hat{p}(\bar{p}^{2M-1}(x)) = y$. Using $\bar{p}^{2M} = 1_{\mathbf{B}}$ it is straightforward to check that $\hat{p}$ is indeed a partial isometry. Now the construction of two extensions that are disjoint over $\mathrm{Orb}_{\mathrm{p}}(x)$ is easy. Take, for example, two different $\varepsilon_1 \leqslant 2\delta$, $\varepsilon_2 \leqslant 2\delta$, $\varepsilon_1 \neq \varepsilon_2$ such that

$$\varepsilon_i \notin \{d_{\mathbf{B}}(x_1, x_2) : x_1, x_2 \in \mathbf{B}\},$$

let $(\mathbf{A}_i, p_i) = (\bar{\mathbf{A}}(\varepsilon_i), \hat{p})$. Then $\mathrm{Orb}_{\mathrm{p}_1}(x)$ and $\mathrm{Orb}_{\mathrm{p}_2}(x)$ are disjoint over $\mathrm{Orb}_{\mathrm{p}}(x)$. $\square$

The main power of Theorem 4.7 is the explicit construction of an extension of a partial isometry to a full isometry of a finite metric space. Moreover, this extension is as independent as possible. For our purposes we only need an extension to a partial isomorphism, but we want to keep the independence. Let us state explicitly a corollary of the theorem that gives everything that we need.

Corollary 4.10. *For any finite metric space* $\mathbf{A}$ *and a partial isometry* p *there is finite metric space* $\mathbf{C}$*, a partial isometry* p_1 *of* $\mathbf{C}$*, which is an extension of* p*, and a natural number* M *such that*

(i) $\mathrm{Z}(p) = \mathrm{Z}(p_1)$*;*
(ii) $\mathbf{A} \cup p_1^M(\mathbf{A})$ *is the amalgam of* $\mathbf{A}$ *and* $p_1^M(\mathbf{A})$ *over* $(\mathrm{Z}(p), id_{\mathrm{Z}(p)}, p_1^M|_{\mathrm{Z}(p)})$*.*
(iii) for any $x \in \mathrm{dom}(p_1)$
$$p_1(x) \in \mathbf{A} \iff x \in \mathrm{dom}(p).$$

Moreover, the distances in $\mathbf{C}$ *are taken from the additive semigroup generated by the distances in* $\mathbf{A}$*, and hence the density is preserved:* $\mathcal{D}(\mathbf{C}) = \mathcal{D}(\mathbf{A})$*.*

Proof. Apply Theorem 4.7 to $\mathbf{A}$ and p to get a metric space $\mathbf{B}$, a full isometry $\bar{p}$ of $\mathbf{B}$ and a natural number M. Now set
$$\mathbf{C} = \mathbf{A} \cup \bar{p}(\mathbf{A}) \cup \ldots \cup \bar{p}^M(\mathbf{A}),$$
and $p_1 = \bar{p}|_{\mathbf{A}\cup\bar{p}(\mathbf{A})\cup\ldots\cup\bar{p}^{M-1}(\mathbf{A})}$. It is trivial to check that such a $\mathbf{C}$ and p_1 satisfy the conditions. □

Definition 4.11. Let (M, d) be a metric space, and let $x, y \in M$. We say that the distance $d(x, y)$ *passes through a point* $z \in M$ if
$$d(x, y) = d(x, z) + d(z, y).$$

We are going to apply Corollary 4.10 to partial isometries that also preserve an ordering. That is why we impose an additional assumption: all periodic points are fixed points, i.e., $\mathrm{Z}(p) = \mathrm{F}(p)$.

Theorem 4.12. *Let* $\mathbf{A}$ *be a finite metric space. Let* p *and* q *be two partial isometries of* $\mathbf{A}$ *such that* $\mathrm{Z}(p) = \mathrm{F}(p)$ *and* $\mathrm{Z}(q) = \mathrm{F}(q)$*. Suppose* $\mathrm{F}(p) \cap \mathrm{F}(q) \neq \emptyset$*. Then there are finite metric space* $\mathbf{B}$*, extensions* $\bar{p}$*,* $\bar{q}$ *of* p *and* q *respectively (these extensions are partial isometries of* $\mathbf{B}$*) and an element* $w = t^K v \in F(s, t)$*,* $K \neq 0$ *such that*

(i) $\mathrm{Z}(\bar{p}) = \mathrm{Z}(p)$ $(= \mathrm{F}(p))$*,* $\mathrm{Z}(\bar{q}) = \mathrm{Z}(q)$ $(= \mathrm{F}(q))$*;*
(ii) $\mathrm{dom}(\bar{p}) \cup w(\bar{p}, \bar{q})(\mathbf{A})$ *is the free amalgam of* $\mathrm{dom}(\bar{p})$ *and* $w(\bar{p}, \bar{q})(\mathbf{A})$ *over* $F(p) \cap F(q)$*.*

Moreover, the distances in $\mathbf{B}$ *are taken from the additive semigroup generated by the distances in* $\mathbf{A}$*, and hence* $\mathcal{D}(\mathrm{dom}(\bar{p}) \cup \mathbf{A}) = \mathcal{D}(\mathbf{A})$*,* $\mathcal{D}(\mathrm{dom}(\bar{q}) \cup \mathbf{A}) = \mathcal{D}(\mathbf{A})$*.*

Proof. Let
$$N = \left\lceil \frac{2diam(\mathbf{A})}{\mathcal{D}(\mathbf{A})} \right\rceil.$$
Define inductively the sequence of elements $w_k \in F(s, t)$, extensions $\bar{p}_k$, $\bar{q}_k$ and metric spaces $\mathbf{A}_k$ as follows:

Step 0: Let $\bar{p}_0 = p$, $\bar{q}_0 = q$, $w_0 =$ empty word, $\mathbf{A}_0 = \mathbf{A}$;

Step k: If k is odd then apply Corollary 4.10 to $\bar{p}_{k-1}$ and $\mathbf{A}_{k-1}$ to get $\bar{p}_k$ and M_k; set $\bar{q}_k = \bar{q}_{k-1}$, $w_k = s^{M_k} w_{k-1}$, $\mathbf{A}_k = \mathbf{A}_{k-1} \cup \mathrm{dom}(\bar{p}_k) \cup \mathrm{ran}(\bar{p}_k)$.

If k is even do the same thing with the roles of p and q interchanged.

We claim that $\bar{p} = \bar{p}_{2N+2}$, $\bar{q} = \bar{q}_{2N+2}$, $\mathbf{B} = \mathbf{A}_{2N+2}$, and $w = w_{2N+2}$ fulfill the requirements of the statement. Let d denote the metric on $\mathbf{B}$. It is obvious that $\mathrm{F}(\bar{p}) = \mathrm{F}(p)$ and $\mathrm{F}(\bar{q}) = \mathrm{F}(q)$ (this is given by Corollary 4.10 at each stage). The moreover part is also obvious, since it is fulfilled at every step of the construction. It remains to show that for any $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and any $y \in \mathrm{dom}(\bar{p})$ one has
$$d(x, y) = \min\{d(x, z) + d(z, y) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}. \tag{1}$$
Note that by the last step of the construction for any $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and $y \in \mathrm{dom}(\bar{p})$ we have
$$d(x, y) = \min\{d(x, z) + d(z, y) : z \in \mathrm{F}(q)\}.$$

We first prove several claims.

Claim 1. It is enough to show that (1) holds for all $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and $y \in \mathrm{F}(q)$.

Proof of Claim 1. Assume (1) holds for all $x \in w(\bar{p}, \bar{q})(\mathbf{A})$ and $y \in \mathrm{F}(q)$. If $y' \in \mathrm{dom}(\bar{p})$, then for some $c \in \mathrm{F}(q)$
$$d(x, y') = d(x, c) + d(c, y') = \min\{d(x, e) + d(e, y') : e \in \mathrm{F}(q)\}. \tag{2}$$
By the assumptions of the claim we get
$$d(x, y') = d(x, z) + d(z, c) + d(c, y') \geqslant d(x, z) + d(z, y'),$$
for some $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$; and so, by (2),
$$d(x, y') = d(x, z) + d(z, y').$$

This proves the claim.

Let $w_i(c)$ denote $w_i(\bar{p}_i, \bar{q}_i)(c)$.

Claim 2. Let $x \in \mathrm{F}(p) \cup \mathrm{F}(q)$, $c \in \mathbf{A}$ and suppose that for some $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$ and for some i the distance between $w_i(c)$ and x passes through z. Then for any $j \geqslant i$ the distance between $w_j(c)$ and x passes through the same point z.

Proof of Claim 2. This follows by induction. Here is an inductive step. Assume for definiteness that $j+1$ is odd (the case when $j+1$ is even, is similar). The distance between x and $w_{j+1}(c)$ passes through a point $z' \in \mathrm{F}(p)$ ($z' \in \mathrm{F}(q)$ if $j+1$ is even). Then

$$d(w_j(c), x) = (w_j(c), z) + d(z, x) \leqslant d(w_j(c), z') + d(z', x),$$
$$d(w_{j+1}(c), x) = d(w_{j+1}(c), z') + d(z', x),$$

but $d(w_{j+1}(c), z') = d(w_j(c), z')$ (this is because $w_{j+1} = s^m w_j$ and z' is fixed by p). Hence

$$d(w_j(c), x) \leqslant d(w_{j+1}(c), x),$$

but also

$$d(w_{j+1}(c), x) \leqslant d(w_{j+1}(c), z) + d(z, x) = d(w_j(c), z) + d(z, x) = d(w_j(c), x),$$

and so $d(w_{j+1}(c), x) = d(w_j(c), x)$. This proves the claim.

Claim 3. Let $x \in \mathrm{F}(p) \bigtriangleup \mathrm{F}(q)$ (here $\bigtriangleup$ is symmetric difference of sets), $c \in \mathbf{A}$. Suppose that the distance between $w_i(c)$ and x does not pass through a point in $\mathrm{F}(p) \cap \mathrm{F}(q)$. Then $d(w_i(c), x) \geqslant \lfloor i/2 \rfloor \mathcal{D}(\mathbf{A})$.

Proof of Claim 3. Suppose first that $x \in \mathrm{F}(p) \setminus \mathrm{F}(q)$. We prove the statement by induction on i. The base of the induction is trivial, so we show the inductive step: assume the statement is true for i and we need to show it for $i+1$. If i is even then, since $\lfloor i/2 \rfloor = \lfloor (i+1)/2 \rfloor$ and because $d(w_{i+1}(c), x) = d(w_i(c), x)$ (this is since i is even and $x \in \mathrm{F}(p)$) the statement follows immediately. So, assume i is odd. Then the distance between $w_{i+1}(c)$ and x passes through a point $z \in \mathrm{F}(q)$. Now two things can happen. Suppose first for some $j \leqslant i$ the distance between $w_j(c)$ and z passes through a point $z' \in \mathrm{F}(p) \cap \mathrm{F}(q)$. Then by Claim 2, the distance between z and $w_{i+1}(c)$ must pass through z'. Now

$$d(w_{i+1}(c), x) = d(w_{i+1}(c), z) + d(z, x) = d(w_{i+1}(c), z') + d(z', z) + d(z, x) \geqslant d(w_{i+1}(c), z') + d(z', x).$$

And so the distance between $w_{i+1}(c)$ and x passes through a point $z' \in \mathrm{F}(p) \cap \mathrm{F}(q)$. This contradicts the assumptions of the claim. So, for no $j \leqslant i$ does the distance between $w_j(c)$ and x pass through a point in $\mathrm{F}(p) \cap \mathrm{F}(q)$. Then, applying induction to $w_i(c)$ and z, we get $d(w_i(c), z) \geqslant \lfloor i/2 \rfloor \mathcal{D}(\mathbf{A})$. But since $d(w_{i+1}(c), z) = d(w_i(c), z)$ and since $d(x, z) \geqslant \mathcal{D}(\mathbf{A})$ we get

$$d(w_{i+1}(c), x) \geqslant \lfloor i/2 \rfloor \mathcal{D}(\mathbf{A}) + \mathcal{D}(\mathbf{A}) \geqslant \lfloor (i+1)/2 \rfloor \mathcal{D}(\mathbf{A}).$$

In the case when $x \in \mathrm{F}(q) \setminus \mathrm{F}(q)$, the distance increases by $\mathcal{D}(\mathbf{A})$ at even stages of the construction, and the rest of the argument for this case is similar. The claim is proved.

Now fix $c \in \mathbf{A}$ and $y \in \mathrm{F}(q)$. It remains to show that

$$d(w_{2N+2}(c), y) = \min\{d(w_{2N+2}(c), z) + d(z, y) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}.$$

We have two cases (we will show, though, that Case 2 is impossible).

Case 1. For some $i \leqslant 2N+2$ the distance between $w_i(c)$ and y passes through a point $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$. Then

$$d(y, w_i(c)) = \min\{d(y, z) + d(z, w_i(c)) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}.$$

Applying Claim 2 for $j = 2N+2$, we get

$$d(y, w_{2N+2}(c)) = \min\{d(y, z) + d(z, w_{2N+2}(c)) : z \in \mathrm{F}(p) \cap \mathrm{F}(q)\}.$$

And the theorem is proved for this case.

Case 2. For no $i \leqslant 2N+2$ does the distance between $w_i(c)$ and y pass through a point in $\mathrm{F}(p) \cap \mathrm{F}(q)$. Then by Claim 3

$$d(w_{2N+2}(c), y) \geqslant (N+1)\mathcal{D}(\mathbf{A}) > 2diam(\mathbf{A}).$$

On the other hand, let $z \in \mathrm{F}(p) \cap \mathrm{F}(q)$ be any common fixed point. Then $d(w_{2N+2}(c), y) \leqslant d(w_{2N+2}(c), z) + d(z, y) = d(c, z) + d(z, y) \leqslant 2diam(\mathbf{A})$. This gives a contradiction. So this case never happens. $\square$

Remark 4.13. Note that the same result is also true for ordered metric spaces. For this one just has to apply Proposition 4.5 at each step of the construction of $\bar{p}$ and $\bar{q}$.

Before we apply this result to the classes of topological similarity let us mention another application. For a subset $\mathbf{A} \subseteq \mathbb{U}$ ($\mathbf{A} \subseteq \mathbb{QU}$) let $\mathrm{Iso}_{\mathbf{A}}(\mathbb{U})$ ($\mathrm{Iso}_{\mathbf{A}}(\mathbb{QU})$, respectively) denote the subgroup of isometries that pointwise fix $\mathbf{A}$. Recall a theorem of Julien Melleray from [Mel10].

Theorem 4.14 (Melleray)**.** *Let $\mathbb{U}$ be the Urysohn space, and let $\mathbf{A}, \mathbf{B} \subset \mathbb{U}$ be two finite subsets. Then*

$$\mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{U}) = \overline{\langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{U}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{U})\rangle}.$$

Let us give an equivalent reformulation of the above result.

Theorem 4.15 (Melleray)**.** *Let $\mathbb{U}$ be the Urysohn space, and let $\mathbf{A}, \mathbf{B} \subset \mathbb{U}$ be two finite subsets. Then for any $\varepsilon > 0$, for any $p \in \mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{U})$, and for any finite $\mathbf{C} \subseteq \mathbb{U}$ there is $q \in \langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{U}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{U})\rangle$ such that*

$$\forall x \in \mathbf{C}\ d(p(x), q(x)) < \varepsilon.$$

We show that one can actually eliminate the epsilon in the above reformulation.

Theorem 4.16. *Let $\mathbb{U}$ be the Urysohn space, and let $\mathbf{A}, \mathbf{B} \subset \mathbb{U}$ be two finite subsets. Then for any $p \in \mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{U})$ and for any finite $\mathbf{C} \subseteq \mathbb{U}$ there is $q \in \langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{U}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{U})\rangle$ such that*

$$\forall x \in \mathbf{C}\ p(x) = q(x).$$

Proof. Without loss of generality we may assume that $\mathbf{A} \subseteq \mathbf{C}$ and $\mathbf{B} \subseteq \mathbf{C}$. If $\mathbf{D} = \mathbf{C} \cup p(\mathbf{C})$, then $p|_{\mathbf{C}}$ is a partial isometry of $\mathbf{D}$. Define two partial isometries p_1 and p_2 of $\mathbf{D}$ by

$$\forall x \in \mathbf{A}\ p_1(x) = x,$$
$$\forall x \in \mathbf{B}\ p_2(x) = x.$$

Now apply Theorem 4.12 to p_1, p_2 and $\mathbf{D}$ to get a metric space $\mathbf{D}'$ and extension q_1 of p_1 and q_2 of p_2, and a word $w \in F_2$. Extend q_1 to q_1' by setting

$$\forall x \in \mathbf{C}\ q_1'(w(q_1, q_2)(x)) = w(q_1, q_2)(p(x)).$$

Such a q_1' is then a partial isometry of $\mathbf{D}'$. This follows from the fact that

$$w(q_1, q_2)(\mathbf{C}) \cup \mathrm{dom}(q_1)$$

is an amalgam of $w(q_1, q_2)(\mathbf{C})$ and $\mathrm{dom}(q_1)$ over $F(p_1) \cap F(p_2) = \mathbf{A} \cap \mathbf{B} \subseteq F(p)$. Indeed, if $y \in \mathrm{dom}(q_1)$ and $x \in \mathbf{C}$ then

$$\begin{aligned}
&d\big(q_1(y), w(q_1, q_2)(p(x))\big) = \\
&\min\Big\{d\big(q_1(y), z\big) + d\big(z, w(q_1, q_2)(p(x))\big) : z \in F(p_1) \cap F(p_2)\Big\} = \\
&\min\Big\{d\big(q_1(y), q_1(z)\big) + d\big(w(q_1, q_2)(p(z)), w(q_1, q_2)(p(x))\big) : z \in F(p_1) \cap F(p_2)\Big\} = \\
&\min\Big\{d(y, z) + d(z, x) : z \in F(p_1) \cap F(p_2)\Big\} = \\
&\min\Big\{d(y, z) + d(z, w(q_1, q_2)(x)) : z \in F(p_1) \cap F(p_2)\Big\} = d(y, w(q_1, q_2)(x)).
\end{aligned}$$

Now extend q_1' and q_2 to full isometries (we still denote them by the same symbols) and set

$$q = w^{-1}(q_1', q_2) q_1' w(q_1', q_2).$$

Then for any $x \in \mathbf{C}$, $p(x) = q(x)$, and $q_1' \in \mathrm{Iso}_{\mathbf{A}}(\mathbb{U})$, $q_2 \in \mathrm{Iso}_{\mathbf{B}}(\mathbb{U})$. □

Note that if we start from metric spaces with rational distances, then the space $\mathbf{D}'$, constructed in the proof, would also have rational distances. And we arrive at the

Corollary 4.17. *Let $\mathbb{QU}$ be the rational Urysohn space, and let $\mathbf{A}, \mathbf{B} \subset \mathbb{QU}$ be two finite subsets. Then*

$$\mathrm{Iso}_{\mathbf{A}\cap\mathbf{B}}(\mathbb{QU}) = \overline{\langle \mathrm{Iso}_{\mathbf{A}}(\mathbb{QU}), \mathrm{Iso}_{\mathbf{B}}(\mathbb{QU})\rangle}.$$

Before showing another application of our extension result we need the following easy observation.

Lemma 4.18. *Let p, q be partial isometries of the Urysohn space $\mathbb{U}$ such that $\mathrm{dom}(p) = \mathrm{dom}(q)$, and let*
$$\{c_i\}_{i=1}^n = \mathrm{dom}(p).$$
For any $\varepsilon > 0$ there are partial isometries $\bar{p}, \bar{q}$ of $\mathbb{U}$ such that
$$\mathrm{dom}(\bar{p}) = \mathrm{dom}(p) = \mathrm{dom}(q) = \mathrm{dom}(\bar{q}),$$
$$\forall i \quad d(\bar{p}(c_i), p(c_i)) < \varepsilon, \quad d(\bar{q}(c_i), q(c_i)) < \varepsilon,$$
and the sets $\mathrm{dom}(p), \bar{p}(\mathrm{dom}(p)), \bar{q}(\mathrm{dom}(p))$ are pairwise disjoint.

Proof. Set $\mathbf{A} = \mathrm{dom}(p) \cup p(\mathrm{dom}(p)) \cup q(\mathrm{dom}(p))$. Let $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ be new symbols, disjoint from all other data. Set
$$\mathbf{B} = \{c_i\} \cup \{p(c_i)\} \cup \{q(c_i)\} \cup \{a_i\} \cup \{b_i\}.$$
Let $\varepsilon > 0$ be given. We may decrease it to ensure that $\varepsilon < \mathcal{D}(\mathbf{A})$. Now define the metric on $\mathbf{B}$ as follows. The metric on $\mathbf{A}$ is the one inherited from $\mathbb{U}$. For $x \in \mathbf{A}$ set

$$d(a_i, x) = \begin{cases} d(p(c_i), x) & \text{if } x \neq p(c_i); \\ \varepsilon & \text{if } x = p(c_i); \end{cases}$$
$$d(b_i, x) = \begin{cases} d(q(c_i), x) & \text{if } x \neq q(c_i); \\ \varepsilon & \text{if } x = q(c_i); \end{cases}$$
$$d(a_i, b_j) = \begin{cases} d(p(c_i), q(c_j)) & \text{if } p(c_i) \neq q(c_j); \\ \varepsilon & \text{if } p(c_i) = q(c_j). \end{cases}$$

It is routine to check that d is indeed a metric on $\mathbf{A}$, and we leave this to the reader. Finally, set
$$\bar{p}(c_i) = a_i \quad \bar{q}(c_i) = b_i.$$
Then $\bar{p}$ and $\bar{q}$ satisfy the conclusions of the lemma. $\square$

One of the corollaries from the results in [Sol05] is that the group $\mathrm{Aut}(\mathbb{U})$ is topologically 2-generated, in other words there is a pair of isometries (f, g) such that the group $\langle f, g\rangle$ is dense in $\mathrm{Aut}(\mathbb{U})$. If Λ is the set of pairs that generate a dense subgroup, then
$$\Lambda = \{(f, g) \in \mathrm{Aut}(\mathbb{U}) \times \mathrm{Aut}(\mathbb{U}) : \forall \varepsilon > 0\ \forall h\ \forall n\ \forall \{c_i\}_{i=1}^n\ \exists w\ \forall i \quad d(w(f, g)(c_i), h(c_i)) < \varepsilon\}.$$

Theorem 4.19. *Λ is a dense G_δ-subset of $\mathrm{Aut}(\mathbb{U}) \times \mathrm{Aut}(\mathbb{U})$.*

Proof. Let $\{h_j\}_{j=1}^\infty$ be a dense subset of $\mathrm{Aut}(\mathbb{U})$, and $\{c_i\}_{i=1}^\infty$ be a dense set of points in $\mathbb{U}$. Set

$$B(n, m, j) = \{(f, g) \in \mathrm{Aut}(\mathbb{U}) \times \mathrm{Aut}(\mathbb{U}) : \exists w\ d(w(f, g)(c_i), h_j(c_i)) < 1/n \text{ for } 1 \leqslant i \leqslant m\}.$$

Each $B(n, m, j)$ is open and
$$\Lambda = \bigcap_{n,m,j} B(n, m, j),$$
hence Λ is G_δ. It remains to check that all of the $B(n, m, j)$ are dense. Fix m, n, and j. Let p, q be partial isometries of $\mathbb{U}$, $\varepsilon > 0$, and without loss of generality we assume that $\mathrm{dom}(p) = \mathrm{dom}(q)$ and that $\{c_i\}_{i=1}^m \subseteq \mathrm{dom}(p)$. Let $\tilde{h}_j$ be the partial isometry given by the restriction of h_j onto $\{c_i\}$. By ultrahomogeneity of $\mathbb{U}$ it is enough to show that there are partial isometries $\tilde{p}, \tilde{q}$ such that
$$d(\tilde{p}(c), p(c)) < \varepsilon, \quad d(\tilde{q}(c), q(c)) < \varepsilon \quad \text{for all } c \in \mathrm{dom}(p)$$
and a word w such that
$$d(w(\tilde{p}, \tilde{q})(c_i), \tilde{h}_j(c_i)) < 1/n$$
for all $i \in \{1, \ldots, m\}$. By Lemma 4.18 we may find $\bar{p}$, $\bar{q}$ such that
$$\mathrm{dom}(\bar{p}) = \mathrm{dom}(p) = \mathrm{dom}(q) = \mathrm{dom}(\bar{q}),$$
$$d(\bar{p}(c), p(c)) < \varepsilon, \quad d(\bar{q}(c), q(c)) < \varepsilon \quad \text{for all } c \in \mathrm{dom}(p)$$
and
$$\mathrm{dom}(p),\ \bar{p}(\mathrm{dom}(p)),\ \bar{q}(\mathrm{dom}(p))$$

are pairwise disjoint. Now add a common fixed point z to $\bar{p}$, $\bar{q}$ and $\tilde{h}_j$ (and denote the new partial isometries still by $\bar{p}$, $\bar{q}$ and $\tilde{h}_j$.)

We can now apply Theorem 4.12 to the partial isometries $\bar{p}, \bar{q}$ and the set

$$\mathbf{A} = \mathrm{dom}(\bar{p}) \cup \bar{p}(\mathrm{dom}(\bar{p})) \cup \bar{q}(\mathrm{dom}(\bar{p})) \cup \tilde{h}_j(\mathrm{dom}(\bar{p})).$$

This gives us partial isometries p', q' that extend $\bar{p}$ and $\bar{q}$ and a word w_1.

The next step is to extend p' to $\tilde{p}$ by setting

$$\tilde{p}\Big((w_1(p', q'))(c_i)\Big) = w_1(p', q')(\tilde{h}_j(c_i)).$$

We claim that $\tilde{p}$ is still a partial isometry. The argument is similar to the one in the proof of Theorem 4.16. We have $\{z\} = F(p') \cap F(q') \cap F(\tilde{h}_j)$. Then for any $y \in \mathrm{dom}(p')$ and any c_i

$$\begin{aligned}
&d\big(p'(y), w_1(p', q')(\tilde{h}_j(c_i))\big) = d(p'(y), z) + d\big(z, w_1(p', q')(\tilde{h}_j(c_i))\big) = \\
&d(p'(y), p'(z)) + d\big(w_1(p', q')(\tilde{h}_j(z)), w_1(p', q')(\tilde{h}_j(c_i))\big) = \\
&d(y, z) + d(z, c_i) = d(y, z) + d(w_1(p', q')(z), w_1(p', q')(c_i)) = \\
&d(y, z) + d(z, w_1(p', q')(c_i)) = d(y, w_1(p', q')(c_i)),
\end{aligned}$$

and hence $d\big(\tilde{p}(y), w_1(p', q')(\tilde{h}_j(c_i))\big) = d(y, w_1(p', q')(c_i))$.

Finally set $w = w_1^{-1} s w_1$ then for $\tilde{q} = q'$

$$w(\tilde{p}, \tilde{q})(c_i) = \tilde{h}_j(c_i) = h_j(c_i) \quad \text{for all } i.$$

So $B(n, m, j)$ is dense and by Baire Theorem Λ is dense G_δ. $\square$

5. Isometries of the Ordered Urysohn Space

There is a rich variety of linearly ordered Fraïssé limits, of which the countable dense linear ordering without endpoints is the simplest example. In fact, as proved in [KPT05], if the group of automorphisms of a particular Fraïssé class $\mathcal{K}$ is extremely amenable, then there is a linear ordering on the Fraïssé limit of $\mathcal{K}$ that is preserved by all automorphisms. Moreover, the ordered limit is still Fraïssé , i.e., is a Fraïssé limit of a Fraïssé class.

We consider another example of a linearly ordered Fraïssé limit: the ordered rational Urysohn space $\mathbb{QU}_\prec$.

Let us briefly recall the definition of this structure. Formally speaking, one has to consider the Fraïssé class $\mathcal{M}$ of finite ordered metric spaces with rational distances. Then $\mathbb{QU}_\prec$ is, by definition, the Fraïssé limit of $\mathcal{M}$. Intuitively one can think of this structure as a classical rational Urysohn space with a linear ordering on top (such that ordering is isomorphic to the ordering of the rationals) and such that this ordering is independent of the metric structure.

Our goal is to prove that every two-dimensional class of topological similarity in the group of automorphisms of $\mathbb{QU}_\prec$ is meager. We would like to emphasize that the structure of conjugacy classes in $\mathrm{Aut}(\mathbb{Q})$ and $\mathrm{Aut}(\mathbb{QU}_\prec)$ is substantially different. As was mentioned earlier there is a generic conjugacy class in $\mathrm{Aut}(\mathbb{Q})$, while it is not hard to derive from results in [KR07], that each conjugacy class in $\mathrm{Aut}(\mathbb{QU}_\prec)$ is meager.

Recall (see [KR07], Definition 3.3)

Definition 5.1. A class $\mathcal{K}$ of finite structures satisfies the *weak amalgamation property* (WAP for short) if for every $\mathbf{A} \in \mathcal{K}$ there are $\mathbf{B} \in \mathcal{K}$ and an embedding $e : \mathbf{A} \to \mathbf{B}$ such that for all $\mathbf{C} \in \mathcal{K}$, $\mathbf{D} \in \mathcal{K}$ and all embeddings $i : \mathbf{B} \to \mathbf{C}$, $j : \mathbf{B} \to \mathbf{D}$ there are $\mathbf{E} \in \mathcal{K}$ and embeddings $k : \mathbf{C} \to \mathbf{E}$, $l : \mathbf{D} \to \mathbf{E}$ such that $k \circ i \circ e = l \circ j \circ e$, i.e. in the following diagram the paths from $\mathbf{A}$ to $\mathbf{E}$ commute (but not necessarily paths from $\mathbf{B}$ to $\mathbf{E}$).

$$\begin{tikzcd}
 & & \mathbf{C} \arrow[dr, dashed, "k"] & \\
\mathbf{A} \arrow[r, dashed, "e"] & \mathbf{B} \arrow[ur, "i"] \arrow[dr, "j"'] & & \mathbf{E} \\
 & & \mathbf{D} \arrow[ur, dashed, "l"'] &
\end{tikzcd}$$

A class $\mathcal{K}$ satisfies the *local weak amalgamation property* if for some $\mathbf{A} \in \mathcal{K}$ weak amalgamation holds for the class of structures $\mathbf{B} \in \mathcal{K}$ that extend $\mathbf{A}$.

Definition 5.2. Let $\mathcal{K}$ be a Fraïssé class. We associate with it a class of structures $\mathcal{K}_p$. Elements of $\mathcal{K}_p$ are partial isomorphisms of $\mathcal{K}$, more precisely tuples

$$(\mathbf{A}; p : \mathbf{A}' \to \mathbf{A}''),$$

where $\mathbf{A}$, $\mathbf{A}'$ and $\mathbf{A}'' \in \mathcal{K}$, $\mathbf{A}'$, $\mathbf{A}'' \subseteq \mathbf{A}$ and p is an isomorphism.

Theorem 5.3 (Kechris–Rosendal, see [KR07], Theorem 3.7). *The group of automorphisms of a Fraïssé class $\mathcal{K}$ has a non-meager conjugacy class if and only if class $\mathcal{K}_p$ satisfies the local weak amalgamation property.*

Proposition 5.4. *Every conjugacy class in $\mathrm{Aut}(\mathbb{QU}_\prec)$ is meager.*

Proof. By Theorem 5.3 it is enough to show that the class $\mathcal{M}_p$ does not have the local WAP. Let $\bar{\mathbf{A}} = (\mathbf{A},\ \phi : \mathbf{A}' \to \mathbf{A}'') \in \mathcal{M}_p$, and assume without loss of generality that ϕ has at least one non-fixed point (otherwise take an extension of ϕ). We claim that the class of structures that extend $\mathbf{A}$ does not have WAP.

Fix $\bar{\mathbf{B}} = (\mathbf{B}, \psi : \mathbf{B}' \to \mathbf{B}'')$ that extends $\mathbf{A}$ and assume for notational simplicity that $\mathbf{A} \subseteq \mathbf{B}$. Let $z \in \mathbf{A}'$ be such that $\phi(z) \neq z$ and let $\mathrm{Orb}_\phi(z)$ be the orbit of z under ϕ. Then $\mathrm{Orb}_\psi(z) \supseteq \mathrm{Orb}_\phi(z)$. Since we have ordering $\phi(z) \neq z$ implies that z is not a periodic point of ψ, because for ordered structures periodic points coincide with fixed points. Let $x \in \mathbf{B}'$ be "the beginning of the orbit of z", that is $x \in \mathrm{Orb}_\psi(z)$ and $x \notin \mathrm{ran}(\psi)$. Such an x exists and is unique. Let $m_0 \in \mathbb{N}$ be such that $x = \psi^{-m_0}(z)$. Now take (by Lemma 4.9 and Proposition 4.5) two structures $\bar{\mathbf{C}} = (\mathbf{C},\ \sigma : \mathbf{C}' \to \mathbf{C}'') \in \mathcal{M}_p$, $\bar{\mathbf{D}} = (\mathbf{D},\ \tau : \mathbf{D}' \to \mathbf{D}'') \in \mathcal{M}_p$ such that $\bar{\mathbf{B}} \subseteq \bar{\mathbf{C}}$ and $\bar{\mathbf{B}} \subseteq \bar{\mathbf{D}}$ such that $x \notin \mathrm{ran}(\sigma)$, $x \notin \mathrm{ran}(\tau)$ and $\mathrm{Orb}_\sigma(x)$ and $\mathrm{Orb}_\tau(x)$ are disjoint over $\mathrm{Orb}_\phi(x)$. We claim that there is no weak amalgamation of $\bar{\mathbf{C}}$ and $\bar{\mathbf{D}}$ over $\bar{\mathbf{B}}$ and $\bar{\mathbf{A}}$. Indeed, suppose there is a structure $(\mathbf{E},\ \xi : \mathbf{E}' \to \mathbf{E}'')$ together with two embeddings $k : \mathbf{C} \to \mathbf{E}$ and $l : \mathbf{D} \to \mathbf{E}$ such that $k(a) = l(a)$ for all $a \in \mathbf{A}'$. In particular, $k(z) = l(z)$. But the maps k, l are not only isometries but also preserve partial isometries ϕ, ψ, σ, τ. Hence

$$k(\sigma^m(z)) = l(\tau^m(z))$$

for any $m \in \mathbb{Z}$ whenever both sides are defined. And thus $k(x) = k(\sigma^{-m_0}(z)) = l(\tau^{-m_0}(z)) = l(x)$. Suppose, for definiteness, that $|\mathrm{Orb}_\sigma(x)| \geqslant |\mathrm{Orb}_\tau(x)|$ or, in other words, there is $m_1 \in \mathbb{N}$ such that $\sigma^{m_1}(x)$ is defined but $\tau^{m_1+1}(x)$ is not (i.e., $\tau^{m_1}(x) \notin \mathrm{dom}(\tau)$). Then $\mathrm{Orb}_\sigma(x)$ extends $\mathrm{Orb}_\tau(x)$ over $\mathrm{Orb}_\psi(x)$. This is because

$$k^{-1}(l(\tau^m(z))) = \sigma^m(z) \quad \forall m \in \{0, \ldots, m_1\}.$$

This contradicts the choice of $\mathrm{Orb}_\sigma(x)$ and $\mathrm{Orb}_\tau(x)$. $\square$

For classes of topological similarity the situation is rather different. All non-trivial elements in $\mathrm{Aut}(\mathbb{QU}_\prec)$ fall into a single class of topological similarity. And more generally, if $\mathbb{K}$ is any countable linearly ordered structure and $\mathrm{Aut}(\mathbb{K})$ is endowed with the topology of pointwise convergence ($\mathbb{K}$ is discrete here), then $\mathrm{Aut}(\mathbb{K})$ has exactly two classes of topological similarity (unless $\mathrm{Aut}(\mathbb{K}) = \{id\}$, then, of course, there is only one): all non-trivial automorphisms generate a discrete copy of $\mathbb{Z}$ and hence fall into a single class. Thus, in spite of the previous proposition, it makes sense to ask if there is a non-meager two-dimensional similarity class in $\mathrm{Aut}(\mathbb{QU}_\prec)$.

We define the notions of elementary and piecewise elementary pairs of partial isomorphisms of $\mathbb{QU}_\prec$ and the notion of liberation exactly as for the partial isomorphisms of the rationals.

It turns out that the analog of Theorem 3.15 for the ordered Urysohn space holds. Let us first briefly sketch the idea of the proof before diving into the details. We will prove that, again, for a generic pair there is a sequence of reduced words, such that this pair converges along it. One can repeat all the arguments up to Lemma 3.12 (only obvious changes are necessary). So one gets for a piecewise elementary pair (p, q) a triple (p', q', w) that liberates p in (p, q). But now, contrary to the case of the rationals, one cannot in general declare that $p'(w(p', q')(c)) = c$ for $c \in \mathrm{Ess(p)} \cup \mathrm{Ess(q)}$, since such a p' may be not an isometry. At this moment we have to take further extensions of p' and q'. But once an analog of Lemma 3.13 is proved for the Urysohn case, the rest of Theorem 3.15 goes unchanged.

If p is a partial isometry, we can use amalgamation of its domain with a one point metric space over the empty set to add a fixed point for p. Using this observation the following two lemmata, which are analogs of Lemma 3.12 and Lemma 3.11, are proved as for the rationals, and we omit the details.

Lemma 5.5. *Let (p,q) be a piecewise elementary pair of partial isomorphisms of $\mathbb{QU}_\prec$ and assume a triple (p',q',w) liberates p [liberates q] in (p,q). Let $u = t^n v$ [$u = s^m v$] be a reduced word such that uw is irreducible. Then there is a triple (p'',q'',uw) that liberates p [liberates q] in (p,q). Moreover, one can take p'' to be an extension of p' and q'' to be an extension of q'.*

Lemma 5.6. *Let (p,q) be a piecewise elementary pair of partial isomorphisms of $\mathbb{QU}_\prec$ and $u \in F(s,t)$ be a reduced word. Then there is a triple (p',q',vu) that liberates p in (p,q) [liberates q] and such that $|vu| = |v| + |u|$.*

Lemma 5.7. *For any pair (p,q) of partial isomorphisms of the $\mathbb{QU}_\prec$ and any word $u \in F(s,t)$ there are extensions p' and q' of p and q respectively and a reduced word $w = *u$ such that $w(p',q')(c) = c$ for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$.*

Proof. We can assume that (p,q) is piecewise elementary. By Lemma 5.6 there are extensions $\tilde{p}$, $\tilde{q}$ of p and q and a reduced word vu such that $(\tilde{p},\tilde{q},vu)$ liberates p in (p,q). Now apply Theorem 4.12 (with Remark 4.13 and Lemma 3.11) to $\tilde{p}$, $\tilde{q}$ and

$$\mathbf{A} = \mathrm{dom}(\bar{p}) \cup \mathrm{ran}(\bar{p}) \cup \mathrm{dom}(\bar{q}) \cup \mathrm{ran}(\bar{q})$$

to get extensions $\bar{p}$ and $\bar{q}$ and a reduced word v'. Note that $v'vu$ is reduced, because v starts from a power of t and v' by construction ends in a power of s. By the item (ii) of Theorem 4.12 we can extend $\bar{p}$ to p' by declaring

$$p'|_{v'vu(\mathrm{dom}(p)\cup\mathrm{dom}(q))} = 1.$$

Set $q' = \bar{q}$ and $w = (v'uv)^{-1}s(v'uv)$. It is easy to see that $w(p',q')(c) = c$ holds for any $c \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$. □

Theorem 5.8. *Every two-dimensional class of topological similarity in $\mathrm{Aut}(\mathbb{QU}_\prec)$ is meager.*

Proof. Repeat the proofs of Lemma 3.14 and Theorem 3.15 using Lemma 5.7 instead of Lemma 3.13. □

Remark 5.9. All the results in this section can be proved for the ordered random graph in the same way, as they were proved for the ordered rational Urysohn space. One can also formally deduce this case from the above results viewing graphs as metric spaces with all the distances in $\{0,1,2\}$.

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