Equivariant Borel liftings in complex analysis and PDE

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Abstract

We establish Borel equivariant analogues of several classical theorems from complex analysis and PDE. The starting point is an equivariant Weierstrass theorem for entire functions: there exists a Borel mapping which assigns to each non-periodic positive divisor d an entire function f_d with divisor of zeros $\mathrm{div}(f_d) = d$ and which commutes with translation, $f_{d-w}(z) = f_d(z + w)$. We also examine the existence of equivariant Borel right inverses for the distributional Laplacian, the heat operator, and the $\bar{\partial}$-operator on the space of smooth functions. We demonstrate that Borel equivariant inverses for these maps exist on the free part of the range (for the heat operator, this holds up to the removal of a null set with respect to any invariant probability measure). In general, the freeness assumptions cannot be omitted and Borelness cannot be strengthened to continuity. Our positive results follow from a theorem establishing sufficient conditions for the existence of equivariant Borel liftings. Two key ingredients are Runge-type approximation theorems and the existence of Borel toasts, which are Borel analogues of Rokhlin towers from ergodic theory.

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1 Introduction

1.1 Equivariant analysis and Borel entire functions

This paper contributes to the field of equivariant analysis, which dates back to the notable 1997 paper [55] of Benjamin Weiss. His work was motivated by a question posed by George Mackey, who asked whether the space $\mathcal{E}$ of entire functions admits any non-trivial translation-invariant probability measures. Weiss answered this question in the affirmative by introducing the concept of a measurable entire function. Specifically, given a free probability measure-preserving (p.m.p. for short) action $\mathbb{C} \curvearrowright X$, $(z, x) \mapsto z \cdot x$, of the additive group of complex numbers on a standard probability space (X, μ), a measurable entire function on X is a measurable map $F : X \to \mathbb{C}$ such that, for μ-almost every $x \in X$, the function F_x, defined by $\mathbb{C} \ni z \mapsto F(z \cdot x) \in \mathbb{C}$, is entire. Weiss showed that non-constant measurable entire functions exist for all p.m.p. actions $\mathbb{C} \curvearrowright X$. The push-forward of μ along the map $x \mapsto F_x$ produces the desired non-trivial invariant measure on $\mathcal{E}$. A related phenomenon was studied by Tsirelson [53], who constructed similarly "paradoxical" stationary random vector fields on $\mathbb{R}^d$ with constant non-zero divergence.

One can view measurable entire functions as those entire functions that can be constructed without the choice of an origin. The perspective of "mathematics without an origin" has recently received significant attention within the descriptive set-theoretic community. For instance, the field of descriptive combinatorics has emerged as a central area, see [33] for an overview. In a sense, equivariant analysis stands in the same relation to classical analysis as descriptive combinatorics does to its classical counterpart. In this paper, we explore the validity of equivariant analogs of several classical results from complex analysis and partial differential equations.

Weiss' work is set in the framework of ergodic theory. An alternative approach, which we choose here, is to consider standard Borel spaces and Borel actions. For "positive" results—those that establish the existence of desired objects—Borel formulations are stronger, and in this paper, we adopt the perspective of Borel dynamics. This choice is motivated by two factors: first, our main results are of this positive nature, and second, the primary examples of actions to which we apply our results naturally carry a Borel structure but lack any distinguished measures to highlight. The proof of the existence of measurable entire functions from [55] adapts to the Borel category with a single modification (cf. Remark 5.5): Instead of relying on a Rokhlin-type lemma to cover orbits with coherent rectangular regions—a tool generally unavailable in Borel dynamics—one employs the concept of Borel toasts (following the terminology introduced in [19]) that cover orbits by compact sets with connected complements, see Definition 3.2 and Figure 1.

A slight shift in perspective allows one to think of Borel entire functions as *factor maps*, i.e., Borel measurable equivariant maps, into the space $\mathcal{E}$ of entire functions. Indeed, the space of entire functions is naturally equipped with the Borel σ-algebra generated by the topology of uniform convergence on compact sets. Starting from a Borel entire $F : X \to \mathbb{C}$, we obtain a Borel $\mathbb{C}$-equivariant map $\psi : X \to \mathcal{E}$, where ψ assigns to each $x \in X$ the function F_x. Conversely, given such a map ψ, F can be reconstructed by evaluating $\psi(x)$ at the origin. This perspective motivates the term *equivariant analysis.*

As a side note, let us illustrate the difference between the Borel and ergodic viewpoints with the following example concerning the growth rates of entire functions. As shown in [8, 21], in the context of ergodic theory, every free p.m.p. action of $\mathbb{C}$ admits non-constant measurable entire functions with a specific bound on their growth. However, this property does not generally hold for Borel actions, as demonstrated in Appendix A: there are free Borel actions $\mathbb{C} \curvearrowright X$ for which any Borel entire $F : X \to \mathbb{C}$ that is non-constant on all orbits must have unbounded growth (Corollary A.3) in the sense that for *any* rate function f, there are $x \in X$ for which $\max_{|z|\leq R} F(z \cdot x) = O(f(R))$ fails.

1.2 The equivariant Weierstrass theorem

Before describing our main results, we begin with an example that captures the essence of the questions we are interested in. It was also the starting point of this paper.

Consider the space $\mathcal{E}_{\neq 0}$ of entire functions which do not vanish identically. To each element $f \in \mathcal{E}_{\neq 0}$, one can associate its divisor of zeros $\operatorname{div}(f)$—a discrete multiset in the complex plane. All such divisors form the space $\mathcal{D}^+$, which carries natural Borel and topological structures, as discussed in detail in Section 5.2. The classical Weierstrass theorem guarantees the existence of entire functions with a prescribed set of zeros, which is equivalent to the surjectivity of the map $\operatorname{div} : \mathcal{E}_{\neq 0} \to \mathcal{D}^+$. In particular, this implies the existence of right-inverses $\xi : \mathcal{D}^+ \to \mathcal{E}_{\neq 0}$ satisfying $\operatorname{div}(\xi(d)) = d$ for all $d \in \mathcal{D}^+$.

The spaces $\mathcal{E}_{\neq 0}$ and $\mathcal{D}^+$ have natural actions of the additive group of complex numbers $\mathbb{C}$ through argument shifts, and the divisor map div is equivariant. A natural question is then whether the right-inverse map ξ can also be chosen to be equivariant. An immediate obstruction arises: such a map ξ must preserve stabilizers (i.e., the group of periods). In particular, the image of a non-trivial doubly-periodic divisor must be a doubly-periodic entire function. However, such functions are necessarily constant, and thus their divisors are trivial. Apart from this issue, the axiom of choice allows us to select a representative from each $\mathbb{C}$-orbit in the remaining part $\mathcal{D}^+ \setminus \mathcal{D}_2^+$, where $\mathcal{D}_2^+$ denotes the space of doubly-periodic divisors. An equivariant map ξ can then be constructed without difficulty.

In practice, it is natural to ask for a map ξ with certain regularity properties. Let us for the moment restrict the discussion to the free parts of the actions, $\operatorname{Free}(\mathcal{E}_{\neq 0})$ and $\operatorname{Free}(\mathcal{D}^+)$, which correspond to the elements with trivial stabilizers. (The significance of this restriction will be clarified shortly.) Both of these spaces are equipped with natural Polish topologies: the topology of uniform convergence on compact sets for $\mathcal{E}_{\neq 0}$, and the topology of weak convergence (sometimes also called vague convergence) of measures, when elements of $\mathcal{D}^+$ are viewed as atomic Radon measures. Interestingly, while *continuous* equivariant inverses to the divisor map do not exist even on the free part (Theorem 6.4), we have the following positive result. We denote by $\mathcal{D}$ the space of signed divisors and by $\mathcal{MR}^\times$ the space of meromorphic functions which do not vanish identically.

Theorem 5.3. *There exists a Borel $\mathbb{C}$-equivariant map $\psi : \operatorname{Free}(\mathcal{D}^+) \to \mathcal{E}_{\neq 0}$ such that $\operatorname{div} \circ \psi = \mathrm{id}_{\operatorname{Free}(\mathcal{D}^+)}$. Furthermore, there exists a Borel $\mathbb{C}$-equivariant map $\psi : \operatorname{Free}(\mathcal{D}) \to \mathcal{MR}^\times$ such that $\operatorname{div} \circ \psi = \mathrm{id}_{\operatorname{Free}(\mathcal{D})}$.*

An immediate and rather curious corollary of Theorem 5.3 is that any translation-invariant non-periodic point process on $\mathbb{C}$ lifts, via the map ξ, to a translation-invariant random entire function with zeros at the point process[1]. This yields, for instance, that one can realize the 2D Poisson process as the zero set of a translation-invariant random entire function.

Let us outline a general proof strategy one might follow to obtain equivariant liftings. Each orbit of a free $\mathbb{C}$-action can be identified with a copy of the acting

[1] This corollary was proven earlier in an unpublished work with Oren Yakir, cf. [50, p. 3].

group itself, allowing us to speak of the geometries and shapes of various regions within the phase space. The space $X = \mathrm{Free}(\mathcal{D}^+)$ can be exhausted by Borel sets, which, on each orbit of the action, decompose into disjoint unions of regions diffeomorphic to disks. These regions are coherently aligned with one another. This formal concept, known as a Borel toast, is discussed in detail in Definition 3.2. The construction of the required equivariant map $\xi : \mathrm{Free}(\mathcal{D}^+) \to \mathrm{Free}(\mathcal{E}_{\neq 0})$ proceeds inductively across these regions. Intermediate steps are only "partially" equivariant, that is, they are equivariant only on parts of the orbits, but these parts eventually exhaust the whole space and full equivariance is achieved in the limit.

The key technical step of this construction relies on the following version of the classical Runge theorem: *Given $\varepsilon > 0$, a compact set $K \subseteq \mathbb{C}$ with connected complement, and a holomorphic g on a neighborhood of K, there exists an entire f such that $\sup_{z \in K} |f(z) - g(z)| < \varepsilon$ and, for $z \in K$, $f(z) = 0$ if and only if $g(z) = 0$. Moreover, the function f can be constructed in a Borel manner with respect to the the compact sets K and functions g.*

The primary part of this statement can be easily derived from the standard Runge theorem. However, verifying the Borelness of the construction is a tedious task (see, for instance, [7, 20, 24]), which may get even more cumbersome for other variants of Runge-type theorems. For example, consider the distributional Poisson equation $\Delta u = \mu$, where $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ is a positive Radon measure on $\mathbb{R}^d$. Similar to the case of divisors, there exists a Borel equivariant inverse to Δ defined on the free part of the range, see Theorem 5.13. The corresponding Runge-type result involves approximating a subharmonic function on a compact set K with a given Riesz measure $\mu|_K$ by a subharmonic function u defined on all of $\mathbb{R}^d$, such that $\Delta u|_K = \mu|_K$. Additionally, the construction must be Borel in all parameters involved. Ensuring Borelness in such cases often requires meticulous bookkeeping. We have therefore developed a framework that allows the direct application of standard analytic versions of Runge-type theorems as black boxes, eliminating the need to establish the Borelness of their constructions. This is particularly advantageous because proofs of Runge-type results oftentimes rely on the Hahn–Banach theorem (see, for instance, [11, Section III.8] and [27, II.3.4]), which is not inherently Borel.

In addition to the argument shift action of $\mathbb{C}$, there is another action of a different group that plays a significant role. Let us revisit the map $\mathrm{div} : \mathcal{E}_{\neq 0} \to \mathcal{D}^+$. Its kernel $\mathcal{E}^\times$, consisting precisely of entire functions without zeros, forms a G_δ

subset of $\mathcal{E}$ and is a Polish group in the topology induced from $\mathcal{E}$. This group $H = \mathcal{E}^\times$ acts on $\mathcal{E}_{\neq 0}$ via multiplication, and the equality $\operatorname{div}(f_1) = \operatorname{div}(f_2)$ holds if and only if $f_1/f_2 \in H$.

For the inductive construction of a Borel equivariant map $\xi : \operatorname{Free}(\mathcal{D}^+) \to \operatorname{Free}(\mathcal{E}^\times)$, it suffices to apply Runge's theorem exclusively to elements of the group H. Specifically, for a given holomorphic function f with no zeros on a compact set K with connected complement, there exists an entire function $g \in H$ that approximates f on K. Crucially, we do not initially require g to depend on f and K in a Borel manner; the mere existence of such a g is sufficient. The construction in the main theorem then automatically produces a Borel version that accommodates functions f with zeros in K and ensures Borel measurability in the relevant parameters. We emphasize that in this variant, we do not need to preserve a prescribed set of zeros; instead, we work with holomorphic functions that have no zeros. This simplification is key to achieving Borel measurability with ease. Intuitively, transitioning from $\mathcal{E}_{\neq 0}$ to $\mathcal{E}^\times$ transforms the Runge-type condition into an *open* condition in the topology of $\mathcal{E}^\times$. Obtaining a Borel version then reduces to finding a Borel uniformization[2] of an open set with non-empty slices, which is accomplished using standard descriptive set-theoretic techniques.

2 Our results

2.1 The main theorem

The scenario described above for the divisor map div is quite representative, and many problems in complex analysis and PDE exhibit a similar structure. For example, in the case of the Poisson equation $\Delta u = \mu$, the relevant group H, in which we apply Runge's theorem, is the group of harmonic functions $\mathcal{H}(\mathbb{R}^d)$. Our main theorem captures this structure in an abstract setting, and gives rather general sufficient conditions for the existence of Borel liftings.

2.1.1 Standing assumptions and setting of the main theorem

Here is an overview of the setting of our main result. The precise definitions are given in Sections 3 and 4. We suppose we are given the following:

[2] By a Borel uniformization of a set $P \subseteq X \times Z$, we mean a Borel function $f : \operatorname{proj}_X(P) \to Z$ such that $(x, f(x)) \in P$ for all $x \in \operatorname{proj}_X(P)$.

1. Standard Borel spaces X, Y and Z, and a locally compact Polish group G acting on each of them in a Borel way. The action $G \curvearrowright X$ is always assumed to be free. In our main applications, G is either of the groups $\mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{T}^p$ of the appropriate dimension, X, Y and Z are spaces of functions, measures, or distributions on G, and G acts on all these spaces by argument shifts. Also, X is taken to be $\mathrm{Free}(Y)$.

2. An *equation* $\pi \circ \psi = \varphi$, where $\pi : Z \to Y$ is an equivariant Borel surjection and where $\varphi : X \to Y$ is a Borel equivariant map, to which we seek a Borel equivariant solution $\psi : X \to Z$. In practice, we often take $X = \mathrm{Free}(Y)$ and $\varphi = \mathrm{id}_X$. For the equation $\mathrm{div} f = d$, Y would be the space $\mathcal{D}^+$, Z the space of entire functions, and π the divisor map. Similarly for the Poisson equation, Y is the space $\mathcal{M}^+(\mathbb{R}^d)$ of Radon measures, Z the space of subharmonic functions, and π the Laplacian.

3. A Polish group H and a Borel action $H \curvearrowright Z$, whose orbit equivalence relation E_H is classified by π. That is, $\pi(x) = \pi(y)$ if and only if x and y lie in the same H-orbit. In the aforementioned examples, this condition is automatically satisfied since H is taken to be the kernel of π, that is, H is either the multiplicative group of entire functions or the additive group of harmonic functions.

4. A continuous action $\tau : G \curvearrowright H$ by automorphisms, yielding a Polish semidirect product $H \rtimes_\tau G$ equipped with the product topology and group operations $(h_1, g_1)(h_2, g_2) = (h_1 \tau^{g_1}(h_2), g_1 g_2)$. The semidirect product is assumed to act on Z in a Borel way, compatible with the given actions $G \curvearrowright Z$ and $H \curvearrowright Z$. That is, we have $g \cdot z = (e_H, g) \cdot z$, $h \cdot z = (h, e_G) \cdot z$ where e_H and e_G are the units in H and G, respectively. In applications, G acts on H by shifting the argument.

5. A cofinal[3] G-invariant class of compact sets $\mathfrak{R} \subseteq \mathcal{K}(G)$. In applications, $\mathfrak{R}$ is a (sub)set of Runge domains for the class of functions H. Oftentimes, $\mathfrak{R}$ is the class of compact sets in $\mathbb{C}$ or $\mathbb{R}^d$ diffeomorphic to the closed unit ball.

This completes the general setup. To show the existence of an equivariant Borel lifting of φ, we need two additional key assumptions: the *Runge approximation property* and the existence of *Borel toasts*.

[3] i.e., exhaustive: for any compact $K \subseteq G$, there exists some $K' \in \mathfrak{R}$ such that $K \subseteq K'$.

6. Let $\mathsf{N} = (\|\cdot\|_K)_{K\in\mathcal{K}(G)}$ be a τ-family of seminorms on H. This notion is formalized in Definition 4.6 below, but the reader may keep in mind that for our applications, we exclusively use $\|f\|_K = \max_K |f|$ or $\|f\|_K = \max_K |\log|f||$.

We say that N satisfies the $\mathfrak{R}$-*Runge property* (Definition 4.8) if, for any pairwise disjoint compact sets $K_1, \dots, K_m \in \mathfrak{R}$, any $h_1, \dots, h_m \in H$, and any given $\varepsilon > 0$, there exists $h \in H$ such that $\max_{1\le i\le m} \|hh_i^{-1}\|_{K_i} < \varepsilon$.

7. The final assumption is that the free action $G \curvearrowright X$ admits a *Borel* $\mathfrak{R}$*-toast*, Definition 3.2. A Borel toast is a Borel version of a Rokhlin tower from ergodic theory, and consists of a sequence $(\mathcal{C}_n)_n$ of Borel sets $\mathcal{C}_n$ together with Borel functions $\lambda_n : \mathcal{C}_n \to \mathfrak{R}$, such that the regions $R_n(c) = \lambda_n(c) \cdot c$ cover the orbits in a tree-like coherent way. The central toast axioms entail that for any $x \in X$ and any compact $K \subseteq G$ there is some region such that $K \cdot x \subseteq R_n(c)$. Regions in the same "generation" are moreover disjoint, and two regions $R_m(c)$ and $R_n(c')$, $m < n$, are either disjoint or $R_m(c)$ is contained $R_n(c')$.

Any free $\mathbb{R}^d$ or $\mathbb{R}^p \times \mathbb{T}^q$-flow admits a Borel $\mathfrak{R}$-toast with $\mathfrak{R}$ being the class of compact sets diffeomorphic to the closed unit ball (Section 3.3).

2.1.2 The main result

In this framework, we establish the existence of Borel liftings.

Theorem 4.9. *Assume that the free action $G \curvearrowright X$ admits a Borel $\mathfrak{R}$-toast, and that the τ-family N on H satisfies the $\mathfrak{R}$-Runge property. Then, for any G-equivariant Borel map $\varphi : X \to Y$, there exists a G-equivariant Borel map $\psi : X \to Z$ such that $\pi \circ \psi = \varphi$, making the following diagram commute:*

$$\begin{tikzcd} G \curvearrowright X \arrow[r, dashed, "\psi"] \arrow[dr, "\varphi"'] & G \curvearrowright Z \arrow[d, "\pi"] \\ & G \curvearrowright Y \end{tikzcd}$$

Specializing to $X = \mathrm{Free}(G \curvearrowright Y)$ and taking φ to be the identity map id_X we get the following corollary.

Corollary 4.10. *Suppose that the free action $G \curvearrowright \mathrm{Free}(Y)$ admits a Borel $\mathfrak{R}$-toast, and that the τ-family N on H satisfies the $\mathfrak{R}$-Runge property. Then there exists a G-equivariant Borel map $\psi : \mathrm{Free}(Y) \to Z$ satisfying $(\pi \circ \psi)(y) = y$ for all $y \in \mathrm{Free}(Y)$.*

2.2 Applications of the main theorem

We apply Theorem 4.9 and Corollary 4.10 to the following maps π:

1. The divisor maps $\mathrm{div} : \mathcal{E}_{\neq 0} \to \mathcal{D}^+$ and $\mathrm{div} : \mathcal{MR}^\times \to \mathcal{D}$ that associate with a non-trivial entire or meromorphic function its divisor.
2. The principal part map $\mathrm{pp} : \mathcal{MR} \to \mathcal{P}$ that associates with a meromorphic function its principal parts at all the poles.
3. The distributional Laplacian $\Delta : \mathcal{SH}(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d)$, where $\mathcal{SH}(\mathbb{R}^d)$ is the space of subharmonic functions and $\mathcal{M}^+(\mathbb{R}^d)$ is the space of Radon measures on $\mathbb{R}^d$.
4. The d-bar operator $\bar{\partial} : C^\infty(\mathbb{C}, \mathbb{C}) \to C^\infty(\mathbb{C}, \mathbb{C})$, where $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$ and $C^\infty(\mathbb{C}, \mathbb{C})$ is the space of smooth complex-valued functions on $\mathbb{C}$.
5. The heat operator $(\frac{\partial}{\partial t} - \Delta) : C^\infty(\mathbb{R}^{d+1}) \to C^\infty(\mathbb{R}^{d+1})$.

In all these cases, the domain and co-domain can be endowed with the structure of a standard Borel space. The corresponding maps between these spaces are Borel and surjective. Additionally, in each case, there is a natural argument-shift action of $\mathbb{R}^d$ (for the appropriate value of d), and the maps π are all equivariant with respect to these actions.

Section 5 applies the main theorem to the cases described above. Specifically, taking $X = \mathrm{Free}(Y)$ and φ as the identity map, Corollary 4.10 is applied to $\pi : \mathcal{MR}^\times \to \mathcal{D}$ and $\pi : \mathcal{MR} \to \mathcal{P}$. This yields the existence of Borel equivariant right-inverses $\xi : \mathrm{Free}(\mathcal{D}) \to \mathrm{Free}(\mathcal{MR}^\times)$ and $\xi : \mathrm{Free}(\mathcal{P}) \to \mathrm{Free}(\mathcal{MR})$, as stated in Theorems 5.3 and 5.7, respectively. These results represent Borel equivariant versions of the Weierstrass and Mittag-Leffler theorems, respectively. The restriction to the free part is essential, as no such map ξ exists on the space $\mathcal{D}_1$ of 1-periodic divisors (Theorem 7.1).

Similarly, Corollary 4.10 applies to $\Delta : \mathcal{SH}(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d)$ and yields a Borel equivariant map $\xi : \mathrm{Free}(\mathcal{M}^+(\mathbb{R}^d)) \to \mathrm{Free}(\mathcal{SH}(\mathbb{R}^d))$ satisfying $\Delta\xi(\mu) = \mu$. In other words, we have an equivariant Brelot–Weierstrass theorem for subharmonic functions. We should stress that the choice of working with subharmonic functions and positive measures is rather arbitrary, and one obtains equivariant right-inverses to the Laplacian on $\mathrm{Free}(C^\infty(\mathbb{R}^d))$ or the space $\mathrm{Free}(\mathscr{D}'(\mathbb{R}^d))$ of (non-periodic) distributions in exactly the same way.

In contrast to the Weierstrass theorem, the situation for the periodic part is more nuanced. Let Γ be a closed subgroup of $\mathbb{R}^d$, and let $\mathcal{M}_\Gamma^+(\mathbb{R}^d)$ denote the space of Radon measures μ with $\mathrm{Stab}(\mu) = \Gamma$, and $\mathcal{SH}_\Gamma(\mathbb{R}^d)$ denote the set of subharmonic functions with stabilizer Γ. A Borel equivariant map $\xi : \mathcal{M}_\Gamma^+(\mathbb{R}^d) \to \mathcal{SH}_\Gamma(\mathbb{R}^d)$ satisfying $\Delta\xi(\mu) = \mu$ exists if and only if $\dim \Gamma \leq d-2$ (Theorems 5.14 and 7.9).

Theorem 5.12 provides an application of Theorem 4.9 to the d-bar equation $\bar{\partial} f = g$. It establishes the existence of Borel equivariant right-inverses

$$\xi : \mathrm{Free}(C^\infty(\mathbb{C},\mathbb{C})) \to C^\infty(\mathbb{C},\mathbb{C}).$$

Theorem 5.17 deals with the heat operator $(\frac{\partial}{\partial t} - \Delta) : C^\infty(\mathbb{R}^{d+1}) \to C^\infty(\mathbb{R}^{d+1})$, which presents a distinct case compared to other applications. Typically, the class $\mathfrak{R}$ is chosen to consist of arbitrary sets diffeomorphic to the unit ball, since this is sufficiently restrictive to ensure that Runge's theorem applies to finite disjoint unions of elements of $\mathfrak{R}$. However, this choice is insufficient for the heat operator. The characterization of Runge domains for the heat operator is known [29]. Here, the class $\mathfrak{R}$ is defined as the collection of compact sets $K \in \mathcal{K}(\mathbb{R}^{d+1})$ for which $P \setminus K$ is connected for every hyperplane P orthogonal to the time axis. While we do not know whether an arbitrary free Borel $\mathbb{R}^{d+1}$-action admits an $\mathfrak{R}$-toast, Rokhlin's lemma for $\mathbb{R}^{d+1}$-actions guarantees that this holds ergodic-theoretically—that is, up to a null set with respect to any given invariant probability measure.

2.3 Lack of equivariant inverses

The remaining sections show that the statement of Theorem 4.9 is, in many respects, optimal. For instance, Section 6 shows that the existence of Borel equivariant inverses on the free part generally cannot be strengthened to the existence of continuous such maps. Here, we focus specifically on the maps $\mathrm{div} : \mathcal{E}_{\neq 0} \to \mathcal{D}^+$ (Theorem 6.4) and $\bar{\partial} : C^\infty(\mathbb{C},\mathbb{C}) \to C^\infty(\mathbb{C},\mathbb{C})$ (Theorem 6.6), primarily because these cases involve spaces endowed with natural Polish topologies.

The restriction to the free part of the range is essential to ensure the existence of Borel equivariant inverses. Such inverses generally fail to exist on the periodic parts of the range. A descriptive set-theoretic perspective is particularly useful here, as it provides a framework to articulate the underlying reasons. For example, we demonstrate that the argument-shift action of $\mathbb{C}$ on the space of entire functions with period 1 admits a Borel transversal (Proposition 7.4), whereas the action of $\mathbb{C}$ on the space of divisors with period 1 does not (Lemma 7.6). This evidently precludes the existence of a Borel equivariant inverse to the divisor map div. A similar argument rules out the existence of a Borel equivariant map $\xi : \mathcal{M}_\Gamma(\mathbb{R}^d) \to \mathcal{SH}_\Gamma(\mathbb{R}^d)$ when $\dim \Gamma = d-1$.

The non-existence of Borel equivariant inverses on the 1-periodic part also applies to the $\bar\partial$-problem (Theorem 7.14), though proving it requires a different approach. Our proof hinges on the existence of non-stationary 1-periodic random $C^\infty(\mathbb{C}, \mathbb{C})$-functions f for which $\bar\partial f$ is stationary.

Section 8 treats another natural case that can be framed in terms of the existence of a lifting ψ in the diagram of Theorem 4.9. This is the question of whether a given Borel entire function admits a Borel entire antiderivative. While this is always true for classical entire functions in complex analysis, there exist Borel entire functions that lack Borel antiderivatives (Theorem 8.1 and Section 8.3). Juxtaposing with the setting of Theorem 4.9, we have $G = \mathbb{C}$ acting on the spaces $Z = Y = \mathcal{E}$ by argument shifts, the group $H = \mathbb{C}$ is identified with the constant functions, and the action $G \curvearrowright H$ is trivial. The issue which prevents application of the corollary arises from the failure of the Runge property among the constants. Consequently, the main theorem does not apply, and indeed the corresponding equivariant Borel liftings may fail to exist. This non-existence proof relies on Birkhoff's classical observation that $\mathbb{C} \curvearrowright \mathcal{E}$ has dense orbits.

Also, the exponential map, the absolute value function, the logarithmic derivative, and a few other maps discussed in Section 8.3 fail to admit equivariant inverses in general. The latter has some interesting consequences for the relation between the Mittag-Leffler and Weierstrass theorems in the equivariant world. The classical Weierstrass theorem is often deduced from Mittag-Leffler's theorem by finding a meromorphic function g with principal part $m(w)/(z-w)$ for each desired zero w of multiplicity $m(w)$. A holomorphic function f with logarithmic derivative $\frac{f'}{f} = g$ has zeros at the poles of g and their multiplicities are precisely $m(w)$. However, this approach fails in equivariant analysis, as the logarithmic derivative might not admit Borel equivariant right-inverses.

2.4 The appendices

The paper includes several appendices.

Appendix A demonstrates that, unlike in ergodic theory, there exist free Borel actions $\mathbb{C} \curvearrowright X$ for which every nowhere constant entire function exhibits unbounded growth (Theorem A.2). The proof relies on a result by Gao, Jackson, Krohne and Seward on the vanishing rate of sequences of markers for $\mathbb{Z}^d$-actions [19, Thm. 1.1].

Appendix B demonstrates that the multiplicative group of meromorphic functions $\mathcal{MR}^\times$ admits no Polish group topologies. In particular, the map $\operatorname{div} : \mathcal{MR}^\times \to \mathcal{D}$ cannot be interpreted as a continuous homomorphism between Polish groups. Our results also resolve a question posed in [22], proving that there are no metrizable complete algebra topologies on $\mathcal{MR}$ (Theorem B.9). The question of whether non-metrizable such topologies exist, also raised in [22], is not addressed by our methods.

The results in Appendices A and B are, to the best of our knowledge, new. However, as they lie outside the main scope of this work, we have included them in the appendices.

Finally, Appendix C establishes a version of Runge's theorem for periodic harmonic functions, which is needed for proving the existence of Borel equivariant inverses to the Laplacian Δ on $\mathcal{M}_\Gamma(\mathbb{R}^d)$ when $\dim \Gamma \le d - 2$. While this result is a special case of the Lax–Malgrange approximation theorem and is certainly well-known to the experts, we provide a self-contained proof for the reader's convenience.

Frequently used notation

X, Y, Z	Standard Borel spaces (see Section 3 for the definition).
G, H	Polish groups (Section 3). Typically, G is assumed locally compact.
$G \curvearrowright X$ etc.	A Borel action (Section 3).
a_Z, a_X, τ	Notation for Borel actions. The action τ is always by automorphisms $G \curvearrowright H$.
π, φ, ψ	Equivariant Borel maps. Typically, $\pi : Z \to Y$ and $\varphi : X \to Y$ are given, and we seek an equivariant lifting ψ satisfying $\pi \circ \psi = \varphi$ (see Section 2.1.1).
$\mathcal{F}(G), \mathcal{K}(G)$	Effros and Vietoris Borel spaces of the group G (Section 3.2).

$\mathfrak{D}_d, \mathfrak{C}_d$	Classes of all compact subsets of $\mathbb{R}^d$ diffeomorphic to the unit ball, and with connected complements, respectively.
$\mathfrak{K}$	A G-invariant cofinal family of compact subsets of G (Section 4.1).
$(\mathcal{C}_n, \lambda_n)_n$	A Borel toast (Section 3.3) on a standard Borel space X.
$R_n(c)$	Regions $R_n(c) = \lambda_n(c) \cdot c$, $c \in \mathcal{C}_n$, on the orbits, forming part of a Borel toast.
D	A family of pseudometrics on Z, satisfying the axioms of an a_Z-family for an action $a_Z : G \curvearrowright Z$ (Definition 4.2).
N	A family of seminorms on the group H satisfying the axioms of a τ-family (Definition 4.6) for an action $\tau : G \curvearrowright H$. Often required to satisfy the $\mathfrak{K}$-Runge property (Definition 4.8).
$\mathcal{E}$, $\mathcal{E}_{\neq 0}$, $\mathcal{E}^\times$	Spaces of entire functions, of entire functions which do not vanish identically, and of entire functions without zeros, respectively (Section 5.1).
$\mathcal{MR}, \mathcal{MR}^\times$	Space of all meromorphic functions, and space of those which do not vanish identically (i.e., invertible elements) (Section 5.1).
$\mathcal{D}, \mathcal{D}^+$	Spaces of signed and positive divisors, respectively (Section 5.2).
$\mathcal{P}$	The space of principal parts (Section 5.5).
div	The divisor map $\mathrm{div} : \mathcal{MR} \to \mathcal{D}$ (Section 5.2), assigning to each meromorphic function its divisor of zeros an poles.
pp	The principal part map $\mathrm{pp} : \mathcal{MR} \to \mathcal{P}$ (Section B.2), assigning to a meromorphic function its principal part.
$L^1_{\mathrm{loc}}(\mathbb{R}^d)$	Space of locally Lebesgue integrable real-valued functions on $\mathbb{R}^d$ (Section 5.10).
$\mathcal{SH}$, $\mathcal{H}$	The spaces of subharmonic and harmonic functions, respectively (Section 5.10).
$\mathcal{M}, \mathcal{M}^+$	The spaces of signed and positive Radon measures, respectively (Section 5.10).

Central definitions

Borel toast $(\mathcal{C}_n, \lambda_n)_n$	–	Definition 3.2
$\mathfrak{R}$-Runge property	–	Definition 4.8
$(\mathfrak{R}, P)$-Runge property	–	Definition 4.4
a_Z-family of pseudometrics	–	Definition 4.2
τ-family of seminorms	–	Definition 4.6
Concrete classifiability	–	Section 3.1
Borel transversal	–	Section 3.1

A note concerning the exposition. This paper uses tools from a number of areas, including descriptive set theory, Borel dynamics, topological vector spaces, complex analysis, and PDE. We have therefore tried to provide explanations of several standard and well-known arguments, hopefully, to the benefit of the reader.

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3 Descriptive set-theoretic preliminaries

3.1 Polish and Borel spaces

A *Polish space* is a separable completely metrizable topological space. A *Polish group* is a topological group whose underlying topology is Polish. For a topological space, its *Borel σ-algebra* is the σ-algebra generated by the open sets. A *standard Borel space* is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra on X that coincides with the Borel σ-algebra for some Polish topology on X. All uncountable standard

Borel spaces are isomorphic. Two standard references for the theory of Polish and standard Borel spaces are the books by Kechris [32] and Srivastava [51] (when readily available, we try to give references to both).

An action $G \curvearrowright X$ of a Polish group on a standard Borel space X is *Borel* if the map $G \times X \ni (g, x) \mapsto g \cdot x \in X$ is Borel measurable. A continuous action of G on a Polish space is defined similarly. (When no confusion should occur, we sometimes simplify notation and write gx for the action.) An important result due to Douglas Miller (see [32, 9.17] or [51, Thm. 4.8.4]) states that for any Borel action $G \curvearrowright X$ of a Polish group and any $x \in X$, the stabilizer $\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}$ is necessarily a closed subgroup of G.

For an action $G \curvearrowright X$, the *free part* of the action, denoted $\mathrm{Free}(G \curvearrowright X)$ or simply $\mathrm{Free}(X)$, is defined as:

$$\mathrm{Free}(X) = \{x \in X : g \cdot x \neq x \text{ for all } g \neq e\} = \{x \in X : \mathrm{Stab}(x) = \{e\}\}.$$

Recall that a subset of a Polish space is G_δ if it is a countable intersection of open sets. The following proposition is well-known, and its proof is included for the reader's convenience.

Proposition 3.1. *Let G be a locally compact Polish group, and let $G \curvearrowright X$ be a continuous action on a Polish space X. Then the set $\mathrm{Free}(G \curvearrowright X)$ is G_δ.*

Proof. Let $K \subseteq G$ be compact, and define

$$Y_K = \{y \in X : g \cdot y = y \text{ for some } g \in K\}.$$

Note that Y_K is closed. Indeed, suppose $y_n \in Y_K$, $n \in \mathbb{N}$, converges to some $y \in X$. Let $g_n \in K$ satisfy $g_n \cdot y_n = y_n$. By passing to a subsequence if necessary, we may assume $g_n \to g$ for some $g \in K$. By continuity of the action, $g_n \cdot y_n \to g \cdot y$, but $g_n \cdot y_n = y_n \to y$, so $g \cdot y = y$ and thus $y \in Y_K$.

Let $(K_n)_n$ be a countable sequence of compact sets such that $\bigcup_n K_n = G \setminus \{e\}$. Then $\mathrm{Free}(G \curvearrowright X) = X \setminus \bigcup_n Y_{K_n} = \bigcap_n (X \setminus Y_{K_n})$, which is G_δ. $\square$

A *Borel equivalence relation* on a standard Borel space X is a Borel subset $E \subseteq X \times X$ that is reflexive, symmetric, and transitive: $(x, x) \in E$, $(x, y) \in E$ implies $(y, x) \in E$, and $(x, y), (y, z) \in E$ implies $(x, z) \in E$ for all $x, y, z \in X$. The notation xEy is equivalent to $(x, y) \in E$.

An action $a : G \curvearrowright X$ generates the *orbit equivalence relation* E_a on X, defined by xE_ay whenever $G \cdot x = G \cdot y$. When the action is clear from the context, we may write E_G instead of E_a. Orbit equivalence relations of Borel actions of locally compact Polish groups are always Borel [4, p. 109].

A map $\rho : E_G \to G$ is called a *cocycle* if it satisfies the *cocycle identity*:

$$\rho(x,y) = \rho(z,y)\rho(x,z) \quad \text{for all } x,y,z \in X \text{ such that } xE_G yE_G z.$$

For example, if the action is free, then for each pair of E_G-equivalent elements x, y, there corresponds a unique $g \in G$ such that $gx = y$; we denote this element by $\rho_G(x,y)$, or simply $\rho(x,y)$ when there is no danger of confusion.

A Borel equivalence relation E is *concretely classifiable*[4] if there exists a standard Borel space Y and a Borel map $f : X \to Y$ such that xEy if and only if $f(x) = f(y)$, for all $x, y \in X$. A Borel *transversal* for E is a Borel set $T \subseteq X$ that intersects each E-class in exactly one point.

It is worth mentioning that an orbit equivalence relation, given by a Borel action of a Polish group, is concretely classifiable if and only if it admits a Borel transversal. One direction (*existence of a transversal implies concrete classifiability*) is straightforward. The other direction is due to Burgess, see [32, Exercise 18.20, cf. p. 360], though we will not use this hereafter.

We will need several times the standard fact that any Borel action $G \curvearrowright X$ of a compact Polish group G on a standard Borel space X has a Borel transversal. For continuous actions on compact spaces this can be seen by noting that the map $X \ni x \mapsto G \cdot x \in \mathcal{K}(X)$ is continuous with respect to the Vietoris topology on the space of compact subsets of X (Section 3.2). If $s : \mathcal{K}(X) \to X$ is a Borel selector given by the Kuratowski–Ryll-Nardzewski theorem (see Section 3.2), then $\{s(G \cdot x) : x \in X\}$ is a Borel transversal for the action. The general case follows by embedding an arbitrary Borel action $G \curvearrowright X$ into a continuous action on a compact set. Pick a Haar measure on G and consider the action $G \curvearrowright L^\infty(G,\mu)$ given by $(g \cdot f)(h) = f(g^{-1}h)$. This is a continuous action when $L^\infty(G,\mu)$ is endowed with the weak*-topology. Furthermore, the unit ball B of $L^\infty(G,\mu)$ is a G-invariant compact set (Alaoglu's theorem). Any Borel action $G \curvearrowright X$ can be embedded into $G \curvearrowright B$. Indeed, if the standard Borel space X is identified with $[0,1]$, then an embedding is given by $G \ni x \mapsto f_x \in B$, $f_x(g) = g^{-1} \cdot x$. Further details, as well as an alternative argument, can be found in [4, pp. 27–28].

Let E_1 and E_2 be Borel equivalence relations on spaces X_1 and X_2, respectively. A Borel *reduction* from E_1 to E_2 is a Borel map $f : X_1 \to X_2$ such that xE_1y if

[4] The terms *smooth* and *tame* are also commonly used in the literature.

and only if $f(x)E_2f(y)$, for all $x, y \in X_1$. An equivalence relation is concretely classifiable if and only if it is Borel reducible to the equality relation on some standard Borel space. If E_1 is Borel reducible to E_2 and E_2 is concretely classifiable, then so is E_1.

Throughout this paper, G denotes a locally compact Polish group, and unless stated otherwise, $G \curvearrowright X$ is a free Borel action on a standard Borel space. (Actions of G on spaces other than X are generally not assumed to be free.) An *exhaustion* of a locally compact Polish group G by compact sets is a sequence of compact sets $(K_n)_n$ such that $K_n \subseteq \operatorname{int} K_{n+1}$, $n \in \mathbb{N}$, and $G = \bigcup_n K_n$. Here $\operatorname{int} K$ denotes the interior of K.

3.2 Effros and Vietoris spaces

Let X be a locally compact Polish space. The space $\mathcal{K}(X)$ of compact subsets of X is equipped with the *Vietoris topology* [32, 4.F], [51, p. 66], whose basis is parametrized by open sets $U_0, U_1, \ldots, U_n \subseteq X$ and is given by

$$\{K \in \mathcal{K}(X) : K \subseteq U_0, K \cap U_1 \neq \varnothing, \ldots, K \cap U_n \neq \varnothing\}.$$

The space $\mathcal{K}(X)$ is Polish [32, 4.25], [51, Cor. 2.4.16].

The *Effros Borel space* $\mathcal{F}(X)$ consists of all closed subsets of X and is endowed with the σ-algebra generated by the sets $\{F \in \mathcal{F}(X) : F \cap U \neq \varnothing\}$, where $U \subseteq X$ is open [32, 12.C], [51, p. 97]. This is a standard Borel space [32, 12.6], [51, Thm. 3.3.10], and the Borel structure of $\mathcal{K}(X)$ coincides with the one induced from $\mathcal{F}(X)$ [32, 12.11i].

An important fact about Effros Borel spaces is the Kuratowski–Ryll-Nardzewski theorem [32, 12.13], [51, Thm. 5.2.1], which guarantees the existence of Borel selectors $s_n : \mathcal{F}(X) \to X$, $n \in \mathbb{N}$, i.e., Borel functions such that $\{s_n(F)\}_n$ is dense in F whenever $F \in \mathcal{F}(X)$ is non-empty.

3.3 Borel toasts

A key tool in many proofs within Borel dynamics is the concept termed "Borel toast" by Gao–Jackson–Krohne–Seward in [19]. It is the descriptive set-theoretic counterpart of Rokhlin towers in ergodic theory. Specific properties that one chooses to require in the definition of a toast vary depending on the application. In it's simplest form, this concept is closely related to the notion of hyperfinitness.

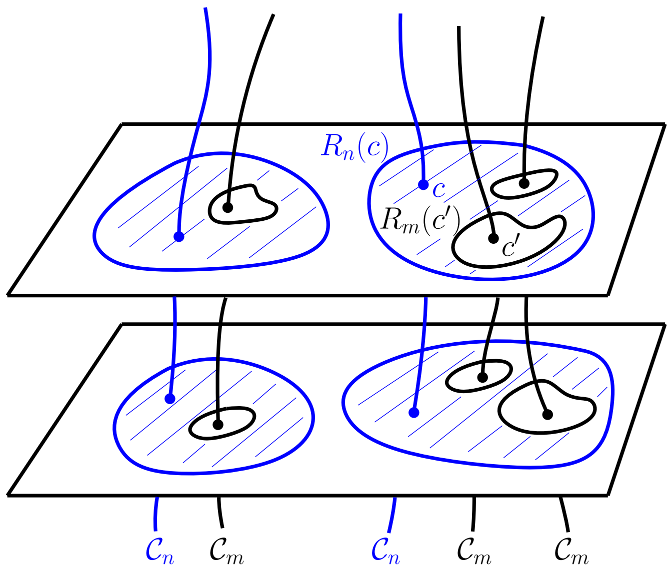

Fig. 1: Illustration of a Borel toast, showing two cross-sections $\mathcal{C}_m$ and $\mathcal{C}_n$, $m < n$. Two points $c \in \mathcal{C}_n$ and $c' \in \mathcal{C}_m$ with regions $R_m(c')$ and $R_n(c)$ around them are highlighted. The two planes correspond to distinct G-orbits.

Borel actions of $\mathbb{Z}^d$ were shown to be hyperfinite by Weiss (unpublished) (see Jackson–Kechris–Louveau [28, Thm. 1.16] for a more general result for groups of polynomial growth). In a more refined setting, the definition of a toast puts restrictions on the shape of the regions or on the mutual location of regions across different levels. A powerful way of constructing toasts is based on the cross-section construction of Boykin–Jackson [6] (see also Marks–Unger [39, Appendix A]). Other applications of the toast idea can be found in Gao–Jackson–Krohne–Seward [17–19] and Slutsky [49].

The following definition is tailored to the needs of Lemma 4.5, and is essentially from [49] (see, however, Remark 3.3). We let $\mathcal{K}_*(G)$ denote the collection of compact subsets of G with non-empty interior.

Definition 3.2. Let $a_X : G \curvearrowright X$ be a free Borel action of a locally compact Polish group on a standard Borel space. A Borel *toast* for a_X is a sequence $(\mathcal{C}_n, \lambda_n)_n$ of Borel sets $\mathcal{C}_n \subseteq X$ and Borel functions $\lambda : \mathcal{C}_n \to \mathcal{K}_*(G)$ satisfying the following conditions. For each n, define $R_n(c) = \lambda_n(c) \cdot c$ for $c \in \mathcal{C}_n$, and let $X_n = \bigcup_{c \in \mathcal{C}_n} R_n(c)$. Then:

1. $R_n(c_n) \cap R_n(c'_n) = \varnothing$ for all distinct $c_n, c'_n \in \mathcal{C}_n$.

2. For all $m < n$, $c_m \in \mathcal{C}_m$, and $c_n \in \mathcal{C}_n$, either $R_m(c_m) \cap R_n(c_n) = \varnothing$ or $R_m(c_m) \subseteq R_n(c_n)$.

3. For all $c_n \in \mathcal{C}_n$ there exists $c_{n+1} \in \mathcal{C}_{n+1}$ such that $R_n(c_n) \subseteq R_{n+1}(c_{n+1})$.

4. For all $c_{m_1} \in \mathcal{C}_{m_1}$ and $c_{m_2} \in \mathcal{C}_{m_2}$, there exist $n > m_1, m_2$ and an element $c_n \in \mathcal{C}_n$ such that $R_{m_i}(c_{m_i}) \subseteq \operatorname{int} R_n(c_n)$ for $i = 1, 2$, where $\operatorname{int} R_n(c_n) = (\operatorname{int} \lambda_n(c_n))c_n$.

5. There exists a neighborhood of the identity $U \subseteq G$ such that $U \subseteq \lambda_n(c_n)$ for all $c_n \in \mathcal{C}_n$ and all n.

6. $\bigcup_n X_n = X$.

7. The range $\operatorname{ran} \lambda_n$ is countable for each n and

$$\{\rho(c_m, c_n) : c_m \in \mathcal{C}_m, c_n \in \mathcal{C}_n, c_m E_{a_X} c_n\}$$

   is countable, where $\rho : E_{a_X} \to G$ is the cocycle defined by the condition $\rho(x_1, x_2)x_1 = x_2$.

We say that $(\mathcal{C}_n, \lambda_n)_n$ is an $\mathfrak{R}$-toast, where $\mathfrak{R} \subseteq \mathcal{K}(G)$, if $\operatorname{ran} \lambda_n \subseteq \mathfrak{R}$ for all n.

Here are a couple of remarks and consequences of this definition.

Remark 3.3. Item (3) should not be confused with the so-called layeredness condition. In some descriptive set-theoretic applications, it is important to know that containment $R_m(c_m) \subseteq R_n(c_n)$, $m < n$, implies that the region $R_m(c_m)$ is far from the boundary of $R_n(c_n)$. In this case, one generally cannot guarantee that for each $c_n \in \mathcal{C}_n$ there is some $c_{n+1} \in \mathcal{C}_{n+1}$ satisfying $R_n(c_n) \subseteq R_{n+1}(c_{n+1})$. However, for the purposes of this chapter, we allow regions at different levels to coincide, which is why item (3) can be easily achieved.

Indeed, if $(\mathcal{C}'_n, \lambda'_n)_n$ satisfies all the items in the definition of the toast except possibly for item (3), then we can set $X'_n = \bigcup_{c_n \in \mathcal{C}'_n} R'_n(c_n)$ and

$$\begin{aligned} \mathcal{C}_n &= \mathcal{C}'_n \cup \bigcup_{k<n} \{c_k \in \mathcal{C}'_k : c_k \notin X'_m \text{ for all } k < m \leq n\}, \\ \lambda_n(c_n) &= \lambda'_k(c_k) \quad \text{for } k \text{ such that } c_n = c_k \in \mathcal{C}'_k. \end{aligned}$$

In plain words, we can repeat regions $R_n(c_n)$ to ensure that X_{n+1} contains X_n.

Remark 3.4. Definition 3.2 does not require sets $\mathcal{C}_n$ to be complete, i.e., $G \cdot \mathcal{C}_n$ may be a proper subset of X. If needed, completeness can be achieved by "re-indexing" the points to ensure that $\mathcal{C}_0$ intersects every orbit. Items (1), (5) and (6) will then ensure that each $\mathcal{C}_n$ is a *Borel cross-sections*—a Borel set that intersects each orbit in a lacunary set.

As Kechris showed, cross-sections exist for arbitrary Borel actions of locally compact groups [31]. On the other hand, the existence of a Borel toast is a considerably stronger condition.

In our applications, we work with the groups $G = \mathbb{R}^d$ and $G = \mathbb{R}^d \times \mathbb{T}^p$, and, except for the section studying the heat operator, it suffices to take the class $\mathfrak{R} \subseteq \mathcal{K}(G)$ to be either the class $\mathfrak{D}_d$ of compact sets diffeomorphic to the d-dimensional ball or the products of such sets with $\mathbb{T}^p$. Free Borel actions of $\mathbb{R}^d$ admit Borel $\mathfrak{D}_d$-toasts in the sense of Definition 3.2. This is essentially [49, Thm. 5]. The formulation therein does not include item (3), but as we explained in Remark 3.3, this property can be easily achieved. Borel actions of $\mathbb{R}^d \times \mathbb{T}^p$ admit Borel $\mathfrak{D}_d \times \mathbb{T}^p$-toasts, as will follow from Lemma 3.7.

Lemma 3.5. *Let $(\mathcal{C}_n, \lambda_n)_n$ be a Borel toast for $G \curvearrowright X$. For all $x \in X$ and $K \in \mathcal{K}(G)$, there exist n and $c_n \in \mathcal{C}_n$ such that $K \cdot x \subseteq \operatorname{int} R_n(c_n)$.*

Proof. Fix $x \in X$, $K \in \mathcal{K}(G)$, and $h \in K$. By item (6) of the toast definition, there exists $c_m \in \mathcal{C}_m$ and $g_h \in \lambda(c_m)$ such that $h \cdot x = g_h \cdot c_m$. Without loss of generality, by item (4), increasing m if necessary, we may assume $g_h \in \operatorname{int} \lambda(c_m)$. Thus, there exists an open neighborhood of the identity U_h such that $U_h g_h \subseteq \operatorname{int} \lambda(c_m)$, and consequently, $U_h h \cdot x \subseteq \operatorname{int} R_m(c_m)$.

The sets $(U_h h)_{h \in K}$ form an open cover of K. By compactness, we may pass to a finite sub-cover, yielding finitely many elements $h_i \in K$, open sets $U_i \subseteq G$, indices m_i, and elements $c_{m_i} \in \mathcal{C}_{m_i}$ such that $K \subseteq \bigcup_i U_i h_i$ and $U_i h_i \cdot x \subseteq \operatorname{int} R_{m_i}(c_{m_i})$. Iterating the property from item (4), we can find a single $c_n \in \mathcal{C}_n$ such that $R_{m_i}(c_{m_i}) \subseteq \operatorname{int} R_n(c_n)$ for all i. This c_n satisfies $\bigcup_i U_i h_i \cdot x \subseteq \operatorname{int} R_n(c_n)$, and thus $K \cdot x \subseteq \operatorname{int} R_n(c_n)$, as required. □

Remark 3.6. Item (5) of Definition 3.2 ensures that for any $c_n \in \mathcal{C}_n$ there are only finitely many $c_m \in \mathcal{C}_m$, $m < n$, satisfying $R_m(c_m) \subseteq R_n(c_n)$. This follows easily by identifying the orbit of c_n with G and using the Haar measure to limit the number of pairwise disjoint regions $R_m(c_m)$ inside $R_n(c_n)$.

Finally, we argue that if every free Borel action of $\mathbb{R}^d$ admits a Borel $\mathfrak{R}$-toast, then free actions of $\mathbb{R}^d \times \mathbb{T}^p$ admit Borel $\mathfrak{R} \times \mathbb{T}^p$-toasts. Let T be a compact Polish group and let, as before, G be a locally compact Polish group. Given a cofinal G-invariant class of compact sets $\mathfrak{R} \subseteq \mathcal{K}(G)$, let $\mathfrak{R} \times T$ denote the class $\{L \times T : L \in \mathfrak{R}\}$. Note that the cofinality of $\mathfrak{R}$ in $\mathcal{K}(G)$ implies the cofinality of $\mathfrak{R} \times T$ in $\mathcal{K}(G \times T)$.

Lemma 3.7. *If every (free) Borel G-action admits a Borel $\mathfrak{R}$-toast, then every (free) Borel $G \times T$-action admits a Borel $(\mathfrak{R} \times T)$-toast.*

Proof. Let $G \times T \curvearrowright X$ be a Borel action and consider its restriction to the action $T \curvearrowright X$. Since T is compact, there exists a Borel transversal $Y \subseteq X$ for E_T. Define a Borel action $a : G \curvearrowright Y$ by setting $a(g, y)$ to be the unique $y' \in Y$ such that gyE_Ty'. Equivalently, if $s : X \to Y$ is the Borel selector for E_T corresponding to the chosen transversal, i.e., $s(x) = y$ for the unique $y \in Y$ such that xE_Ty, then $a(g, y) = s(gy)$. Note that a is free if the original action $G \times T \curvearrowright X$ is free, since $a(g, y) = y$ implies $gy = ty$ for some $t \in T$, which necessitates $g = e$.

By assumption, a admits a Borel $\mathfrak{R}$-toast $(\mathcal{C}_n, \lambda_n)_n$. We transform it into a Borel $(\mathfrak{R} \times T)$-toast $(\mathcal{C}_n, \mu_n)_n$ by setting $\mu_n(c_n) = \lambda_n(c_n) \times T$. The toast axioms are straightforward to verify. $\square$

4 Equivariant Borel liftings

The primary goal of this section is to establish Theorem 4.9. Let G be a locally compact Polish group acting in a Borel way on standard Borel spaces Z and Y, and let $\pi : Z \to Y$ be a Borel G-equivariant surjection. The theorem provides sufficient conditions under which, for a free Borel G-action $G \curvearrowright X$ and a G-equivariant map $\varphi : X \to Y$, there exists a Borel G-equivariant lifting $\psi : X \to Z$ satisfying $\pi \circ \psi = \varphi$:

$$\begin{tikzcd} G \curvearrowright X \arrow[r, dashed, "\psi"] \arrow[dr, "\varphi"'] & G \curvearrowright Z \arrow[d, "\pi"] \\ & G \curvearrowright Y \end{tikzcd} \tag{4.1}$$

We begin in Section 4.1 with a preliminary result (Lemma 4.5), which gives sufficient conditions under which a Borel set $P \subseteq X \times Z$ admits an *equivariant Borel uniformization*, which we will define a few lines below. This result requires a measurable version of the Runge property (Definition 4.4) in the space Z.

As outlined in the introduction, we wish to refrain from establishing such measurable Runge theorems in our applications. For that reason, we develop a framework to transfer the application of the Runge property from Z to the Polish group H, where an a priori weaker Runge-type property suffices (Definition 4.8). The reader may wish to keep in mind the relation between H and Z in the motivating examples from the introduction: in the case of entire functions we take Z to be the space $\mathcal{E}_{\neq 0}$ of entire function which do not vanish identically, while H is the multiplicative group $\mathcal{E}^{\times}$ of zero-free entire functions. For our application to the Poisson equation, Z is the space of subharmonic functions on $\mathbb{R}^d$, while H is the additive group of harmonic functions. The framework to accomplish the transfer between H and Z is laid out in Section 4.2. The key result is Lemma 4.7, which shows that a certain family of seminorms on H induces a family of pseudometrics on Z with desirable properties.

In Section 4.3, we state and prove our main result. This will be derived rather directly from the equivariant uniformization of Lemma 4.5. The key technical step is to show that the $\mathfrak{R}$-Runge property in H implies the a priori stronger measurable Runge property in the space Z.

4.1 Equivariant uniformizations

Consider standard Borel spaces X and Z equipped with Borel G-actions $a_X : G \curvearrowright X$ and $a_Z : G \curvearrowright Z$ of a locally compact Polish group G. Let $P \subseteq X \times Z$ be a Borel subset. We say that P is (a_X, a_Z)-*equivariant* if for all $(x, z) \in P$ and $g \in G$, the pair (gx, gz) also lies in P. When the actions a_X and a_Z are clear from the context, we may simply say that P is *equivariant*. A *uniformization* of P is a function $f : \mathrm{proj}_X(P) \to Z$ such that $(x, f(x)) \in P$ for all $x \in \mathrm{proj}_X(P)$. A uniformization f is said to be (a_X, a_Z)-*equivariant* if its graph is equivariant, or equivalently, if $f(g \cdot x) = g \cdot f(x)$ for all $x \in \mathrm{proj}_X(P)$ and $g \in G$.

Conditions ensuring the existence of Borel uniformizations for a Borel set $P \subseteq X \times Z$ are well established (see [32, Sec. 18]). Our goal is to provide an instance of conditions under which a set $P \subseteq X \times Z$ admits an (a_X, a_Z)-equivariant Borel uniformization (Lemma 4.5 and Theorem 4.9). Prior work in this area includes the recent work of Kechris and Wolman [34], which, when restricted to the framework of orbit equivalence relations, examines the existence of equivariant uniformizations with the trivial action a_Z.

Let us look at an example before we proceed:

Example 4.1. Let $P = \{(x, z) \in X \times Z : \varphi(x) = \pi(z)\}$, where X, Z, φ and π are as in Diagram 4.1. By the equivariance of π and φ, P is readily seen to be an equivariant Borel subset of $X \times Z$. The set P consists of all candidate pairs (x, z) for which we could have $\psi(x) = z$ for the lifting $\psi : X \to Z$ in Diagram 4.1. An equivariant Borel uniformization of P is then nothing but the desired equivariant lifting.

Throughout the rest of this section we assume that the action $a_X : G \curvearrowright X$ is free. (Actions of G on other spaces are generally not assumed to be free.)

We recall that a *pseudometric* on a set Z is a map $d : Z \times Z \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ that is symmetric, satisfies the triangle inequality, and is zero on the diagonal, $d(z, z) = 0$ for all $z \in Z$. Unlike a genuine metric, a pseudometric may assign distance 0 to distinct points. We also remind that $\mathcal{K}(G)$ denotes the space of compact subsets of G.

Definition 4.2. Let $a_Z : G \curvearrowright Z$ be a Borel G-action. A family $\mathsf{D} = (d_K)_{K \in \mathcal{K}(G)}$ of pseudometrics on Z is said to be an *a_Z-family* if it satisfies the following conditions:

1. $d_K(z_1, z_2) \leq d_{K'}(z_1, z_2)$ for all $z_1, z_2 \in Z$ and $K, K' \in \mathcal{K}(G)$ satisfying $K \subseteq K'$.

2. $d_K(gz_1, gz_2) = d_{Kg}(z_1, z_2)$ for all $z_1, z_2 \in Z$, $g \in G$, and $K \in \mathcal{K}(G)$.

3. The family D separates points: for any distinct $z_1, z_2 \in Z$ there exists $K \in \mathcal{K}(G)$ such that $d_K(z_1, z_2) > 0$.

4. The uniformity generated by D is complete: if $(z_n)_n$ is D-Cauchy (i.e., d_K-Cauchy for every $K \in \mathcal{K}(G)$), then there exists some $z \in Z$ such that $d_K(z_n, z) \to 0$ for all $K \in \mathcal{K}(G)$.

An a_Z-family is *Borel* if each d_K is Borel as a map $d_K : Z \times Z \to \mathbb{R}^{\geq 0} \cup \{\infty\}$.

In applications, Z is often a space of functions on the acting group G, and the relevant families of pseudometrics will oftentimes be of the form $d_K(f, g) = \sup_{x \in K} |f(x) - g(x)|$.

Definition 4.3. Let $\mathfrak{R} \subseteq \mathcal{K}(G)$ be a class of compact subsets of G. We say that $\mathfrak{R}$ is

- *cofinal* if for all $K \in \mathcal{K}(G)$ there exists $K' \in \mathfrak{R}$ such that $K \subseteq K'$;
- *invariant*[5] if $Kg \in \mathfrak{R}$ whenever $K \in \mathfrak{R}$ and $g \in G$.

Given a set $P \subseteq X \times Z$ and $m \in \mathbb{N}$, we use the notation $P^{(m)}$ for the set

$$P^{(m)} = \{(x, z_1, \ldots, z_m) : (x, z_i) \in P \text{ for } 1 \leq i \leq m\} \subseteq X \times Z^m.$$

In particular, $P^{(1)} = P$.

Given a set $P \subseteq X \times Z$ and $x \in X$, we let P_x stand for $\{z \in Z : (x, z) \in P\}$.

Definition 4.4. Let $\mathfrak{R} \subseteq \mathcal{K}(G)$ be a cofinal invariant class of compact sets and let $P \subseteq X \times Z$ be an (a_X, a_Z)-equivariant Borel set. An a_Z-family $\mathsf{D} = (d_K)_{K \in \mathcal{K}(G)}$ of pseudometrics on Z is said to satisfy the $(\mathfrak{R}, P)$-*Runge property* if for all pairwise disjoint $K_1, \ldots, K_m \in \mathfrak{R}$ and all $\varepsilon > 0$ there exists a Borel map $f : P^{(m)} \to Z$ such that $d_{K_i}(z_i, f(x, z_1, \ldots, z_m)) < \varepsilon$ for all $1 \leq i \leq m$ and all $(x, z_1, \ldots, z_m) \in P^{(m)}$.

On its face, the $(\mathfrak{R}, P)$-Runge property of Definition 4.4 is a stronger condition than the $\mathfrak{R}$-Runge property which appears in the main theorem. Indeed, the former requires measurability of the "Runge map" $f : P^{(m)} \to Z$ while the latter merely requires the existence of approximating elements. However, in the proof of Theorem 4.9 we show that the $\mathfrak{R}$-Runge property (in the Polish group H) which appears in Theorem 4.9 implies the $(\mathfrak{R}, P)$-Runge property for a particular choice of P.

The following lemma provides the key technical statement that ensures the existence of an equivariant uniformization. For its formulation, we fix Borel actions $a_X : G \curvearrowright X$ and $a_Z : G \curvearrowright Z$ of a locally compact Polish group G. Suppose that a_X is free. Let $P \subseteq X \times Z$ be a Borel (a_X, a_Z)-equivariant set that admits a Borel uniformization. Let $\mathfrak{R} \subseteq \mathcal{K}(G)$ be a cofinal invariant class and $\mathsf{D} = (d_K)_{K \in \mathcal{K}(G)}$ be an a_Z-family of pseudometrics on Z. Recall the notion of a Borel $\mathfrak{R}$-toast from Definition 3.2 from the previous section.

Lemma 4.5. *Suppose that* D *satisfies the* $(\mathfrak{R}, P)$*-Runge property and assume that* a_X *admits a Borel* $\mathfrak{R}$*-toast. If each fiber* P_x*,* $x \in \operatorname{proj}_X(P)$*, is* D*-closed, then* P *admits an* (a_X, a_Z)*-equivariant Borel uniformization.*

[5] More precisely, such a class is *right-invariant*, but left-invariance will not be used in this section.

Proof. By the assumption, the set P admits a Borel uniformization, which implies that $\mathrm{proj}_X(P)$ is a Borel subset of X. Additionally, $\mathrm{proj}_X(P)$ is a_X-invariant. Without loss of generality, we may substitute $\mathrm{proj}_X(P)$ for X, thereby assuming that $\mathrm{proj}_X(P) = X$.

Recall that a_X is assumed to be free, which allows us to define the cocycle map $\rho : E_G^X \to G$ by the condition $\rho(x, y)x = y$ for $xE_G^X y$. Let $(\mathcal{C}_n, \lambda_n)_n$ be a Borel $\mathfrak{R}$-toast for a_X.

We inductively construct a sequence of Borel maps $\xi_n : \mathcal{C}_n \to Z$ such that $(c_n, \xi_n(c_n)) \in P$ and the following condition holds:

$$d_{\lambda_{n-1}(c_{n-1})\rho(c_n,c_{n-1})}(\xi_n(c_n), \rho(c_{n-1}, c_n) \cdot \xi_{n-1}(c_{n-1})) < 2^{-n} \tag{4.2}$$

for all $c_{n-1} \in \mathcal{C}_{n-1}$ and $c_n \in \mathcal{C}_n$ such that c_{n-1} is a predecessor of c_n, i.e., $R_{n-1}(c_{n-1}) \subseteq R_n(c_n)$.

For the base case, define $\xi_0 : \mathcal{C}_0 \to Z$ to be the restriction of any Borel uniformization of P to $\mathcal{C}_0$. Now, assume that ξ_{n-1} has already been constructed. Choose an element $c_n \in \mathcal{C}_n$, and let $c_{n-1}^1, \ldots, c_{n-1}^m \in \mathcal{C}_{n-1}$ denote all the elements of $\mathcal{C}_{n-1}$ satisfying $R_{n-1}(c_{n-1}^i) \subseteq R_n(c_n)$. Define compact sets $K_i = \lambda_{n-1}(c_{n-1}^i)\rho(c_n, c_{n-1}^i)$ for $1 \le i \le m$. By the invariance of $\mathfrak{R}$, each K_i lies in $\mathfrak{R}$, and the definition of the toast ensures that the sets K_i, $1 \le i \le m$, are pairwise disjoint.

Let $f : P^{(m)} \to Z$ be the map provided by the $(\mathfrak{R}, P)$-Runge property for the sets $K_1, \ldots, K_m$ and $\varepsilon = 2^{-n}$. Observe that, for each $1 \le i \le m$, $(c_{n-1}^i, \xi_{n-1}(c_{n-1}^i)) \in P$ implies $(c_n, \rho(c_{n-1}^i, c_n) \cdot \xi_{n-1}(c_{n-1}^i)) \in P$. We can thus define

$$\xi_n(c_n) = f(c_n, \rho(c_{n-1}^1, c_n) \cdot \xi_{n-1}(c_{n-1}^1), \ldots, \rho(c_{n-1}^m, c_n) \cdot \xi_{n-1}(c_{n-1}^m)).$$

The defining property of f ensures that

$$d_{K_i}(\xi_n(c_n), \rho(c_{n-1}^i, c_n) \cdot \xi_{n-1}(c_{n-1}^i)) < 2^{-n}$$

for all $1 \le i \le m$. Thus, ξ_n satisfies Eq. (4.2). Note that item (7) of the definition of a Borel toast ensures that there are only countably many distinct possibilities for the sets K_i, which allows us to perform the construction above in a Borel way over all $\mathcal{C}_n$.

Let $X_n = \bigcup_{c_n \in \mathcal{C}_n} R_n(c_n)$, and define functions $\beta_n : X_n \to \mathcal{C}_n$ by the condition $x \in R_n(\beta_n(x))$ for all $x \in X_n$. Next, define functions $\psi_n : X_n \to Z$ as

$$\psi_n(x) = \rho(\beta_n(x), x) \cdot \xi_n(\beta_n(x)).$$

Since $(c_n, \xi_n(c_n)) \in P$ for all $c_n \in \mathcal{C}_n$ and using the (a_X, a_Z)-equivariance of P, it follows that

$$P \ni \Big(\rho(\beta_n(x), x) \cdot \beta_n(x), \rho(\beta_n(x), x) \cdot \xi_n(\beta_n(x))\Big) = (x, \psi_n(x)).$$

Thus, each ψ_n serves as a Borel uniformization of $P \cap (X_n \times Z)$.

We claim that the sequence $(\psi_n(x))_n$ is D-Cauchy for every $x \in X$. To verify this, it is enough to show that for each $K \in \mathcal{K}(G)$, every $x \in X$, and all sufficiently large n, the inequality $d_K(\psi_n(x), \psi_{n-1}(x)) < 2^{-n}$ holds. Fix $K \in \mathcal{K}(G)$ and define $c_{n-1} = \beta_{n-1}(x)$ and $c_n = \beta_n(x)$. For sufficiently large n, Lemma 3.5 gives

$$K \cdot x \subseteq \lambda_{n-1}(c_{n-1}) \cdot c_{n-1} \subseteq \lambda_n(c_n) \cdot c_n.$$

Since the action a_X is free, it follows that

$$K\rho(c_n, x) = K\rho(c_{n-1}, x)\rho(c_n, c_{n-1}) \subseteq \lambda_{n-1}(c_{n-1})\rho(c_n, c_{n-1}). \tag{4.3}$$

We now have the following equalities

$$\begin{aligned} d_K(\psi_n(x), \psi_{n-1}(x)) &= d_K(\rho(c_n, x) \cdot \xi_n(c_n), \rho(c_{n-1}, x) \cdot \xi_{n-1}(c_{n-1})) \\ &= d_{K\rho(c_n,x)}(\xi_n(c_n), \rho(x, c_n)\rho(c_{n-1}, x) \cdot \xi_{n-1}(c_{n-1})) \\ &= d_{K\rho(c_n,x)}(\xi_n(c_n), \rho(c_{n-1}, c_n) \cdot \xi_{n-1}(c_{n-1})) \end{aligned}$$

which yields the estimate

$$\begin{aligned} d_K(\psi_n(x), \psi_{n-1}(x)) &= d_{K\rho(c_n,x)}(\xi_n(c_n), \rho(c_{n-1}, c_n) \cdot \xi_{n-1}(c_{n-1})) \\ &\leq d_{\lambda_{n-1}(c_{n-1})\rho(c_n,c_{n-1})}(\xi_n(c_n), \rho(c_{n-1}, c_n) \cdot \xi_{n-1}(c_{n-1})) \\ &< 2^{-n}, \end{aligned} \tag{4.4}$$

where the first inequality is due to Eq. (4.3) and the second follows from Eq. (4.2).

We have established that $(\psi_n(x))_n$ is D-Cauchy for every $x \in X$. By item (4) of Definition 4.2, the family of pseudometrics D is complete. Consequently, for each $x \in X$, the limit $\psi(x) = \mathsf{D}\text{-}\lim_n \psi_n(x)$ exists and is unique, as D separates points. Furthermore, ψ is a Borel map because D is Borel. To verify this, take an exhaustion $(K_n)_n$ of G by compact sets. The graph of ψ can be written as

$$\Big\{(x, y) : \forall k\ \forall m\ \exists N\ (\forall n \geq N)\ [d_{K_m}(\psi_n(x), y) < 1/k]\Big\},$$

which is Borel since the pseudometrics d_{K_m} and the functions ψ_n are Borel.

We claim that ψ is (a_X, a_Z)-equivariant. Observe that $\rho(y, g \cdot x) = g\rho(y, x)$ for all $x, y \in X$ and $g \in G$. Moreover, for all sufficiently large n, we have $\beta_n(g \cdot x) = \beta_n(x)$ and consequently

$$\psi_n(g \cdot x) = \rho(\beta_n(g \cdot x), g \cdot x) \cdot \xi_n(\beta_n(g \cdot x)) = g\rho(\beta_n(x), x) \cdot \xi_n(\beta_n(x)) = g \cdot \psi_n(x).$$

Taking the limit as $n \to \infty$, we obtain

$$\psi(g \cdot x) = \mathsf{D}\text{-}\lim_n \psi_n(g \cdot x) = \mathsf{D}\text{-}\lim_n g \cdot \psi_n(x) = g \cdot (\mathsf{D}\text{-}\lim_n \psi_n(x)) = g \cdot \psi(x),$$

where the penultimate equality follows from the continuity of the map $z \mapsto g \cdot z$ in the topology of D, given by item (2) of Definition 4.2. Thus, ψ is G-equivariant.

Recall that each ψ_n is a Borel uniformization of $P \cap (X_n \times Z)$. Since slices P_x are assumed to be D-closed, we conclude that $(x, \psi(x)) \in P$ holds for all $x \in X$ and ψ is therefore a Borel (a_X, a_Z)-equivariant uniformization of P. $\square$

4.2 Orbital action families

We now examine a specific class of examples of a_Z-families. Consider an action $H \curvearrowright Z$ of some group H. When the action is free, each orbit in Z can be identified with an affine copy of H. More precisely, for any $x_0 \in Z$, the map $H \ni h \mapsto hx_0 \in Z$ establishes a bijection between H and the H-orbit of x_0.

Given a pseudometric d on H, we can induce a pseudometric on the orbit of x_0 by defining $d_{x_0}(z_1, z_2) = d(h_1, h_2)$, where h_i is the unique element in H such that $h_i x_0 = z_i$. If we select a different point x_1 in the same orbit, the resulting pseudometric d_{x_1} on the orbit may differ from d_{x_0}. Specifically, if $fx_1 = x_0$ for some $f \in H$, then $d_{x_1} = d_{x_0}$ holds if and only if $d(h_1 f, h_2 f) = d(h_1, h_2)$ for all $h_1, h_2 \in H$. Thus, the pseudometric d_{x_0} is independent of the choice of the orbit representative x_0 precisely when d is right-invariant.

Even when the H-action is not free, a right-invariant pseudometric d on H induces a pseudometric on each orbit of the action. This is defined by the formula

$$d(z_0, z_1) = \inf\{||h|| : hz_0 = z_1\},$$

where $||h|| = d(h, e)$, and the infimum of an empty set is interpreted as $+\infty$. Notably, $d(z_0, z_1) < \infty$ holds if and only if $z_0 E_H^Z z_1$.

Recall that a seminorm on a group H is a function $||\cdot|| : H \to \mathbb{R}^{\geq 0}$ satisfying the following properties for all $h, h_1, h_2 \in H$:

- $||e|| = 0$,
- $||h|| = ||h^{-1}||$,
- $||h_1 h_2|| \leq ||h_1|| + ||h_2||$.

Now, suppose H is equipped with the structure of a standard Borel space (for example, H is a Polish group), and let $\tau : G \curvearrowright H$ be a Borel action. Consider a τ-family $\mathsf{D}^H = (d_K^H)_{K \in \mathcal{K}(G)}$ of pseudometrics on H. We say that D^H is *right-invariant* if each pseudometric d_K^H satisfies $d_K^H(h_1 f, h_2 f) = d_K^H(h_1, h_2)$ for all $h_1, h_2, f \in H$.

Note that right-invariant pseudometrics are in one-to-one correspondence with seminorms. Specifically, if d is a right-invariant pseudometric on H, then the function $||h|| = d(h, e)$ defines a seminorm. Conversely, d can be recovered from the seminorm via the formula $d(h_1, h_2) = ||h_1 h_2^{-1}||$. As a result, any right-invariant τ-family of pseudometrics on H (Definition 4.2) for the action $G \curvearrowright H$ can be uniquely determined by specifying a τ-family $\mathsf{N} = (|| \cdot ||_K)_{K \in \mathcal{K}(G)}$ of seminorms on H. The properties of pseudometrics given in Definition 4.2 translate into the following properties of the seminorms N.

Definition 4.6. A family of seminorms $\mathsf{N} = (|| \cdot ||_K)_{K \in \mathcal{K}(G)}$ on H is a τ-family if the following holds for all $h \in H$ and $g \in G$.

1. $||h||_K \leq ||h||_{K'}$ for all $K, K' \in \mathcal{K}(G)$ satisfying $K \subseteq K'$.
2. $||\tau^g(h)||_K = ||h||_{Kg}$ for all $K \in \mathcal{K}(G)$.
3. If $h \neq e$ then $||h||_K \neq 0$ for some $K \in \mathcal{K}(G)$.
4. The right uniformity generated by N is complete: if $(h_n)_n$ is N-Cauchy in the sense that $||h_n h_m^{-1}||_K \to 0$ as $m, n \to \infty$, then there exists some $h_\infty \in H$ such that $||h_\infty h_n^{-1}||_K \to 0$ for all $K \in \mathcal{K}(G)$.

Let H be a Polish group, and let $\tau : G \curvearrowright H$ be a continuous action by automorphisms. The semidirect product $H \rtimes_\tau G$ is defined by the multiplication rule

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \tau^{g_1}(h_2), g_1 g_2), \tag{4.5}$$

and it is Polish in the product topology. Consider a Borel action $H \rtimes_\tau G \curvearrowright Z$ on a standard Borel space Z, and let $a_Z : G \curvearrowright Z$ and $H \curvearrowright Z$ denote the corresponding restrictions of this action.

Suppose we are given a right-invariant τ-family, defined by seminorms $\mathsf{N} = (||\cdot||_K)_{K\in\mathcal{K}(G)}$ on H. Assume further that N generates the Polish topology on H. Under these assumptions, we can construct an a_Z-family $\mathsf{D} = (d_K)_{K\in\mathcal{K}(G)}$ of pseudometrics on Z by defining

$$d_K(z_1, z_2) = \inf\Big\{||h||_K : h \in H \text{ satisfies } hz_1 = z_2\Big\},$$

where, as usual, the infimum over the empty set is taken to be $+\infty$.

Lemma 4.7. *Let N and D be as above. Then $\mathsf{D} = (d_K)_{K\in\mathcal{K}(G)}$ is an a_Z-family. Moreover, D is Borel if and only if the orbit equivalence relation E_H^Z is Borel.*

Proof. Item (1) of Definition 4.2 follows directly from the corresponding property of the seminorms. For (2), observe that the definition of the semidirect product Eq. (4.5) gives

$$hgz_1 = gz_2 \iff g\tau^{g^{-1}}(h)z_1 = gz_2 \iff \tau^{g^{-1}}(h)z_1 = z_2.$$

Consequently,

$$\begin{aligned} d_K(gz_1, gz_2) &= \inf\{||h||_K : hgz_1 = gz_2\} \\ &= \inf\{||h||_K : \tau^{g^{-1}}(h)z_1 = z_2\} \\ [h' = \tau^{g^{-1}}(h)] &= \inf\{||\tau^g(h')||_K : h'z_1 = z_2\} \\ &= \inf\{||h'||_{Kg} : h'z_1 = z_2\} = d_{Kg}(z_1, z_2). \end{aligned}$$

The family D separates points. To see this, let $(K_n)_n$ be an exhaustion of G, and suppose $d_{K_n}(z_1, z_2) = 0$ for all n. Then there exist $h_n \in H$ such that $h_n z_1 = z_2$ and $||h_n||_{K_n} < 2^{-n}$ for all n. The sequence $(h_n)_n$ converges to the identity element e of H because N generates the given Polish topology of H. Since $H_{z_1,z_2} = \{h : hz_0 = z_1\}$ is closed in H for any Borel action (by Miller's theorem [32, 9.17] or [51, Thm. 4.8.4]), it follows that $z_2 = (\lim_n h_n)z_1 = ez_1$, and thus $z_1 = z_2$.

To verify completeness (item (4) of Definition 4.2), suppose $(z_n)_n$ in Z is d_K-Cauchy for each $K \in \mathcal{K}(G)$. In particular, $z_m E_H^Z z_n$ for all sufficiently large m, n. By passing to a subsequence of $(z_n)_n$ if necessary, we may assume that $d_{K_m}(z_m, z_n) < 2^{-m}$ for all $n \geq m$. Let $h_m \in H$ satisfy $||h_m||_{K_m} < 2^{-m}$ and $h_m z_m = z_{m+1}$. The sequence $(h_n h_{n-1} \cdots h_m)_n$ is N-Cauchy. Indeed, for all $m \leq$

$k \leq n$,

$$||(h_n h_{n-1} \cdots h_m)(h_k h_{k-1} \cdots h_m)^{-1}||_K = ||h_n h_{n-1} \cdots h_{k+1}||_K \\ \leq \sum_{i=k+1}^{n} ||h_i||_K \leq \sum_{i=k+1}^{n} 2^{-i} < 2^{-k},$$

where the penultimate inequality holds for k sufficiently large so that $K \subseteq K_{k+1}$. Thus, for each m, the limit $f_m = \lim_n h_n \cdots h_m$ exists. Moreover,

$$||f_m||_K = \lim_n ||h_n \cdots h_m||_K \leq \sum_{i=m}^{n} ||h_i||_K \leq 2^{-m+1},$$

provided $K \subseteq K_m$. In particular, $||f_m||_K \to 0$ as $m \to \infty$ for all $K \in \mathcal{K}(G)$. Since H is a topological group in the topology of N,

$$f_m h_{m-1} \cdots h_0 = (\lim_n h_n \cdots h_m) h_{m-1} \cdots h_0 = \lim_n h_n \cdots h_0 = f_0,$$

and thus $f_0 z_0 = f_m h_{m-1} \cdots h_0 z_0 = f_m z_m$. The point $f_m z_m$ is independent of m; let z denote this common value. Finally, note that $d_K(z_n, z) \leq ||f_n||_K \to 0$ as $n \to \infty$, thus showing that z is the limit of $(z_n)_n$.

We have established that D is an a_Z-family. If D is Borel, then

$$E_H = \{(z_1, z_2) : \exists n \; [d_{K_n}(z_1, z_n) < \infty]\}$$

is also Borel. Conversely, if E_H is Borel, then by Becker–Kechris [4, Thm. 7.1.2], the map

$$E_H \ni (z_1, z_2) \mapsto H_{z_1,z_2} = \{h \in H : hz_1 = z_2\} \in \mathcal{F}(H)$$

is Borel with respect to the Effros Borel structure on $\mathcal{F}(H)$. By the Kuratowski–Ryll-Nardzewski selection theorem, there exist Borel functions $s_n : \mathcal{F}(H) \to H$ such that $\{s_n(F)\}_n$ is dense in F for every non-empty $F \in \mathcal{F}(H)$. For $z_1, z_2 \in E_H$, the pseudometric d_K can then be expressed as

$$d_K(z_1, z_2) = \alpha \iff \forall n \; ||s_n(H_{z_1,z_2})||_K \geq \alpha \text{ and} \\ \forall k \; \exists n \; [\, ||s_n(H_{z_1,z_2})||_K < \alpha + 1/k \,].$$

This demonstrates that d_K has a Borel graph and is therefore a Borel function [32, 14.12], [51, Thm. 4.5.2]. $\square$

4.3 Equivariant Borel liftings for semidirect product actions

We are now ready to state and prove our main theorem. First, let us recall the relevant notation and assumptions (cf. Section 2.1.1).

Let G be a locally compact Polish group acting in a Borel way on standard Borel spaces Z and Y, and let $\pi : Z \to Y$ be a Borel G-equivariant surjection. Recall that our goal is to provide sufficient conditions under which, for a free Borel G-action $G \curvearrowright X$ and a G-equivariant map $\varphi : X \to Y$, there exists a Borel G-equivariant lifting $\psi : X \to Z$ satisfying $\pi \circ \psi = \varphi$.

Next, let's assume that we have a Polish group action $H \curvearrowright Z$ whose orbit equivalence relation E_H is classified by π, i.e., $\pi(z_1) = \pi(z_2)$ if and only if $z_1 E_H z_2$. We also have a *continuous* action $\tau : G \curvearrowright H$ by automorphisms, giving rise to a Polish semidirect product $H \rtimes_\tau G$ equipped with the product topology and group operations $(h_1, g_1)(h_2, g_2) = (h_1 \tau^{g_1}(h_2), g_1 g_2)$. The semidirect product is assumed to act on Z in a Borel way, inducing actions of the subgroups G and H on Z. We require that the induced actions coincide with the actions $G \curvearrowright Z$ and $H \curvearrowright Z$ already present. (In our applications, this is absolutely automatic.)

We denote by $\mathfrak{K} \subseteq \mathcal{K}(G)$ a cofinal G-invariant class of compact subsets of G and let $\mathsf{N} = (||\cdot||_K)_{K \in \mathcal{K}(G)}$ be a τ-family of seminorms on H (Definition 4.6). The following approximation property is a central requirement for our main theorem.

Definition 4.8. A τ-family N of seminorms on H is said to satisfy the $\mathfrak{K}$-Runge property if for any pairwise disjoint compact sets $K_1, \ldots, K_m \in \mathfrak{K}$, any $h_1, \ldots, h_m \in H$, and any $\varepsilon > 0$, there exists an element $h \in H$ such that $\|hh_i^{-1}\|_{K_i} < \varepsilon$ for $i = 1, \ldots, m$.

Recall also the definition of a Borel $\mathfrak{K}$-toast from Definition 3.2. We are now ready to state our main result.

Theorem 4.9. *Assume that the free action $G \curvearrowright X$ admits a Borel $\mathfrak{K}$-toast, and that the τ-family N on H satisfies the $\mathfrak{K}$-Runge property. Then, for any G-equivariant Borel map $\varphi : X \to Y$, there exists a G-equivariant Borel map $\psi : X \to Z$ such that $\pi \circ \psi = \varphi$, making the following diagram commute:*

$$\begin{tikzcd} G \curvearrowright X \arrow[r, dashed, "\psi"] \arrow[dr, "\varphi"'] & G \curvearrowright Z \arrow[d, "\pi"] \\ & G \curvearrowright Y \end{tikzcd}$$

Proof. Let $P \subseteq X \times Z$ be defined as $P = \{(x,z) : \varphi(x) = \pi(z)\}$. Since φ and π are equivariant, the set P is also equivariant. An equivariant uniformization of P will yield the desired map ψ. The existence of such uniformizations will be proved using Lemma 4.5.

The map π shows that the action $H \curvearrowright Z$ is concretely classifiable, i.e., its orbit equivalence relation is Borel reducible to the equality relation. A theorem of Burgess (see, for example, [32, 18.20iii]) guarantees the existence of a Borel inverse $\delta_Y : Y \to Z$ satisfying $\pi(\delta_Y(y)) = y$ for every $y \in Y$. Setting $\delta(x) = \delta_Y(\varphi(x))$, we see that P admits a Borel uniformization. Let D be the a_Z-family for $G \curvearrowright Z$ as given in Lemma 4.7.

It remains to verify that D satisfies the $(\mathfrak{R}, P)$-Runge property and that the slices P_x are D-closed. The latter is straightforward: if $(x, z_n) \in P$ and $z = \mathsf{D}\text{-}\lim_n z_n$, then $z_n E^Z_H z$ holds for all sufficiently large n. In particular, $\pi(z_n) = \pi(z)$ eventually, which implies $(x,z) \in P$ by the definition of P.

We now verify the $(\mathfrak{R}, P)$-Runge property. Let $s : \mathcal{F}(H) \to H$ be a Borel selector provided by the Kuratowski–Ryll-Nardzewski theorem. Define $\beta : P \to H$ by $\beta(x,z) = s(H_{\delta(x),z})$, where $H_{z_1,z_2} = \{h \in H : hz_1 = z_2\}$. Note that $H_{\delta(x),z}$ is non-empty since $\pi(\delta(x)) = \varphi(x) = \pi(z)$ and π classifies H-orbits. Recall the notation $P^{(m)} = \{(x, z_1, \dots, z_m) : \forall i\ (x, z_i) \in P\}$.

Since N satisfies the $\mathfrak{R}$-Runge property, for any pairwise disjoint compact sets $K_1, \dots, K_m \in \mathfrak{R}$, any $\varepsilon > 0$, and any $h_1, \dots, h_m \in H$, there exists $h \in H$ such that $||hh_i^{-1}||_{K_i} < \varepsilon$. Let $(\mathfrak{h}_n)_n$ be a dense sequence in H, and set

$$f(x, z_1, \dots, z_m) = \mathfrak{h}_n \cdot \delta(x) \text{ for the minimal } n \text{ such that } ||\beta(x,z_i)\mathfrak{h}_n^{-1}||_{K_i} < \varepsilon \text{ for all } 1 \le i \le m.$$

We finally claim that the index $n = n(x, z_1, \dots, z_m)$ is a Borel function of x and z_i, which guarantees Borelness of $f : P^{(m)} \to Z$ and thus shows the $(\mathfrak{R}, P)$-Runge property for D.

To see that $n : P^{(m)} \to H$ is Borel, note first that for each k and i, the function $(x, z_i) \mapsto \|\beta(x,z_i)\mathfrak{h}_k^{-1}\|_{K_i}$ is Borel. This is because all the involved ingredients, that is, the map $\beta : P \to H$, the seminorms $\|\cdot\|_{K_i} : H \to \mathbb{R}^{\geq 0}$ and the group operations in H, are Borel. We define for each $k \in \mathbb{N}$ the set

$$\begin{aligned} A_k &= \Big\{(x, z_1, \dots, z_m) \in P^{(m)} : \|\beta(x,z_i)\mathfrak{h}_k^{-1}\|_{K_i} < \varepsilon \text{ for } i = 1, \dots, m\Big\} \\ &= \bigcap_{i=1}^{m} \Big\{(x, z_1, \dots, z_m) \in P^{(m)} : \|\beta(x,z_i)\mathfrak{h}_k^{-1}\|_{K_i} < \varepsilon\Big\}, \end{aligned}$$

which, in view of the Borelness of $(x, z_i) \mapsto \|\beta(x, z_i)\mathfrak{h}_k^{-1}\|_{K_i}$, is a Borel subset of $P^{(m)}$. The minimal index function n is nothing but

$$n(x, z_1, \dots, z_m) = \inf\Big\{k \in \mathbb{N} : (x, z_1, \dots, z_m) \in A_k\Big\},$$

and hence for each $\ell \in \mathbb{N}$, the pre-image $n^{-1}(\{\ell\})$ can be written as

$$n^{-1}(\{\ell\}) = A_\ell \setminus \bigcup_{j=1}^{\ell-1} A_j.$$

Since we just established that all the sets A_k are Borel, so is the minimal index map n. This completes the proof. □

The following corollary is obtained by applying Theorem 4.9 to $X = \mathrm{Free}(Y)$ and taking φ to be the identity map $\mathrm{id} : \mathrm{Free}(Y) \hookrightarrow Y$.

Corollary 4.10. *Suppose that the free action $G \curvearrowright \mathrm{Free}(Y)$ admits a Borel $\mathfrak{R}$-toast, and that the τ-family N on H satisfies the $\mathfrak{R}$-Runge property. Then there exists a G-equivariant Borel map $\psi : \mathrm{Free}(Y) \to Z$ satisfying $(\pi \circ \psi)(y) = y$ for all $y \in \mathrm{Free}(Y)$.*

5 Applications of the main theorem

We now specialize Theorem 4.9 and present several applications to complex analysis and PDEs. Sections 5.3 and 5.5 give several applications to entire and meromorphic functions. But first, we introduce the relevant spaces and maps and establish their measurability properties.

5.1 Entire and meromorphic functions

Let $\mathcal{MR}$ denote the space of meromorphic functions on $\mathbb{C}$ (note that we do not consider the constant ∞ as a meromorphic function). This space carries a natural Polish topology $\tau_{\mathcal{MR}}$, defined as the topology of uniform convergence on compact sets with respect to, say, the spherical metric on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (see, for instance, [10, VII.3]). The spherical metric on $\overline{\mathbb{C}}$ is given by

$$\mathrm{dist}(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}} \quad \text{for } z_1, z_2 \in \mathbb{C} \quad \text{and}$$
$$\mathrm{dist}(z, \infty) = \frac{2}{\sqrt{1+|z|^2}} \quad \text{for } z \in \mathbb{C}.$$

Let $\mathcal{MR}^\times = \mathcal{MR} \setminus \{0\}$ denote the subspace of meromorphic functions which do not vanish identically. With pointwise operations, $\mathcal{MR}$ forms an associative division algebra over $\mathbb{C}$, and $\mathcal{MR}^\times$ is its multiplicative group of invertible elements.

As observed by Cima and Schober in [9], addition of meromorphic functions is discontinuous with respect to $\tau_{\mathcal{MR}}$. Furthermore, no comparable topology makes $\mathcal{MR}$ a locally convex topological vector space [9, Prop. 4]. Multiplication on $\mathcal{MR}$ is also discontinuous. To illustrate this, consider the functions $f_n(z) = z - 1/n$ and $h_n(z) = 1/(z + 1/n)$. Sequences $(f_n)_n$ and $(h_n)_n$ converge to the functions $z \mapsto z$ and $z \mapsto 1/z$, respectively. If multiplication were continuous, then $f_n \cdot h_n$ would converge to the constant function $z \mapsto 1$. However, any function in a sufficiently small neighborhood of $z \mapsto 1$ cannot have zeros or poles in, say, the closed disk $\overline{\mathbb{D}}$, whereas $f_n \cdot h_n$ has both a zero and a pole in $\overline{\mathbb{D}}$ for each n.

This phenomenon is not due to the choice of topology on $\mathcal{MR}$ but rather a consequence of its algebraic structure. In Appendix B, we use an automatic continuity result due to Dudley [14] to show in Corollary B.8 that $\mathcal{MR}^\times$ admits no Polish group topologies. Incidentally, in Theorem B.9, we prove that there is no metrizable topology on $\mathcal{MR}$ that makes it a complete topological algebra, answering a question raised by Grosse-Erdmann in [22]. These results, though tangential to our main focus, justify the generality of the setup in Section 4. Had we had the luxury of working with continuous homomorphisms between Polish groups, some arguments therein could have been significantly simpler.

Nonetheless, algebraic operations on $\mathcal{MR}$ are Borel.

Proposition 5.1. *$\mathcal{MR}$ is a standard Borel algebra in the sense that addition, multiplication, and scalar multiplication are Borel maps with respect to the Borel σ-algebra of $\tau_{\mathcal{MR}}$.*

Proof. Since $\tau_{\mathcal{MR}}$ is finer than the topology of pointwise convergence, the evaluation maps $f \mapsto f(w) \in \overline{\mathbb{C}}$ are $\tau_{\mathcal{MR}}$-continuous for all $w \in \mathbb{C}$.

Let $\mathbb{C}^* = \mathbb{C}^\times \sqcup \{*\}$ be the quotient of $\overline{\mathbb{C}}$, where 0 and ∞ are identified and denoted by $*$. Let $\pi : \overline{\mathbb{C}} \to \mathbb{C}^*$ be the corresponding projection, with $\pi(0) = * = \pi(\infty)$ and $\pi(z) = z$ for $z \in \mathbb{C}^\times$. We extend the algebraic operations from $\mathbb{C}^\times$ to all of $\mathbb{C}^*$ by setting $z \cdot * = * = * \cdot z$ and $z + * = * = * + z$ for all $z \in \mathbb{C}^*$ and $w \cdot * = *$ for all $w \in \mathbb{C}$. Choose a countable dense set $\{\omega_n\}_n$ of complex numbers. Restricted to functions in $\mathcal{MR}^\times$, the graphs of multiplication, addition, and scalar multiplication are given by the following expressions:

$$\begin{aligned}
&\{(f_1, f_2, g) : f_1 f_2 = g\} = \\
&\qquad \Big\{(f_1, f_2, g) : \forall n \, \Big[\text{ either } (\pi \circ f_1(\omega_n)) \cdot (\pi \circ f_2(\omega_n)) = \pi \circ g(\omega_n) \text{ or} \\
&\qquad\qquad \pi \circ f_1(\omega_n) = * \text{ or } \pi \circ f_2(\omega_n) = * \Big] \Big\}, \\
&\{(f_1, f_2, g) : f_1 + f_2 = g\} = \\
&\qquad \Big\{(f_1, f_2, g) : \forall n \, \Big[\text{ either } (\pi \circ f_1(\omega_n)) + (\pi \circ f_2(\omega_n)) = \pi \circ g(\omega_n) \text{ or} \\
&\qquad\qquad \pi \circ f_1(\omega_n) = * \text{ or } \pi \circ f_2(\omega_n) = * \Big] \Big\}, \\
&\{(w, f, g) : w f_2 = g\} = \{(w, f, g) : \forall n \, [w \cdot (\pi \circ f(\omega_n)) = \pi \circ g(\omega_n)]\}.
\end{aligned}$$

Since functions with Borel graphs are themselves Borel [32, 14.12], the proposition follows. □

The Borelness of the product implies the Borelness of the inverse map on $\mathcal{MR}^\times$, which when combined with Corollary B.8 leads to the following.

Corollary 5.2. *$\mathcal{MR}^\times$ is a non-Polishable standard Borel group.*

The subspace of entire functions on $\mathbb{C}$ is denoted by $\mathcal{E}$, and $\mathcal{E}^\times$ denotes the subspace of entire functions with no zeros. Note that $\mathcal{E}^\times$ consists precisely of the entire functions that admit entire inverses. The restriction of $\tau_{\mathcal{MR}}$ to $\mathcal{E}$ coincides with the topology $\tau_{\mathcal{E}}$ of uniform convergence on compact subsets of the complex plane. $\mathcal{E}$ is closed in $\mathcal{MR}$, and $\tau_{\mathcal{E}}$ is Polish. A basis of neighborhoods of an entire function f is parametrized by $K \in \mathcal{K}(\mathbb{C})$ and $\varepsilon > 0$, and is given by the sets

$$\{h \in \mathcal{E} : \sup_{z \in K} |f(z) - h(z)| < \varepsilon\}.$$

The subspace $\mathcal{E}^\times$ is G_δ in $\mathcal{E}$. Indeed, Hurwitz's theorem [10, VII.2.5] states that if a sequence of entire functions $(f_n)_n$ converges uniformly on compact sets to some (necessarily entire) f, then either f is identically zero, or each zero of f is a limit of zeros of f_n. Thus $\mathcal{E}^\times \cup \{0\}$ is closed in $\mathcal{E}$, hence $\mathcal{E}^\times$ is G_δ and therefore Polish in the induced topology [32, 3.11].

Unlike the whole space of meromorphic functions, addition and multiplication are continuous on $\mathcal{E}$. In fact, $\mathcal{E}$ is a separable Fréchet space, and $\mathcal{E}^\times$ is a multiplicative Polish group.

5.2 Spaces of divisors

A *(signed) divisor* on $\mathbb{C}$ is a map $d : \mathbb{C} \to \mathbb{Z}$ such that the support $d^{-1}(\mathbb{Z} \setminus \{0\})$ is a closed discrete subset of $\mathbb{C}$. The space of divisors can be defined in two equivalent ways. First, any divisor can be identified with an atomic Radon measure, thus viewed as an element of $\mathcal{M}(\mathbb{C})$ (see Section 5.9). The space of divisors $\mathcal{D}$ then corresponds to the following Borel subset of $\mathcal{M}(\mathbb{C})$:

$$\mathcal{D} = \{\mu \in \mathcal{M}(\mathbb{C}) : \forall n \; \exists m \in \mathbb{Z} \; [\mu(U_n) = m]\},$$

where $(U_n)_n$ is a countable basis of open bounded subsets of $\mathbb{C}$.

Alternatively, divisors can be viewed as discrete subsets of $\mathbb{C}$ labeled by non-zero integers (see Section 5.4 below). These two perspectives induce the same Borel structure on $\mathcal{D}$. The space of divisors inherits the structure of an abelian group from $\mathcal{M}(\mathbb{C})$. While it is a Borel group, it is not Polishable, as discussed in Corollary B.8.

A *positive divisor* on $\mathbb{C}$ is a map $d : \mathbb{C} \to \mathbb{N}$ such that $d^{-1}(\mathbb{N} \setminus \{0\})$ is a closed discrete subset of $\mathbb{C}$. Equivalently, a positive divisor is an element $d \in \mathcal{D} \cap \mathcal{M}^+(\mathbb{C})$. The space of positive divisors $\mathcal{D}^+$ is closed in $\mathcal{M}^+(\mathbb{C})$, hence Polish in the induced topology. A specific metric for this topology is discussed in [12, p. 403] (see also [40]).

Poles and zeros of a non-zero meromorphic function f naturally define a divisor $\mathrm{div}(f)$ given by

$$\mathrm{div}(f)(z) = \begin{cases} m & \text{if } z \text{ is a zero of order } m \text{ for } f, \\ -m & \text{if } z \text{ is a pole of order } m \text{ for } f, \\ 0 & \text{otherwise.} \end{cases}$$

The map $\mathrm{div} : \mathcal{MR}^\times \to \mathcal{D}$ is a homomorphism, $\mathrm{div}(f_1 f_2) = \mathrm{div}(f_1) + \mathrm{div}(f_2)$. Its surjectivity follows from the Weierstrass theorem on the existence of entire functions with prescribed zeros. The standard proof of this theorem yields a Borel function $\xi : \mathcal{D} \to \mathcal{MR}^\times$ such that $\mathrm{div}(\xi(d)) = d$ for all $d \in \mathcal{D}$. We call any such ξ a *Borel right-inverse* for div. Moreover, the argument can be adapted to produce a *continuous* right-inverse $\xi : \mathcal{D}^+ \to \mathcal{E}$ (see [44]).

The homomorphism $\mathrm{div} : \mathcal{MR}^\times \to \mathcal{D}$ is itself Borel. Indeed, its graph

$$\{(f, d) \in \mathcal{MR}^\times \times \mathcal{D} : \mathrm{div}(f) = d\}$$

is Borel, as it equals $\{(f,d) : f/\xi(d) \in \mathcal{E}^\times\}$, where ξ is any Borel right-inverse to div.

5.3 Weierstrass' theorem

We can now formulate and prove an equivariant version of the Weierstrass theorem. Recall that $\mathcal{D}^+$ denotes the space of positive divisors, and that $\mathcal{E}_{\neq 0}$ denotes the space of holomorphic functions which do not vanish identically. Both of these are Polish spaces on which the complex plane acts continuously by argument shifts and translation, respectively.

Theorem 5.3. *There exists a Borel $\mathbb{C}$-equivariant map $\psi : \mathrm{Free}(\mathcal{D}^+) \to \mathcal{E}_{\neq 0}$ such that $\operatorname{div} \circ \psi = \mathrm{id}_{\mathrm{Free}(\mathcal{D}^+)}$. Furthermore, there exists a Borel $\mathbb{C}$-equivariant map $\psi : \mathrm{Free}(\mathcal{D}) \to \mathcal{MR}^\times$ such that $\operatorname{div} \circ \psi = \mathrm{id}_{\mathrm{Free}(\mathcal{D})}$.*

Remark 5.4. In classical complex analysis, the Weierstrass theorem for meromorphic functions follows from the corresponding result for entire functions by taking quotients. A signed divisor d admits a canonical decomposition $d = d^+ - d^-$, $d^+, d^- \in \mathcal{D}^+$, which is characterized by d^+ and d^- having disjoint supports. One can therefore get the equivariant Weierstrass theorem for meromorphic functions on the subspace of those divisors $d \in \mathcal{D}$, for which both d^+, d^- are non-periodic, simply by applying the equivariant Weierstrass theorem for entire functions.

In general, a non-periodic $d \in \mathcal{D}$ may have periodic decomposition, as can be seen by taking d to be the difference of two periodic positive divisors with incommensurable periods. This can be circumvented by taking a decomposition of the form $d = (d^+ + d') - (d^- + d')$ for a suitably chosen positive divisor d', which can be selected in a Borel way for each d. Instead of this, we will construct the right-inverse $\psi : \mathrm{Free}(\mathcal{D}) \to \mathcal{MR}^\times$ via a direct application of Corollary 4.10, and obtain the theorem for entire functions by restricting to the positive divisors.

Proof. To obtain Theorem 5.3, we apply Corollary 4.10 in the following context. The group G is the additive group of $\mathbb{C}$, and the class $\mathfrak{R} = \mathfrak{D}_2$ consists of compact subsets of $\mathbb{C}$ which are diffeomorphic to the closed unit disk. Recall that every free Borel $\mathbb{C}$-action admits a Borel $\mathfrak{D}_2$-toast (Section 3.3).

Let $H = \mathcal{E}^\times$ be the multiplicative group of entire functions without zeros, let $\mathbb{C}$ act on $\mathcal{E}^\times$ via the argument shift, and the semidirect product $H \rtimes G = \mathcal{E}^\times \rtimes \mathbb{C}$

acts on $Z = \mathcal{MR}^\times$ according to the rule

$$((f, z) \cdot h)(w) = f(w + z)h(w + z).$$

The family of seminorms $\mathsf{N} = (|| \cdot ||)_{K \in \mathcal{K}(\mathbb{C})}$ on $\mathcal{E}^\times$ is given by

$$\begin{aligned} ||f||_K &= \max\{\log(\sup_{w \in K} |f(w)|), -\log(\inf_{w \in K} |f(w)|)\} \\ &= \sup_{w \in K} \Big| \log |f(w)| \Big|. \end{aligned}$$

We claim that it satisfies the $\mathfrak{D}_2$-Runge property. Indeed, consider pairwise disjoint sets $K_1, \ldots, K_m \in \mathfrak{D}_2$ and nowhere zero entire functions $f_1, \ldots, f_m \in \mathcal{E}^\times$. Note that $\mathbb{C} \setminus (K_1 \cup \cdots \cup K_m)$ is connected. Since $\mathbb{C}$ is connected and simply connected, there exist entire functions g_i such that $f_i = e^{g_i}$. By Runge's approximation theorem, for any $\delta > 0$, there exists an entire function g (in fact, a polynomial) satisfying $\sup_{z \in K_i} |g(z) - g_i(z)| < \delta$ for all i. Setting $f = e^g$, we observe that $||f f_i^{-1}||_{K_i} < \varepsilon$ for some $\varepsilon = \varepsilon(\delta, K_i)$, where $\varepsilon(\delta, K_i) \to 0$ as $\delta \to 0$.

The group $\mathbb{C}$ also acts on the space of divisors $Y = \mathcal{D}$ by translations, and the map $\operatorname{div} : \mathcal{MR}^\times \to \mathcal{D}$ satisfies $f_1 E_{\mathcal{E}^\times} f_2$ if and only if $\operatorname{div}(f_1) = \operatorname{div}(f_2)$. Consequently, Theorem 4.9 guarantees the existence of the required map ψ and the proof of Theorem 5.3 is complete. □

Remark 5.5. Using Theorem 4.9 instead of Corollary 4.10 in the proof of Theorem 5.3, one shows that for any free Borel action $\mathbb{C} \curvearrowright X$ and any equivariant Borel map $\varphi : X \to \mathcal{D}$ there exists a Borel equivariant lifting $\psi : X \to \mathcal{MR}^\times$ such that $\operatorname{div} \circ \psi = \varphi$. Let $\mathcal{C} \subseteq X$ be a Borel set that intersects each orbit in a countable discrete set. With such $\mathcal{C}$, we can associate a Borel equivariant map $\varphi_\mathcal{C} : X \to \mathcal{D}^+$ given by

$$\varphi_\mathcal{C}(x)(z) = \begin{cases} 1 & \text{if } (-z) \cdot x \in \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

A Borel equivariant lifting $\psi : X \to \mathcal{E}$ is then a Borel entire function. Furthermore, if $\mathcal{C}$ has a non-empty intersection with each orbit, then ψ is necessarily non-constant, because it has zeros within each orbit. This way one establishes the existence of non-constant Borel entire functions for any free Borel action $\mathbb{C} \curvearrowright X$, which is a Borel counterpart of Weiss's result [55]. (As we mentioned earlier, it is also possible to modify Weiss's original argument to work in the descriptive set-theoretic context directly.)

5.4 Spaces of labeled discrete subsets

Our next application will be a Borel equivariant analogue of the Mittag-Leffler theorem for meromorphic functions. Before we can formulate and prove it, we need to establish several properties of the space of principal parts, which we can think of as labeled discrete subsets of $\mathbb{C}$.

Recall that $\mathcal{F}(\mathbb{C})$ denotes the Effros Borel space of closed subsets of $\mathbb{C}$. Let $\mathcal{F}_{\mathrm{dis}}(\mathbb{C}) \subseteq \mathcal{F}(\mathbb{C})$ denote the subspace of closed discrete subsets of $\mathbb{C}$, i.e., those $F \in \mathcal{F}(\mathbb{C})$ that have finite intersection with every compact subset of $\mathbb{C}$. This is a Borel subset of $\mathcal{F}(\mathbb{C})$. Indeed, $\mathcal{K}(K)$ is a Borel subset of $\mathcal{F}(\mathbb{C})$ for each $K \in \mathcal{K}(\mathbb{C})$, and the set $\mathcal{K}_{\mathrm{fin}}(K)$ of finite subsets of $\mathcal{K}(K)$ is Borel[6]. The intersection map $\mathcal{F}(\mathbb{C})^2 \ni (F_1, F_2) \mapsto F_1 \cap F_2 \in \mathcal{F}(\mathbb{C})$ is Borel due to the local compactness of $\mathbb{C}$[7]. Finally, if $(K_n)_n$ is an exhaustion of $\mathbb{C}$ by compact sets, then $F \in \mathcal{F}_{\mathrm{dis}}(\mathbb{C})$ if and only if $F \cap K_n \in \mathcal{K}_{\mathrm{fin}}(K_n)$ for all n. Thus, $\mathcal{F}_{\mathrm{dis}}(\mathbb{C})$ is indeed a Borel subset of $\mathcal{F}(\mathbb{C})$.

We now consider a generalization of $\mathcal{F}_{\mathrm{dis}}(\mathbb{C})$ where points of discrete subsets of $\mathbb{C}$ are labeled. This arises naturally in the context of the space of principal parts of meromorphic functions, where the set of poles can be viewed as an element of $\mathcal{F}_{\mathrm{dis}}(\mathbb{C})$, and each pole is labeled by the finite tuple of coefficients determining the corresponding principal part.

One way to define the required space would be as follows. Let Y be a standard Borel space. Choose a Polish topology on Y compatible with its Borel structure, and consider the space $\mathcal{F}(\mathbb{C} \times Y)$. The projection $\mathrm{proj}_{\mathbb{C}} : \mathbb{C} \times Y \to \mathbb{C}$ is continuous, so the map $\mathcal{F}(\mathbb{C} \times Y) \ni F \mapsto \overline{\mathrm{proj}_{\mathbb{C}}(F)} \in \mathcal{F}(\mathbb{C})$ is Borel. In fact, $\mathrm{proj}_{\mathbb{C}}(F) \cap U \neq \varnothing$ if and only if $F \cap (U \times Y) \neq \varnothing$. Also, note that $\overline{\mathrm{proj}_{\mathbb{C}}(F)} = \mathrm{proj}_{\mathbb{C}}(F)$ whenever $\overline{\mathrm{proj}_{\mathbb{C}}(F)} \in \mathcal{F}_{\mathrm{dis}}(\mathbb{C})$, since elements of $\mathcal{F}_{\mathrm{dis}}(\mathbb{C})$ have no proper dense subsets. In particular, the set $\{F \in \mathcal{F}(\mathbb{C} \times Y) : \mathrm{proj}_{\mathbb{C}}(F) \in \mathcal{F}_{\mathrm{dis}}(\mathbb{C})\}$ is Borel. We need to pass to a further subset that consists of those $F \in \mathcal{F}(\mathbb{C} \times Y)$ for which the projection map

[6] In fact, the set of finite subsets of size at most n is closed, because a set has at least $(n+1)$-many points if and only if it intersects a collection of $(n+1)$-many pairwise disjoint open sets.

[7] More specifically, let $(V_n)_n$ be a precompact basis for the topology on $\mathbb{C}$. For each n, pick a sequence $(\mathcal{U}_{n,m})_m$, $\mathcal{U}_{n,m} = (U^i_{n,m})_{i=1}^{p_{n,m}}$, of finite open covers of $\overline{V}_n$ by sets $U^i_{n,m}$ satisfying $\mathrm{diam}(U^i_{n,m}) < 1/m$ for all $1 \leq i \leq p_{n,m}$. Then for an open U, we have $F_1 \cap F_2 \cap U \neq \varnothing$ if and only if there exists n such that $\overline{V}_n \subseteq U$ and

$$\forall m \; \exists 1 \leq i \leq p_{m,n} \; \big[F_1 \cap U^i_{n,m} \neq \varnothing \text{ and } F_2 \cap U^i_{n,m} \neq \varnothing\big].$$

is injective. Let $s_n : \mathcal{F}(\mathbb{C} \times Y) \to \mathbb{C} \times Y$, $n \in \mathbb{N}$, be Kuratowski–Ryll-Nardzewski Borel selectors, and define

$$\mathcal{F}_{\text{dis}}(\mathbb{C}; Y) = \{\varnothing\} \cup \Big\{ F \in \mathcal{F}(\mathbb{C} \times Y) : \operatorname{proj}_{\mathbb{C}}(F) \in \mathcal{F}_{\text{dis}}(\mathbb{C}) \text{ and } \\ \forall n\, \forall m \left[\operatorname{proj}_{\mathbb{C}}(s_n(F)) = \operatorname{proj}_{\mathbb{C}}(s_m(F)) \implies s_n(F) = s_m(F) \right] \Big\}.$$

The condition $\operatorname{proj}_{\mathbb{C}}(F) \in \mathcal{F}_{\text{dis}}(\mathbb{C})$ ensures that $\{\operatorname{proj}_{\mathbb{C}}(s_n(F))\}_n = \operatorname{proj}_{\mathbb{C}}(F)$. Thus, $F \in \mathcal{F}(\mathbb{C} \times Y)$ belongs to $\mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$ if and only if $\operatorname{proj}_{\mathbb{C}}(F) \in \mathcal{F}_{\text{dis}}(\mathbb{C})$ and the projection map $\operatorname{proj}_{\mathbb{C}} : F \to \mathbb{C}$ is injective. This allows us to view $\mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$ as the collection of discrete subsets of $\mathbb{C}$ whose points are labeled by elements of Y. Note that if $Y = \{*\}$ consists of a single point, then $\mathcal{F}_{\text{dis}}(\mathbb{C}; \{*\})$ is naturally isomorphic to $\mathcal{F}_{\text{dis}}(\mathbb{C})$.

The Borel structure on $\mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$ is independent of the choice of a compatible Polish topology on Y. This can be verified directly or deduced from the following alternative presentation of $\mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$. Let $(K_n)_n$ be an exhaustion of $\mathbb{C}$ by compact sets. Define

$$\begin{aligned} \mathbb{C} \boxdot_n Y &= \{(x_k, y_k)_{k<n} \in (\mathbb{C} \times Y)^n : \forall k < n\, \forall m < n\, [m \neq k \implies x_k \neq x_m]\}, \\ \mathbb{C} \boxdot_\infty Y &= \Big\{(x_k, y_k)_k \in (\mathbb{C} \times Y)^\infty : \forall k\, \forall m\, [m \neq k \implies x_k \neq x_m] \text{ and} \\ &\qquad \forall n\, \exists N\, \forall k \geq N\, [x_k \notin K_n]\Big\}. \end{aligned}$$

Each sequence in $\mathbb{C} \boxdot_n Y$ provides an injective enumeration of an n-element set $\{x_k\}_{k<n}$, with y_k as the label of x_k. Sequences in $\mathbb{C} \boxdot_\infty Y$ enumerate infinite discrete subsets of $\mathbb{C}$, and every discrete subset of $\mathbb{C}$ is enumerated by some sequence in $\mathbb{C} \boxdot Y = \mathbb{C} \boxdot_\infty Y \sqcup \left(\bigsqcup_n \mathbb{C} \boxdot_n Y\right)$.

Two sequences of the same length, say $(x_k, y_k)_k$ and $(x'_k, y'_k)_k$, encode the same labeled set if and only if $(x'_k, y'_k)_k$ is a permutation of $(x_k, y_k)_k$. For $\kappa \in \mathbb{N} \cup \{\infty\}$, let S_κ denote the group of permutations of a κ-element set. The space of labeled discrete κ-element subsets of $\mathbb{C}$ can be identified with the factor space of the action $S_\kappa \curvearrowright \mathbb{C} \boxdot_\kappa Y$ via coordinate permutations.

The action of S_κ on $\mathbb{C} \boxdot_\kappa Y$ is concretely classifiable. To see this, choose a Borel linear order $\prec$ on $\mathbb{C}$ and let $T^\kappa \subseteq \mathbb{C} \boxdot_\kappa Y$ consist of those $(x_k, y_k)_{k<\kappa} \in \mathbb{C} \boxdot_\kappa Y$ satisfying

$$\forall n\, \forall m < \kappa\, \forall k < m \Big[x_m \in K_n \implies x_k \in K_n \text{ and} \\ x_k, x_m \in K_n \setminus K_{n-1} \implies x_k \prec x_m \Big].$$

In other words, the sequence $(x_k, y_k)_{k<\kappa}$ is in T^κ if it lists elements $\{x_k\}$ of K_0 first, followed by those in $K_1 \setminus K_0$, $K_2 \setminus K_1$, etc., with elements in each block ordered by $\prec$. The set T^κ is a Borel transversal for $S_\kappa \curvearrowright \mathbb{C} \boxdot_\kappa Y$, so the factor space $\mathbb{C} \boxdot_\kappa Y/E_{S_\kappa}$ is standard Borel.

Let E_S be the union of the orbit equivalence relations E_{S_κ}: for $z_1, z_2 \in \mathbb{C} \boxdot Y$,

$$z_1 E_S z_2 \iff \exists \kappa \in \mathbb{N} \cup \{\infty\} \left[z_1, z_2 \in \mathbb{C} \boxdot_\kappa Y \text{ and } z_1 E_{S_\kappa} z_2\right].$$

The factor space $\mathbb{C} \boxdot Y/E_S$ is standard Borel and encodes discrete subsets of $\mathbb{C}$ whose points are labeled by elements of Y.

The spaces $\mathbb{C} \boxdot Y/E_S$ and $\mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$ are Borel isomorphic. To see this, let $\pi : \mathbb{C} \boxdot Y \to \mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$ be the map sending a sequence $(x_k, y_k)_{k<\kappa} \in \mathbb{C} \boxdot Y$ to the set $\{(x_k, y_k)\}_{k<\kappa} \in \mathcal{F}(\mathbb{C} \times Y)$. Since for any open $U \subseteq \mathbb{C} \times Y$ (in fact, for any Borel $U \subseteq \mathbb{C} \times Y$), the set

$$\{(x_k, y_k)_{k<\kappa} \in \mathbb{C} \boxdot_\kappa Y : \exists k < \kappa \ (x_k, y_k) \in U\}$$

is Borel, the map π is also Borel. Moreover, $\pi : \mathbb{C} \boxdot Y \to \mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$ is surjective, and $\pi((x_k, y_k)_{k<\kappa}) = \pi((x'_k, y'_k)_{k<\kappa'})$ if and only if $(x_k, y_k)_{k<\kappa} E_S (x'_k, y'_k)_{k<\kappa'}$. Thus, π factors into a Borel injection from $\mathbb{C} \boxdot Y/E_S$ onto $\mathcal{F}_{\text{dis}}(\mathbb{C}; Y)$, which must be an isomorphism between the standard Borel spaces [32, 15.2].

5.5 Principal parts and the Mittag-Leffler theorem

We next introduce the space $\mathcal{P}$, which describes the principal parts of meromorphic functions, and establish some of its basic properties. Near a pole w of order m, a meromorphic function f admits an expansion of the form

$$f(z) = \frac{c_1}{z-w} + \frac{c_2}{(z-w)^2} + \cdots + \frac{c_m}{(z-w)^m} + \sum_{n=0}^{\infty} a_n z^n = p_w(z) + \sum_{n=0}^{\infty} a_n z^n,$$

where $p_w(z)$ is the *principal part* at w, encoded by the m-tuple $(c_1, \ldots, c_m)$. Let $\mathbb{C}^{<\infty} = \bigsqcup_{n=1}^{\infty} \mathbb{C}^n$ denote the space of finite non-empty sequences of complex numbers. The *space of principal parts* $\mathcal{P} = \mathcal{F}_{\text{dis}}(\mathbb{C}; \mathbb{C}^{<\infty})$ consists of discrete subsets of $\mathbb{C}$ labeled by finite sequences of complex numbers.

Let $\mathrm{pp} : \mathcal{MR} \to \mathcal{P}$ be the map associating each meromorphic function f with the labeled set of principal parts at its poles. This map is Borel. To see this, it is best to view $\mathcal{P} = \mathcal{F}_{\text{dis}}(\mathbb{C}; \mathbb{C}^{<\infty})$ as the quotient space of $\mathbb{C} \boxdot \mathbb{C}^{<\infty}$, as described at the end of Section 5.4. Consider the set A of pairs $(f, (w_k, y_k)_k)$, where f is a meromorphic function and $(w_k, y_k)_k$ is a finite or infinite sequence with pairwise distinct $w_k \in \mathbb{C}$ and labels $y_k \in \mathbb{C}^{<\infty}$ such that $(w_k)_k$ enumerates the poles of f and $y_k \in \mathbb{C}^{<\infty}$ encodes the principal part of f at w_k. To show that pp is Borel, it suffices to verify that A is Borel. For (w_k, y_k) with $y_k = (c_1^k, \dots, c_{m_k}^k) \in \mathbb{C}^{<\infty}$, define

$$p_k(z) = \frac{c_1^k}{z - w_k} + \cdots + \frac{c_{m_k}^k}{(z - w_k)^{m_k}}.$$

The condition $f(n\overline{\mathbb{D}}) \subseteq \mathbb{C}$ is equivalent to f having no poles in $n\overline{\mathbb{D}}$, which is an open condition in $\tau_{\mathcal{MR}}$. Finally, $(f, (w_k, y_k)_k) \in A$ if and only if for each n

$$\Big(f - \sum_{w_k \in n\overline{\mathbb{D}}} p_k\Big)(n\overline{\mathbb{D}}) \subseteq \mathbb{C},$$

which is Borel since the algebraic operations in $\mathcal{MR}$ are Borel by Proposition 5.1.

Mittag-Leffler's theorem on the existence of meromorphic functions with prescribed principal parts implies that $\mathrm{pp} : \mathcal{MR} \to \mathcal{P}$ is surjective. We summarize these results as follows:

Proposition 5.6. *The map* $\mathrm{pp} : \mathcal{MR} \to \mathcal{P}$ *is a Borel surjection.*

We are now ready to state and prove the equivariant Mittag-Leffler theorem.

Theorem 5.7. *There exists a Borel* $\mathbb{C}$*-equivariant map* $\psi : \mathrm{Free}(\mathcal{P}) \to \mathcal{MR}$ *that is a right-inverse to* pp.

Proof. Just as in the proof of Theorem 5.3, we apply Corollary 4.10 in the following context. The group G is the additive group of $\mathbb{C}$, and the class $\mathfrak{R} = \mathfrak{D}_2$ consists of compact subsets of $\mathbb{C}$ diffeomorphic to the unit disk. Note that every free Borel $\mathbb{C}$-action admits a Borel $\mathfrak{D}_2$-toast (Section 3.3).

We next let $H = \mathcal{E}$ denote the additive group of entire functions, and let $\mathbb{C}$ act on $\mathcal{E}$ via the argument shift. Define Z to be the space of meromorphic functions $\mathcal{MR}$. The group $\mathcal{E}$ acts on Z additively, $(h \cdot f)(w) = f(w) + h(w)$. The semidirect product $H \rtimes G = \mathcal{E} \rtimes \mathbb{C}$ acts on Z as

$$((h, z) \cdot f)(w) = f(w + z) + h(w + z).$$

The family of seminorms $\mathsf{N} = (||\cdot||_K)_{K \in \mathcal{K}(\mathbb{C})}$ on $\mathcal{E}$ is given by $||f||_K = \sup_{z \in K} |f(z)|$. It satisfies the $\mathfrak{D}_2$-Runge property in view of the standard Runge theorem. The space $Y = \mathcal{P}$ is the space of principal parts and $\pi = \mathrm{pp}$. The equivariant Mittag-Leffler's theorem is then an instance of Corollary 4.10. $\square$

5.6 Luzin spaces and LF-spaces

Our next application uses the notion and basic properties of Luzin spaces. A *Luzin space* is a Hausdorff topological space (X, τ) whose topology can be refined into a Polish topology. A systematic treatment of Luzin spaces can be found in [47, Ch. II]. A key property for our purposes is that the Borel σ-algebras of Luzin spaces are standard. Specifically, if (X, τ) is a Luzin space and (X, τ') is a Polish refinement of τ, then $\mathcal{B}_\tau = \mathcal{B}_{\tau'}$ [47, p. 101], where $\mathcal{B}_\tau$ and $\mathcal{B}_{\tau'}$ are the σ-algebras generated by the open sets in the corresponding topologies.

As observed in [47, p. 90], most of the separable topological spaces encountered in analysis are Luzin. Recall that a *Fréchet space* is a completely metrizable locally convex topological vector space. An *LF-space* is a locally convex topological vector space E that admits an increasing and exhaustive sequence of subspaces, $E_0 \subseteq E_1 \subseteq \cdots \subseteq E = \bigcup_n E_n$, where each E_n is a Fréchet space in the induced topology, and the topology of E coincides with the inductive limit topology of $(E_n)_n$ [42, p. 428]. (Sometimes the term strict LF-space is used instead, while for general LF-spaces, the requirement on E_n is relaxed to being Fréchet in a topology coarser than the induced one.)

Theorem 5.8. *(Schwartz [47, p. 112 Thm. 7]) If E and F are separable LF-spaces, then $\mathcal{L}_c(E, F)$—the space of continuous linear operators $E \to F$ equipped with the compact-open topology—is a Luzin space.*

Remark 5.9. Theorem 5.8 is stated in [47, p. 112 Thm. 7] in a more general form, where E is assumed to be the inductive limit of a sequence of separable Fréchet spaces E_n satisfying:

1. E_n increases with n, and
2. every compact subset of E is a compact subset of E_n, for some n.

When E is a separable LF-space, these conditions are automatically satisfied by [42, Thm. 12.1.7(c)]. Additionally, the separability of E implies the separability of the Fréchet subspaces E_n [54] (see also [37] for a more general argument). Furthermore, the conditions on F can be relaxed to requiring that F is a Hausdorff topological vector space that is the countable union of images (under linear continuous maps) of separable Fréchet spaces.

For the purposes of this work, the generality of Theorem 5.8 is sufficient.

Corollary 5.10. *The Borel σ-algebras of separable LF-spaces and their weak* duals are standard.*

Proof. Let F be the field of scalars. An LF-space E and its dual E^* with the weak* topology can be viewed as spaces of linear operators $\mathcal{L}(F, E)$ and $\mathcal{L}(E, F)$ endowed with the pointwise convergence topology. The topology of pointwise convergence is coarser than the compact-open topology. Theorem 5.8 implies that $\mathcal{L}(F, E)$ and $\mathcal{L}(E, F)$ are Luzin, and thus their Borel σ-algebras are standard. □

Remark 5.11. The standardness of the Borel σ-algebras of separable LF-spaces also follows from a deeper classification theorem due to Mankiewicz [38], which states that, up to homeomorphism, there are only three distinct infinite-dimensional separable LF-spaces.

We note that LF-spaces are typically non-metrizable [42, 12.1.8].

5.7 Continuous and smooth functions

Let $C^\infty(\mathbb{R}^d)$ denote the space of infinitely differentiable real-valued functions on $\mathbb{R}^d$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, let ∂^α stand for the corresponding partial derivative operator $\dfrac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$. For a compact set $K \in \mathcal{K}(\mathbb{R}^d)$, define the sup-norm of the αth derivative over K as $||f||_{\alpha,K} = \sup_{x \in K} |\partial^\alpha f(x)|$. The family of seminorms $|| \cdot ||_{\alpha,K}$ endows $C^\infty(\mathbb{R}^d)$ with the structure of a separable Fréchet space [52, p. 85].

Let $C_c(\mathbb{R}^d)$ denote the space of compactly supported continuous functions on $\mathbb{R}^d$. Consider an exhaustion $(K_n)_n$ of $\mathbb{R}^d$ by a sequence of compact sets. Then $C_c(\mathbb{R}^d)$ can be expressed as the increasing union of spaces

$$\{f \in C_c(\mathbb{R}^d) : \operatorname{supp} f \subseteq K_n\},$$

each of which is a Banach space with respect to the sup-norm. Equipped with the inductive limit topology, $C_c(\mathbb{R}^d)$ becomes an LF-space [52, p. 131].

Let $C_c^\infty(\mathbb{R}^d)$ denote the space of compactly supported smooth functions on $\mathbb{R}^d$, also known as the *space of test functions*. For a compact set $K \subseteq \mathbb{R}^d$, the subspace $\{f \in C_c^\infty(\mathbb{R}^d) : \operatorname{supp}(f) \subseteq K\}$ is closed in $C^\infty(\mathbb{R}^d)$ and thus inherits the structure of a separable Fréchet space. The space of test functions, equipped with the inductive limit topology of the union $\bigcup_n \{f \in C_c^\infty(\mathbb{R}^d) : \operatorname{supp}(f) \subseteq K_n\}$, is also an LF-space [52, p. 132]. The inductive topologies on $C_c(\mathbb{R}^d)$ and $C_c^\infty(\mathbb{R}^d)$

are not metrizable [42, 12.1.8], but they are separable [42, 12.109]. Consequently, their Borel σ-algebras are standard by Corollary 5.10. Furthermore, the inclusion $C_c^\infty(\mathbb{R}^d) \subseteq C_c(\mathbb{R}^d)$ is continuous, and $C_c^\infty(\mathbb{R}^d)$ is therefore a Borel subset of $C_c(\mathbb{R}^d)$.

5.8 $\bar{\partial}$-problem

We now discuss the existence of Borel equivariant inverses to the $\bar{\partial}$ map. Let $C^\infty(\mathbb{C}, \mathbb{C})$ denote the space of complex-valued smooth functions on $\mathbb{C}$. We can identify $C^\infty(\mathbb{C}, \mathbb{C})$ with $C^\infty(\mathbb{C}) \times C^\infty(\mathbb{C})$ through the mapping $(u, v) \mapsto u + \mathrm{i}v$, where $C^\infty(\mathbb{C})$ represents the space of real-valued smooth functions on $\mathbb{C}$, as discussed in Section 5.7. The $\bar{\partial}$-operator on this space is defined as $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})$, i.e.,

$$\frac{\partial}{\partial \bar{z}}(u + \mathrm{i}v) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}\mathrm{i}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

In particular, the kernel of $\bar{\partial}$ consists of the functions that satisfy the Cauchy–Riemann equations, allowing us to identify it with the space $\mathcal{E}$ of entire functions. Moreover, the topology induced on $\mathcal{E}$ by $C^\infty(\mathbb{C}, \mathbb{C})$ is precisely the topology of uniform convergence on compact sets.

The $\bar{\partial}$-problem on the space $C^\infty(\mathbb{C}, \mathbb{C})$ involves finding a function $f \in C^\infty(\mathbb{C}, \mathbb{C})$ that satisfies the equation $\bar{\partial} f = g$ for a given $g \in C^\infty(\mathbb{C}, \mathbb{C})$. It is known that $\bar{\partial} : C^\infty(\mathbb{C}, \mathbb{C}) \to C^\infty(\mathbb{C}, \mathbb{C})$ is surjective, ensuring that a solution to the $\bar{\partial}$-problem exists for any smooth function g. This follows from the fact that linear partial differential operators with constant coefficients are surjective as maps of the space $\mathscr{D}'(\mathbb{C})$ of distributions into itself ([26, Cor. 3.6.2]), together with Weyl's lemma, which implies that distributional solutions $\bar{\partial} u = f$ with f smooth are themselves smooth ([26, Cor. 4.1.2]).

Corollary 4.10 shows that, on the free part $\mathrm{Free}(C^\infty(\mathbb{C}, \mathbb{C}))$, solutions the d-bar equation can be constructed in a Borel and equivariant manner. More precisely, let $G = \mathbb{C}$ and let $H = \mathcal{E}$ be the group of entire functions. The class $\mathfrak{R}$ is defined, as before, to be the collection $\mathfrak{D}_2$ of compact sets of $\mathbb{C}$ diffeomorphic to the unit disk The action of $\mathbb{C}$ on $\mathcal{E}$ is via the argument shift, and the semidirect product $\mathcal{E} \rtimes \mathbb{C}$ acts on the space of smooth functions $C^\infty(\mathbb{C}, \mathbb{C})$ as $((f, z) \cdot h)(w) = f(w + z) + h(w + z)$. The family of seminorms N with the $\mathfrak{D}_2$-Runge property is given, as earlier, by $||f||_K = \sup_{z \in K} |f(z)|$. Let Y also denote the space $C^\infty(\mathbb{C}, \mathbb{C})$ of complex-valued smooth functions on $\mathbb{C}$, with the same action $\mathbb{C} \curvearrowright Y$ by the argument shift and let π be the $\bar{\partial}$ operator. Corollary 4.10 applies and yields the following result.

Theorem 5.12. *There exists a Borel $\bar{\partial}$-equivariant map $\psi : \mathrm{Free}(C^\infty(\mathbb{C},\mathbb{C})) \to C^\infty(\mathbb{C},\mathbb{C})$ which is a right-inverse to $\bar{\partial}$.*

The corresponding statement holds true for the Laplacian on $\mathbb{R}^d$ (as will be discussed in Section 5.11) as well as for any other elliptic partial differential operator P with constant coefficients. We will not pursue the details here, but the key analytic results are surjectivity of the operator P on the space of non-periodic smooth functions ([26, Cor. 3.6.2] and [26, Cor. 4.1.2]) and the Lax–Malgrange approximation theorem for $\ker P$ ([26, Thm. 4.3.1]), which replaces Runge's theorem.

5.9 Radon measures and distributions

We will next turn to proving an equivariant version of the Brelot–Weierstrass theorem, which is an analogue of Weierstrass theorem for subharmonic functions. We begin by introducing the relevant spaces and maps.

The Riesz representation theorem [46, 2.14] establishes an isomorphism between the weak*-dual $C_c(\mathbb{R}^d)'$ of the space of compactly supported continuous functions and the space $\mathcal{M}(\mathbb{R}^d)$ of (generalized) *Radon measures* on $\mathbb{R}^d$ endowed with the topology of weak convergence (sometimes also called vague convergence), i.e., the weakest topology under which maps $\mathcal{M}(\mathbb{R}^d) \ni \mu \mapsto \int_{\mathbb{R}^d} f\,d\mu$ are continuous for every compactly supported continuous function f. A basis of neighborhoods for a measure μ is given by finite collections $f_1,\dots,f_n \in C_c(\mathbb{R}^d)$ and $\varepsilon > 0$:

$$U(\mu; f_1,\dots,f_n,\varepsilon) = \left\{\nu \in \mathcal{M}(\mathbb{R}^d) : \left|\int f_i\,d\mu - \int f_i\,d\nu\right| < \varepsilon \text{ for } i = 1,\dots,n\right\}.$$

This topology makes $\mathcal{M}(\mathbb{R}^d)$ a separable, non-metrizable, locally convex topological vector space, and the weak*-dual to the separable LF-space $C_c(\mathbb{R}^d)$. Hence, its Borel σ-algebra is standard, as shown by Corollary 5.10.

For a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ and a Borel set $A \subseteq \mathbb{R}^d$, we say that A is *μ-null* if $\mu(A') = 0$ for every Borel subset $A' \subseteq A$. Notably, an open set U is μ-null if and only if $\int f\,d\mu = 0$ for all $f \in C_c(\mathbb{R}^d)$ with $\operatorname{supp} f \subseteq U$. The *support* of a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is defined as the closed set $\operatorname{supp}\mu = \mathbb{R}^d \setminus \bigcup U$, where the union is taken over all open μ-null sets U.

The map $\mathcal{M}(\mathbb{R}^d) \ni \mu \mapsto \operatorname{supp}\mu \in \mathcal{F}(\mathbb{R}^d)$ is Borel. To verify this, consider a countable basis of bounded open sets $(U_n)_n$ in $\mathbb{R}^d$. For each n, select a countable family $(f_{n,m})_m$ of continuous functions supported on U_n that is dense in the space

$\{f \in C_c(\mathbb{R}^d) : \operatorname{supp} f \subseteq U_n\}$. Recall that the Effros Borel structure is generated by sets of the form $\{F \in \mathcal{F}(\mathbb{R}^d) : F \cap U \neq \varnothing\}$, where $U \subseteq \mathbb{R}^d$ is open. Since

$$\operatorname{supp} \mu \cap U \neq \varnothing \iff \exists n\, \exists m \left[U_n \subseteq U \text{ and } \int f_{n,m}\, d\mu \neq 0\right],$$

the set $\{\mu \in \mathcal{M}(\mathbb{R}^d) : \operatorname{supp} \mu \cap U \neq \varnothing\}$ is open, hence the map $\mu \mapsto \operatorname{supp} \mu$ is Borel.

For any bounded Borel set $A \subseteq \mathbb{R}^d$, the evaluation $\mu \mapsto \mu(A)$ is Borel. To see this, consider a bounded open set U and define continuous compactly supported functions $f_n(x) = \min\{1, n \cdot \operatorname{dist}(x, \mathbb{R}^d \setminus U)\}$, where dist is any proper metric on $\mathbb{R}^d$. Since $(f_n)_n$ converges monotonically to the characteristic function $\mathbb{1}_U$ of U, the monotone convergence theorem implies $\int_{\mathbb{R}^d} f_n\, d\mu \to \int_{\mathbb{R}^d} \mathbb{1}_U\, d\mu = \mu(U)$. Thus, $\mu \mapsto \mu(U)$ is a pointwise limit of Borel functions and is therefore Borel [32, 11.2i]. The class of Borel subsets A of U for which $\mu \mapsto \mu(A)$ is Borel includes all open subsets and is closed under complements and countable disjoint unions, forming a Dynkin system. By the π-λ theorem (see, for instance, [5, Thm. 3.2]), this class contains all Borel subsets of U.

Let $\mathcal{M}^+(\mathbb{R}^d)$ denote the subspace of $\mathcal{M}(\mathbb{R}^d)$ consisting of non-negative functionals, i.e., $\varphi \in C_c(\mathbb{R}^d)'$ such that $\varphi(f) \geq 0$ whenever f is non-negative. The Riesz representation theorem identifies $\mathcal{M}^+(\mathbb{R}^d)$ with the space of positive Radon measures on $\mathbb{R}^d$. When restricted to $\mathcal{M}^+(\mathbb{R}^d)$, the weak convergence topology becomes metrizable and, in fact, Polish (see, for instance, [30, Thm. 4.2]).

The weak*-dual of $C_c^\infty(\mathbb{R}^d)$ is known as the space of *distributions* and is denoted by $\mathscr{D}'(\mathbb{R}^d)$. The continuity of the inclusion $C_c^\infty(\mathbb{R}^d) \hookrightarrow C_c(\mathbb{R}^d)$ implies that the restriction $\mathcal{M}(\mathbb{R}^d) = C_c(\mathbb{R}^d)' \ni \mu \mapsto \mu|_{C_c^\infty(\mathbb{R}^d)} \in \mathscr{D}'(\mathbb{R}^d)$ is continuous. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $C_c(\mathbb{R}^d)$, the restriction map is injective. Consequently, $\mathcal{M}(\mathbb{R}^d)$ is a Borel subset of $\mathscr{D}'(\mathbb{R}^d)$.

5.10 Harmonic and subharmonic functions

Another space relevant to this work is the space of subharmonic functions on $\mathbb{R}^d$, denoted by $\mathcal{SH}(\mathbb{R}^d)$. A function $u : \mathbb{R}^d \to [-\infty, \infty)$ is called *subharmonic* if it is upper semicontinuous and satisfies, for all $x \in \mathbb{R}^d$ and $r > 0$,

$$u(x) \leq \frac{1}{\omega(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} u(x + r\omega)\, d\omega, \tag{5.1}$$

where ω denotes the spherical area measure on the sphere $\mathbb{S}^{d-1}$. For an introduction to subharmonic functions, see [26, Ch. 4].

Let $\mathrm{L}^1_{\mathrm{loc}} = \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ denote the space of locally Lebesgue integrable real-valued functions on $\mathbb{R}^d$. Equipped with the seminorms $||f||_{B_n} = \int_{B_n} |f|\, d\lambda$, where B_n is the ball of radius n centered at the origin, $\mathrm{L}^1_{\mathrm{loc}}$ forms a separable Fréchet space. Every subharmonic function belongs to $\mathrm{L}^1_{\mathrm{loc}}$, and distinct elements of $\mathcal{SH}(\mathbb{R}^d)$ correspond to distinct elements of $\mathrm{L}^1_{\mathrm{loc}}$ [26, Thm. 4.1.8]. Thus, $\mathcal{SH}(\mathbb{R}^d)$ can be viewed as a subset of $\mathrm{L}^1_{\mathrm{loc}}$, inheriting a separable metrizable topology. Moreover, $\mathcal{SH}(\mathbb{R}^d)$ is closed in $\mathrm{L}^1_{\mathrm{loc}}$, making it a Polish space. To see this, consider a sequence $u_n \in \mathcal{SH}(\mathbb{R}^d)$ converging in $\mathrm{L}^1_{\mathrm{loc}}$ to some $u \in \mathrm{L}^1_{\mathrm{loc}}$. Then $u_n \to u$ as distributions in $\mathscr{D}'(\mathbb{R}^d)$. Since the Laplacian of any subharmonic function is non-negative, $\Delta u_n \geq 0$ for all n, and Δ is continuous on $\mathscr{D}'(\mathbb{R}^d)$, it follows that $\Delta u \geq 0$. By [26, Thm. 4.1.8], this implies that u is subharmonic. Hence, $\mathcal{SH}(\mathbb{R}^d)$ is a Polish space under the topology of $\mathrm{L}^1_{\mathrm{loc}}$ convergence.

A function u is called *harmonic* if it satisfies $\Delta u = 0$. Equivalently, u is harmonic if and only if both u and $-u$ are subharmonic. The space of harmonic functions on $\mathbb{R}^d$ is denoted by $\mathcal{H}(\mathbb{R}^d)$. On $\mathcal{H}(\mathbb{R}^d)$, the $\mathrm{L}^1_{\mathrm{loc}}$ topology coincides with the topology of uniform convergence on compact sets, making $\mathcal{H}(\mathbb{R}^d)$ a separable Fréchet space and, consequently, a Polish group.

5.11 Poisson equation

A distinct application of Theorem 4.9 arises from PDE. Consider the Poisson equation $\Delta f = g$. Multiple natural spaces of potential solutions can be considered. For instance, we can view $\Delta : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ between the spaces of smooth functions or as a map $\Delta : \mathscr{D}'(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ between the spaces of distributions. Both maps are surjective ([27, Ch. X]). Another common alternative considered in potential theory is to view $\Delta : \mathcal{SH}(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d)$ as a map between the space of subharmonic functions and (non-negative) Radon measures on $\mathbb{R}^d$. The latter is also surjective (see Hayman–Kennedy [23, Thm. 4.1]). In view of Weyl's lemma, the kernel of the Laplacian Δ can be identified with the space of harmonic functions in all these cases. Consequently, Theorem 4.9 can be applied for all these choices of the solution space. For definiteness, we view the Laplacian as a map $\Delta : \mathcal{SH}(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d)$.

Let $H = \mathcal{H}(\mathbb{R}^d)$ be the additive group of harmonic functions on $\mathbb{R}^d$. It acts on $\mathcal{SH}(\mathbb{R}^d)$ by $(h \cdot f)(z) = f(z) + h(z)$, and $\Delta f_1 = \Delta f_2$ if and only if $f_2 - f_1$ is harmonic. The family of seminorms on H is given by

$$||f||_K = \sup_{z\in K} |f(z)|$$

and it satisfies the $\mathfrak{D}_d$-Runge property [2, Cor. 2.6.5] for the class $\mathfrak{D}_d$ of compact subsets diffeomorphic to the closed unit ball. Corollary 4.10 gives the following.

Theorem 5.13. *There exists a Borel $\mathbb{R}^d$-equivariant map $\psi : \mathrm{Free}(\mathcal{M}^+(\mathbb{R}^d)) \to \mathcal{SH}(\mathbb{R}^d)$ that is a right-inverse to Δ.*

We now turn our attention to the existence of equivariant right-inverses to Δ on the non-free part of $\mathcal{M}^+(\mathbb{R}^d)$. Suppose that Γ is a proper closed subgroup of $\mathbb{R}^d$. Note that Γ is isomorphic to a group of the form $\mathbb{Z}^p \times \mathbb{R}^q$, and consequently, the quotient group $\mathbb{R}^d/\Gamma$ is isomorphic to $\mathbb{R}^{d-p-q} \times \mathbb{T}^p$, where $\mathbb{T}$ denotes the circle group $\mathbb{R}/\mathbb{Z}$. Let $d' = d - p - q$.

Define

$$Y = \mathcal{M}^+_\Gamma(\mathbb{R}^d) = \{\mu \in \mathcal{M}^+(\mathbb{R}^d) : \mathrm{Stab}(\mu) = \Gamma\},$$
$$Z = \mathcal{SH}_\Gamma(\mathbb{R}^d) = \{f \in \mathcal{SH}(\mathbb{R}^d) : \mathrm{Stab}(f) = \Gamma\}.$$

Let also $\mathcal{H}_\Gamma(\mathbb{R}^d)$ be the group of harmonic functions whose stabilizer *contains* Γ. We have the natural action $\mathcal{H}_\Gamma(\mathbb{R}^d) \curvearrowright Z$ given by $(h, f) \mapsto f + h$. Observe that the argument shift actions of $\mathbb{R}^d$ on Y and Z induce *free* actions of $G = \mathbb{R}^{d'} \times \mathbb{T}^p$ on these spaces.

Theorem 5.14. *Let $\Gamma \le \mathbb{R}^d$ be a closed subgroup such that $\mathbb{R}^d/\Gamma$ is isomorphic to $\mathbb{R}^{d'} \times \mathbb{T}^p$. If $d' \ge 2$ or, equivalently, $\dim \Gamma \le d - 2$, then there exists a Borel equivariant map $\psi : \mathcal{M}^+_\Gamma(\mathbb{R}^d) \to \mathcal{SH}_\Gamma(\mathbb{R}^d)$ such that $\Delta \circ \psi = \mathrm{id}_{\mathcal{M}^+_\Gamma(\mathbb{R}^d)}$.*

Proof. By the dimension of Γ we mean the dimension of the vector space spanned by Γ. The previous discussion can be adapted to show the existence of such a map ψ for $d' \ge 2$. Here, we have $G = \mathbb{R}^{d'} \times \mathbb{T}^p$ and $H = \mathcal{H}_\Gamma(\mathbb{R}^d)$, which is naturally isomorphic to $\mathcal{H}(G)$. The class $\mathfrak{R}$ comprises the family $\mathfrak{D}_{d'} \times \mathbb{T}^p$ of compact sets of the form $K \times \mathbb{T}^p$ for $K \in \mathfrak{D}_{d'}$, where $\mathfrak{D}_{d'}$ is the class of compact subsets of $\mathbb{R}^{d'}$ diffeomorphic to the unit ball. Note that if $K \in \mathfrak{C}_{d'}$—the class of compact subsets of $\mathbb{R}^d$ with connected complements—then $K \times \mathbb{T}^p$ has connected complement in $\mathbb{R}^{d'} \times \mathbb{T}^p$. We need a Runge-type theorem for the harmonic functions in H for sets in the class $\mathfrak{D}_{d'} \times \mathbb{T}^p$. This result can be derived from the Lax–Malgrange approximation theorem, see [41, Sec. 3.10]. A simple direct proof is presented in Appendix C.

Finally, in order to utilize Corollary 4.10, we need to argue that free $\mathbb{R}^{d'} \times \mathbb{T}^p$-actions admit Borel $\mathfrak{D}_{d'} \times \mathbb{T}^p$-toasts. This was established in Lemma 3.7. □

Remark 5.15. Conversely, if $d' = 0$ or $d' = 1$ in the statement of Theorem 5.14, there exist Borel equivariant maps φ for which the corresponding equivariant Borel liftings ψ fail to exist. This conclusion will be substantiated in Section 7.2.

Remark 5.16. Theorem 5.14 guarantees the existence of a Borel equivariant map $\xi_\Gamma : \mathcal{M}^+_\Gamma(\mathbb{R}^d) \to \mathcal{SH}_\Gamma(\mathbb{R}^d)$ for each subgroup Γ of dimension $\dim \Gamma \leq d-2$. While we do not encounter any apparent obstructions to combining these individual maps ξ_Γ into a single Borel map ξ defined on the set $\{\mu : \dim(\mathrm{Stab}(\mu)) \leq d-2\}$, we do not pursue this direction in the current work.

5.12 Heat equation

Our final application pertains to the existence of equivariant solutions for the inhomogeneous heat equation

$$\frac{\partial f}{\partial t} - \Delta f = g. \tag{5.2}$$

We consider the heat operator $(\frac{\partial}{\partial t} - \Delta) : C^\infty(\mathbb{R}^{d+1}) \to C^\infty(\mathbb{R}^{d+1})$, $d \geq 2$, on the space of smooth functions. Being a linear partial differential operator with constant coefficients, it is a surjective map on the space of smooth functions ([26, Ch. 3]). Elements of the kernel of $(\frac{\partial}{\partial t} - \Delta)$ form an abelian Polish group and are known as *caloric* functions.

To apply Theorem 4.9, we set $Z = C^\infty(\mathbb{R}^{d+1})$, $G = \mathbb{R}^{d+1}$, and $H = \ker(\frac{\partial}{\partial t} - \Delta)$. As usual, G acts by the argument shift and $H \curvearrowright C^\infty(\mathbb{R}^{d+1})$ via $(h, \varphi) \mapsto \varphi + h$. The topology induced by $C^\infty(\mathbb{R}^{d+1})$ on H coincides with the (seemingly weaker) topology of uniform convergence on compact subsets. Therefore, the family of seminorms on H can be expressed using the same formula $||f||_K = \sup_{z \in K} |f(z)|$.

The final component needed for the application of Theorem 4.9 involves selecting a class of compact sets $\mathfrak{R} \subseteq \mathcal{K}(\mathbb{R}^{d+1})$ such that the action $\mathbb{R}^{d+1} \curvearrowright H$ satisfies $\mathfrak{R}$-Runge property. Choosing $\mathfrak{R}$ to be the class of compact sets with connected complements will not suffice, since the Runge-type approximation property does not hold for caloric functions on such sets. Instead, we define $\mathfrak{R}$ to be the class of compact $K \in \mathcal{K}(\mathbb{R}^{d+1})$ that satisfy the following stronger condition: for any hyperplane $P \subseteq \mathbb{R}^{d+1}$ orthogonal to the time axis, the complement $P \setminus K$ is connected. A caloric function on such a set K can be approximately extended to a caloric function on all of $\mathbb{R}^{d+1}$ [29] (see also [13] for a complete classification of Runge pairs for the heat operator).

We do not know whether every free Borel $\mathbb{R}^{d+1}$-action admits a Borel $\mathfrak{R}$-toast with this choice of $\mathfrak{R}$. However, the corresponding statement holds within the realm of ergodic theory, where one can employ the Lind version of Rokhlin's lemma [36] to construct a toast composed entirely of multidimensional boxes.

Theorem 5.17. *Let $\mathbb{R}^{d+1} \curvearrowright X$ be a free Borel action on a standard Borel space X that admits a Borel $\mathfrak{R}$-toast for the class $\mathfrak{R}$ of compact sets whose slices by time-hyperplanes have connected complements. For any Borel $\mathbb{R}^{d+1}$-equivariant function $\varphi : X \to C^\infty(\mathbb{R}^{d+1})$, there exists an $\mathbb{R}^{d+1}$-equivariant Borel map $\psi : X \to C^\infty(\mathbb{R}^{d+1})$ such that $(\frac{\partial}{\partial t} - \Delta) \circ \psi = \varphi$.*

$$\begin{tikzcd}
\mathbb{R}^{d+1} \curvearrowright X \arrow[r, dashed, "\psi"] \arrow[dr, "\varphi"'] & \mathbb{R}^{d+1} \curvearrowright C^\infty(\mathbb{R}^{d+1}) \arrow[d, "\frac{\partial}{\partial t} - \Delta"] \\
& \mathbb{R}^{d+1} \curvearrowright C^\infty(\mathbb{R}^{d+1}).
\end{tikzcd}$$

A natural question is whether the initial value problem

$$\begin{cases} \partial_t u = \Delta u & \text{on } \mathbb{R}^d \times \mathbb{R}^{>0}, \\ u = f & \text{on } \mathbb{R}^d \times \{0\} \end{cases} \tag{5.3}$$

admits a Borel equivariant solution. That is, does there exist a Borel map

$$\xi : C^\infty(\mathbb{R}^d \times \{0\}) \to C^\infty(\mathbb{R}^d \times \mathbb{R}^{\geq 0})$$

which is equivariant with respect to $\mathbb{R}^d \times \{0\}$-shifts, such that $u = \xi(f)$ solves the problem (5.3)? There is no obvious Runge property to make use of, and we do not know what the answer is.

As an aside note, we mention that the corresponding question for the Poisson extension problem in half-spaces does in fact have a positive answer. That is, there exists a Borel $(\mathbb{R}^d \times \{0\})$-equivariant map which assigns to each $f \in C^\infty(\mathbb{R}^d \times \{0\})$ a harmonic function in the upper half-space $\mathbb{R}^d \times \mathbb{R}^{>0}$ with continuous boundary values f.

6 Lack of continuous equivariant inverses

In Section 5, we showed the existence of Borel equivariant right-inverses to several maps, including $\operatorname{div} : \operatorname{Free}(\mathcal{D}^+) \to \mathcal{E}_{\neq 0}$, and $\bar\partial : \operatorname{Free}(C^\infty(\mathbb{C}, \mathbb{C})) \to C^\infty(\mathbb{C}, \mathbb{C})$.

Both the domains and the co-domains of these functions have natural Polish topologies. A considerable strengthening would be therefore to find *continuous* equivariant inverses. The purpose of this section is to show that this is not possible.

6.1 Divisors of entire functions

Let $\mathrm{Free}(\mathcal{D}^+)$ denote the space of non-periodic positive divisors. We claim that there are no continuous equivariant functions $\xi : \mathrm{Free}(\mathcal{D}^+) \to \mathcal{E}_{\neq 0}$ such that $\mathrm{div}(\xi(d)) = d$ for all $d \in \mathrm{Free}(\mathcal{D}^+)$. Note that a periodic entire function has a periodic divisor, so any such ξ would necessarily take values in $\mathrm{Free}(\mathcal{E}) = \mathrm{Free}(\mathcal{E}_{\neq 0})$.

The easiest obstruction is, perhaps, the fact that $\mathrm{Free}(\mathcal{D}^+)$ has plenty of non-trivial $\mathbb{C}$-invariant compact subsets, whereas $\mathrm{Free}(\mathcal{E})$ has no such subspaces.

Lemma 6.1. *There are (non-empty) compact invariant subsets of* $\mathrm{Free}(\mathcal{D}^+)$.

Proof. The simplest way to construct such compact invariant subsets is, arguably, by taking a non-periodic almost periodic subset of $\mathbb{C}$ with respect to the uniform distance [35] (see also [15]). Recall that given two discrete sets $\{a_n\}_n$ and $\{b_n\}_n$ in $\mathbb{C}$, the (extended) uniform transportation distance $\mathrm{dist}(\{a_n\}_n, \{b_n\}_n)$ between them is given by $\inf_\sigma \sup_n |a_n - b_{\sigma(n)}|$, where the infimum is taken over all bijections $\sigma : \mathbb{N} \to \mathbb{N}$. An element $z \in \mathbb{C}$ is an ε-period of $\{a_n\}_n$ if $\mathrm{dist}(\{a_n\}_n, \{a_n + z\}_n) < \varepsilon$. Finally, $\{a_n\}_n$ is *uniformly almost periodic* if for any $\varepsilon > 0$ the set of ε-periods is relatively dense in $\mathbb{C}$. That is, the distance from any point in $\mathbb{C}$ to the set of ε-periods is uniformly bounded.

A discrete set $\{a_n\}_n$ can be viewed as a $\{0,1\}$-valued divisor. If it is almost periodic in the above sense, the closure of its orbit, call it K, is compact even in the much stronger topology of uniform convergence on all of $\mathbb{C}$, and is therefore also compact in the topology of uniform convergence on bounded sets. Furthermore, if $\{a_n\}_n$ is not periodic then $K \subseteq \mathrm{Free}(\mathcal{D}^+)$, for if z is a period of some element of K, then for any $\varepsilon > 0$, z is an ε-period of $\{a_n\}_n$, hence $\{a_n\}_n$ itself is periodic.

Here is an explicit example of an almost periodic non-periodic subset $\mathbb{C}$. For integers $m, n \in \mathbb{Z}$, let $\alpha(m,n)$ denote the power of 2 in the prime decomposition of $\gcd(m,n)$. In other words, $\alpha = \alpha(m,n) \in \mathbb{N}$ satisfies $m = 2^\alpha m_0$, $n = 2^\alpha n_0$, where either n_0 or m_0 is odd. For an element $m + in \in \mathbb{Z} + \mathrm{i}\mathbb{Z}$, let

$$g(m + \mathrm{i}n) = m + \mathrm{i}n + 2^{-1} \sum_{i=1}^{\alpha(m,n)} 2^{-i} = m + \mathrm{i}n + 1/2 - 2^{-\alpha(m,n)-1}$$

and set $Z = \{g(m + in) : m, n \in \mathbb{Z}\}$. Vectors of the form $(2^k m, 2^k n)$, $m, n \in \mathbb{Z}$ form 2^{-k}-periods of Z, hence it is almost periodic. □

Lemma 6.2. *If $K \subseteq \mathcal{E}$ is invariant and compact then $K \subseteq \{constants\}$.*

Proof. Let $K \subset \mathcal{E}$ be compact and invariant. The evaluation map

$$\mathcal{E} \ni f \mapsto f(0) \in \mathbb{C}$$

is a continuous function, and therefore, by compactness, $\sup_{f\in K} |f(0)| \leq A$ for some A. Since K is invariant under the shift of the argument, this is equivalent to $\sup_{f\in K} \sup_{z\in\mathbb{C}} |f(z)| \leq A$. We conclude that all $f \in K$ are bounded entire functions, and therefore must be constant by the Liouville's theorem. □

Corollary 6.3. *$\mathcal{E} \setminus \{constants\}$ has no non-trivial compact invariant subsets.*

Theorem 6.4. *There are no continuous equivariant maps $\beta : \mathrm{Free}(\mathcal{D}^+) \to \mathcal{E} \setminus \{constants\}$. In particular, there are no continuous equivariant right-inverses to* div *defined on all of* $\mathrm{Free}(\mathcal{D}^+)$.

Proof. Suppose such a function β did exist. By Lemma 6.1, there is a non-empty compact invariant $K \subseteq \mathrm{Free}(\mathcal{D}^+)$. Its image, $\beta(K)$, would be a non-empty, compact (by the continuity of β), and invariant (by the equivariance of β) subset of $\mathcal{E} \setminus \{\text{constants}\}$, contradicting Corollary 6.3. □

6.2 $\bar{\partial}$-problem

There are no continuous equivariant right-inverses to $\bar{\partial} : \mathrm{Free}(C^\infty(\mathbb{C}, \mathbb{C})) \to C^\infty(\mathbb{C}, \mathbb{C})$ either, but this argument requires a little more work. The reason for the difference is that there are plenty of compact invariant sets in $\mathrm{Free}(C^\infty(\mathbb{C}, \mathbb{C}))$.

Before we proceed to prove this claim, we record an elementary topological transitivity property in the space $C^\infty(\mathbb{C}, \mathbb{C}) \times C^\infty(\mathbb{C}, \mathbb{C})$. For brevity, we write simply C^∞ for $C^\infty(\mathbb{C}, \mathbb{C})$.

Lemma 6.5. *There is a pair $(f_1, f_2) \in C^\infty \times C^\infty$ whose $\mathbb{C}$-orbit is dense in $C^\infty \times C^\infty$.*

Proof. Choose a sequence $K_n \subset \mathbb{C}$ of compact sets and a sequence of complex numbers $|w_n| \geq 2$, such that

- $K_n \cap (w_n \cdot K_n) = \varnothing$,

- $K_n \cup (w_n \cdot K_n) \subseteq K_{n+1}$,
- $\bigcup_n K_n = \mathbb{C}$.

Here and below, $w \cdot K$ and $w \cdot f$ denote the translated set $\{z - w : z \in K\}$ and function $w \cdot f(z) = f(z+w)$, respectively. We also fix a dense sequence of elements $(h_{1,n}, h_{2,n})_n$ in $C^\infty \times C^\infty$, such that each element of the sequence occurs infinitely many times.

Next, we construct a sequence $f_n = (f_{1,n}, f_{2,n})_n$ which converges to an element (f_1, f_2) with dense $\mathbb{C}$-orbit. We start with an arbitrary pair $(f_{1,0}, f_{2,0})$. Supposing that $(f_{1,n}, f_{2,n})$ has been chosen, we pick $f_{i,n+1} \in C^\infty$, $i = 1, 2$, with the properties that

$$f_{i,n+1} = \begin{cases} f_{i,n} & \text{on } K_n \\ (-w_n) \cdot h_{i,n} & \text{on } w_n \cdot K_n. \end{cases}$$

Then, for $n > m$, $f_{i,n} = f_{i,m}$ on K_m, and hence the sequence converges to a smooth function f_i. It remains to establish that the limiting pair (f_1, f_2) has dense orbits.

To this end, fix K_m, a natural number $p \in \mathbb{N}$, and an arbitrary element $h = (h_1, h_2) \in C^\infty \times C^\infty$. For $g = (g_1, g_2) \in C^\infty \times C^\infty$, we denote by $\|g\|_{m,p}$ the norms

$$\|g\|_{m,p} = \max_{i=1,2} \max_{0 \le j \le p} \max_{K_m} |g_i^{(j)}|.$$

We will show that

$$\liminf_{n \to \infty} \|w_n \cdot f - h\|_{m,p} = 0. \tag{6.1}$$

Given $\varepsilon > 0$, we let $\mathcal{J} \subset \mathbb{N}$ be all indices n for which $\|h_n - h\|_{m,p} < \varepsilon$. By the density of the sequence $(h_n)_n$ and by the fact that each element occurs infinitely often in the sequence, we conclude that $\mathcal{J}$ is an infinite set. For any index n we have that

$$\|w_n \cdot f - h\|_{m,p} \le \|w_n \cdot (f - f_{n+1})\|_{m,p} + \|w_n \cdot f_{n+1} - h_n\|_{m,p} + \|h_n - h\|_{m,p}.$$

For $n > m$, we have that $w_n \cdot f_{n+1} = h_n$ on $K_n \supseteq K_m$, so the second term vanishes. For $n > m$ with $n \in \mathcal{J}$, the third term is moreover bounded above by ε, so for such indices we get

$$\begin{aligned} \|w_n \cdot f - g\|_{m,p} &\le \|w_n \cdot (f - f_{n+1})\|_{m,p} + \varepsilon \\ &\le \|f - f_{n+1}\|_{m+1,p} + \varepsilon. \end{aligned}$$

Taking the limit $n \to \infty$ along $\mathcal{J}$, the claim (6.1) follows. □

We proceed to the main result of this section.

Theorem 6.6. *There are no continuous equivariant maps*

$$\xi : \mathrm{Free}(C^\infty(\mathbb{C}, \mathbb{C})) \to C^\infty(\mathbb{C}, \mathbb{C})$$

such that $\bar{\partial}(\xi(f)) = f$ *for all* $f \in \mathrm{Free}(C^\infty(\mathbb{C}, \mathbb{C}))$.

Proof. Suppose towards a contradiction that $\xi : \mathrm{Free}(C^\infty) \to C^\infty$ is continuous, equivariant, and satisfies $\bar{\partial}(\xi(f)) = f$ for all $f \in \mathrm{Free}(C^\infty)$. We note that there is some freedom in the choice of ξ: we may add any Borel map $c : C^\infty \to \mathbb{C}$ which is constant on the orbits of the action $\mathbb{C} \curvearrowright C^\infty$, and obtain a new map with the same properties.

Consider the map $\alpha(f_1, f_2) = \xi(f_1 + f_2) - \xi(f_1) - \xi(f_2)$, defined for those pairs $(f_1, f_2) \in \mathrm{Free}(C^\infty)^2$ that satisfy $f_1 + f_2 \in \mathrm{Free}(C^\infty)$. Note that α is continuous and equivariant with respect to the diagonal action $\mathbb{C} \curvearrowright \mathrm{Free}(C^\infty) \times \mathrm{Free}(C^\infty)$. Since $\bar{\partial}$ is linear, we have $\bar{\partial}(\alpha(f_1, f_2)) = 0$, showing that $\alpha(f_1, f_2) \in \mathcal{E}$ for all $(f_1, f_2) \in \operatorname{dom} \alpha$.

The proof will consist of three main steps:

1. Show that the map α is constant, so with an appropriate choice of the additive constant, ξ is linear.

2. Show that the map ξ has to be given by a Cauchy transform

$$\xi(f)(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{f(z)}{z - \zeta} dz \wedge d\bar{z}.$$

3. Finally, we argue that the Cauchy transform cannot be continuously extended to the whole of $\mathrm{Free}(C^\infty)$, thus reaching a contradiction.

We proceed with this plan. Suppose that $(f_1, f_2) \in \operatorname{dom} \alpha$ is such that the closure K of its orbit under the diagonal action is compact and satisfies $K \subseteq \operatorname{dom} \alpha$. Then $\alpha(K)$ is also compact and shift invariant, hence $\alpha(K) \subseteq \{\text{constants}\}$ by Liouville's theorem. One can take for such f_1, f_2 non-periodic uniformly almost periodic functions and check that such pairs are dense in $\operatorname{dom} \alpha$. We conclude that $\alpha(f_1, f_2) \in \{\text{constants}\}$ for all $(f_1, f_2) \in \operatorname{dom} \alpha$.

Lemma 6.5 implies that the action $\mathbb{C} \curvearrowright C^\infty \times C^\infty$ is topologically transitive. If $\omega_0 \in \mathbb{C}$ corresponds to the constant function such that $\alpha(f_1, f_2) = \omega_0$, then $\alpha(z \cdot f_1, z \cdot f_2) = z \cdot \alpha(f_1, f_2) = w_0$ for all z. Hence $\alpha(g_1, g_2) = \omega_0$ for all g_1, g_2 in the orbit of (f_1, f_2) and therefore also for all $(g_1, g_2) \in \operatorname{dom} \alpha$ by continuity. We conclude that α is identically equal to the constant function ω_0.

Changing ξ to $\xi'(f) = \xi(f) + \omega_0$ we may therefore assume without loss of generality that ξ is additive in the sense that $\xi(f_1 + f_2) = \xi(f_1) + \xi(f_2)$ holds whenever all the functions belong to the free part $\operatorname{Free}(C^\infty)$.

If $f \in C^\infty$ has compact support and $z_n \to \infty$, then $z_n \cdot f \to 0$ in the topology of C^∞. In particular, $\overline{\mathbb{C} \cdot f} = (\mathbb{C} \cdot f) \cup \{0\}$ is compact in C^∞. Unfortunately, $0 \notin \operatorname{Free}(C^\infty)$, so $\xi(0)$ is not defined. We claim, however, that for any compactly supported $f \in C^\infty$ and any sequence $z_n \to \infty$ one has $\xi(z_n \cdot f) \to 0$, showing that ξ can be extended to a continuous function on $\overline{\mathbb{C} \cdot f}$ by setting $\xi(0) = 0$. It suffices to show that any $z_n \to \infty$ has a subsequence satisfying $\xi(z_{n_k} \cdot f) \to 0$. Take h to be a non-periodic uniformly almost periodic function on $\mathbb{C}$. Then $h, h+f \in \operatorname{Free}(C^\infty)$. Passing to a subsequence and using the almost periodicity of h, we may ensure that $\lim_n (z_n \cdot h)$ exists; let us denote it by $\widetilde{h}$. Using the established additivity and continuity of ξ, we get

$$\lim_n (\xi(z_n \cdot h) + \xi(z_n \cdot f)) = \lim_n \xi(z_n \cdot h + z_n \cdot f) = \xi(\widetilde{h}) = \lim_n \xi(z_n \cdot h),$$

and therefore $\lim_n \xi(z_n \cdot f) = 0$, as claimed.

For any $f \in C^\infty$ with compact support, $\xi(f)$ is necessarily a bounded function. Indeed, $\mathbb{C} \cdot f$ is precompact, and we have just shown that so is $\mathbb{C} \cdot \xi(f) = \xi(\mathbb{C} \cdot f)$. The "evaluation at 0" map $f \mapsto f(0)$ is continuous on C^∞, hence bounded on $\overline{\xi(\mathbb{C} \cdot f)}$. Since the latter is shift-invariant, this is equivalent to $\sup_h \sup_{z \in \mathbb{C}} |h(z)| < \infty$, where the first supremum is over $h \in \overline{\xi(\mathbb{C} \cdot f)}$. Thus $\xi(f)$ is a bounded function.

There is a "natural candidate" for what the map ξ could be. Given a compactly supported $f \in C^\infty$, let

$$\widetilde{\xi}(f)(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} \frac{f(z)}{z - \zeta} dz \wedge d\bar{z}.$$

One has $\widetilde{\xi}(f) \in C^\infty$ and $\bar{\partial} \widetilde{\xi}(f) = f$ (see [25, 1.2.1]). Furthermore, we have that $\lim_{\zeta \to \infty} \widetilde{\xi}(f)(\zeta) = 0$. Indeed, if $M = \max_{z \in \operatorname{supp} f} |f(z)|$, then

$$|\widetilde{\xi}(f)(\zeta)| \leq \frac{\mu(\operatorname{supp} f)}{2\pi} \cdot \frac{M}{\operatorname{dist}(\zeta, \operatorname{supp} f)} \to 0 \text{ as } \zeta \to \infty. \tag{6.2}$$

In particular, $\widetilde{\xi}(f)$ is also a bounded function.

Since $\bar{\partial} \widetilde{\xi}(f) = f$ and $\bar{\partial} \xi(f) = f$ for all compactly supported f, we conclude that $\xi(f) - \widetilde{\xi}(f)$ is entire. Since each of $\xi(f), \widetilde{\xi}(f)$ is bounded, so is their difference $c(f) = \widetilde{\xi}(f) - \xi(f)$ which thus must be constant.

In fact, $c(f)$ is identically zero. Indeed, if $z_n \to \infty$, then

$$c(f) = z_n \cdot c(f) = z_n \cdot \tilde{\xi}(f) - z_n \cdot \xi(f) = z_n \cdot \tilde{\xi}(f) - \xi(z_n \cdot f).$$

Since $\xi(z_n \cdot f) \to 0$, we conclude that $z_n \cdot \tilde{\xi}(f) \to c(f)$ whenever $z_n \to \infty$. However, Eq. (6.2) gives $(z_n \cdot \tilde{\xi}(f))(0) = \tilde{\xi}(f)(z_n) \to 0$, hence $c(f) = 0$. In other words, $\xi(f) = \tilde{\xi}(f)$ for all compactly supported f.

To show that continuous equivariant right-inverse to $\bar{\partial}$ cannot exist, it therefore suffices to verify that $\tilde{\xi}$ does not extend continuously to all of $\mathrm{Free}(C^\infty)$. Let f_n be an element of C^∞ such that

1. $f_n(z) = z$ for all $z \in n\mathbb{D}$
2. $f_n(z) = 0$ for all $z \in \mathbb{C} \setminus (n+1)\mathbb{D}$
3. $|f_n(z)| \leq n+1$ for all $z \in \mathbb{C}$.

Clearly f_n converges to the identity map $z \mapsto z$. However, $\tilde{\xi}(f_n)$ diverges, and in fact, $\tilde{\xi}(f_n)(0) \to \infty$ as $n \to \infty$. Indeed,

$$\begin{aligned}
|2\pi \mathrm{i}\, \tilde{\xi}(f_n)(0)| &= \left| \int_{n\mathbb{D}} \frac{z}{z} + \int_{(n+1)\mathbb{D}\setminus n\mathbb{D}} \frac{f_n}{z} \right| \geq \pi n^2 - \int_{(n+1)\mathbb{D}\setminus n\mathbb{D}} \frac{n+1}{|z|} \\
&\geq \pi n^2 - 2(\pi(n+1)^2 - \pi n^2) \\
&= \pi n^2 - 4\pi n - 2\pi \to \infty \text{ as } n \to \infty.
\end{aligned}$$

This completes the proof. □

7 Lack of periodic equivariant inverses

In most of the applications considered so far, we have concentrated on free $\mathbb{R}^d$-actions, with Section 5.11 being a notable exception. The situation with the periodic parts of the actions discussed in Section 5 is more subtle. In this section we show that equivariant right-inverses to div, Δ and $\bar{\partial}$ generally do not exist on the periodic parts of their domain.

7.1 Divisors and entire functions

The stabilizer of a divisor under the argument shift action must be a closed subgroup of $\mathbb{C}$. Furthermore, since the support of any divisor is a discrete subset of $\mathbb{C}$, the stabilizer of a non-trivial divisor is a discrete subgroup of $\mathbb{C}$, hence isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}^2$. Let $\mathcal{D}_1^+$ be the collection of those non-negative divisors whose stabilizer is of the form $\lambda\mathbb{Z}$ for some non-zero $\lambda \in \mathbb{C}$ and $\mathcal{D}_2^+$ be those non-trivial non-negative divisors that have two independent periods. We therefore get a partition

$$\mathcal{D}^+ = \mathrm{Free}(\mathcal{D}^+) \sqcup \mathcal{D}_1^+ \sqcup \mathcal{D}_2^+ \sqcup \{\varnothing\}.$$

Similarly, the stabilizer of a non-constant entire function must also be a discrete subgroup of $\mathbb{C}$, which gives us a similar partition of $\mathcal{E}$ into

$$\mathcal{E} = \mathrm{Free}(\mathcal{E}) \sqcup \mathcal{E}_1 \sqcup \mathcal{E}_2 \sqcup \{\text{constants}\}.$$

Any equivariant right-inverse ξ to div must respect these partitions and thus satisfy $\xi(\mathrm{Free}(\mathcal{D}^+)) \subseteq \xi(\mathrm{Free}(\mathcal{E}))$ and $\xi(\mathcal{D}_i^+) \subseteq \xi(\mathcal{E}_i)$, $i = 1, 2$. However, $\mathcal{E}_2$—the space of non-constant doubly periodic entire functions—is empty, whereas $\mathcal{D}_2^+$ is not. In particular, there can't be any equivariant right-inverses to div on $\mathcal{D}_2^+$.

The set $\mathcal{E}_1$ is non-empty, and the divisor map

$$\mathrm{div} : \mathcal{E}_1 \to \mathcal{D}_1^+$$

is a Borel surjection. To see this, let $\mathcal{E}_{1,1}$ and $\mathcal{D}_{1,1}^+$ denote the subspaces of (non-constant) holomorphic functions and divisors, respectively, which have period 1. These are Polish subspaces of $\mathcal{E}_1$ and $\mathcal{D}_1^+$, correspondingly. It suffices to see that the restriction $\mathrm{div} : \mathcal{E}_{1,1} \to \mathcal{D}_{1,1}^+$ is surjective. The exponential map $w = e^{2\pi i z}$ transfers the question to surjectivity of the divisor map for analytic functions in the punctured plane $\mathbb{C} \setminus \{0\}$, which is classical for arbitrary domains ([45, Ch. 4]).

A non-measurable equivariant inverse to $\mathrm{div}\colon \mathcal{E}_1 \to \mathcal{D}_1^+$ can easily be constructed using the axiom of choice. It turns out, however, that contrary to the free part of the action, there is no Borel equivariant inverse.

Theorem 7.1. *There is no Borel equivariant map $\xi : \mathcal{D}_1^+ \to \mathcal{E}_1$ such that $\mathrm{div}(\xi(d)) = d$ for all $d \in \mathcal{D}_1^+$.*

We begin with a lemma which is an easy consequence of the Lindelöf maximum principle. For the reader's convenience, we include the proof.

Lemma 7.2. *Let $s < 0 < t$ be reals, $S_{s,t}$ be the strip $s < \mathrm{Im}(z) < t$, and $f : \bar{S}_{s,t} \to \mathbb{C}$ be a bounded continuous function holomorphic on $S_{s,t}$. If $|f(z)| \le M$ for all $z \in \partial S_{s,t}$, then $|f(z)| \le M$ for all $z \in S_{s,t}$.*

Proof. Set $S = S_{s,t}$ and let $z_0 \in \mathbb{C} \setminus \bar{S}$ be such that $|z - z_0| \geq 1$ for all $z \in S$. For instance, we may take $z_0 = \mathrm{i}(t+1)$. Given $\varepsilon > 0$, let $g_\varepsilon(z) = \frac{f(z)}{(z-z_0)^\varepsilon}$, where we take the branch cut from z_0 parallel to the imaginary axes that avoids the strip S. The function $g_\varepsilon(z)$ is holomorphic on S and continuous on $\bar{S}$. Furthermore, the choice of z_0 ensures that $|g_\varepsilon(z)| \leq M$ for all $z \in \partial S$. Since f is bounded by assumption, $\lim_{|z|\to\infty} |g_\varepsilon(z)| = 0$. In particular, for a sufficiently large $R > 0$ and $S^R = S \cap \{z : |\mathrm{Re}(z)| \leq R\}$, we have $|g_\varepsilon(z)| \leq M$ for all $z \in \partial S^R$. Maximum modulus principle applies and shows that $|g_\varepsilon(z)| \leq M$ for all $z \in S^R$ and therefore also for all $z \in S$ as R can be arbitrarily large. We conclude that $|f(z)| \leq M|z - z_0|^\varepsilon$ for all $\varepsilon > 0$ and all $z \in S$. Sending $\varepsilon \to 0$ gives the desired $|f(z)| \leq M$ on all of S. □

Lemma 7.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant periodic entire function with period $z = 1$. Let*

$$M_f(s) = \max\{|f(\mathrm{i}s + t)| : t \in \mathbb{R}\} = \max\{|f(\mathrm{i}s + t)| : t \in [0,1]\}$$

be the maximum modulus of f on the line $\{\mathrm{i}s + t : t \in \mathbb{R}\}$, where $s \in \mathbb{R}$. Then

$$\textit{either} \quad \lim_{s\to+\infty} M_f(s) = +\infty \quad \textit{or} \quad \lim_{s\to-\infty} M_f(s) = +\infty.$$

Proof. Suppose towards a contradiction that there are $s_n \to -\infty$ and $t_n \to +\infty$ such that $M_f(s_n) \leq M$ and $M_f(t_n) \leq M$ for some $M \in \mathbb{R}^{\geq 0}$ and all n. Lemma 7.2 applies to strips S_{s_n,t_n} and functions $f|_{\bar{S}_{s_n,t_n}}$. We conclude that $|f(z)| \leq M$ on each string S_{s_n,t_n} and therefore also on all of $\mathbb{C} = \bigcup_n S_{s_n,t_n}$. Liouville's theorem assures that f must be constant, contrary to the assumption. □

Proposition 7.4. *The action $\mathbb{C} \curvearrowright \mathcal{E}_{1,1}$ has a Borel transversal, i.e., a Borel set $T \subset \mathcal{E}_{1,1}$ that intersects each orbit in exactly one point.*

Proof. Consider a 1-periodic $f \in \mathcal{E}_{1,1}$ and let k be large enough to ensure that the set $W_{f,k} = \{s \in \mathbb{R} : M_f(s) \leq k\}$ is non-empty. Note that $W_{f,k}$ is necessarily closed, whereas Lemma 7.3 guarantees that it must be bounded from above or from below, and possibly both. Let's take the set $\widetilde{T}_k \subseteq \mathcal{E}_{1,1}$ of 1-periodic entire functions for which $M_f(0) \leq k$ and $s = 0$ is either the smallest or the largest element of $W_{f,k}$. This set is Borel. In fact, $\widetilde{T}_k = (\widetilde{T}_k^0 \cap \widetilde{T}_k^+) \cup (\widetilde{T}_k^0 \cap \widetilde{T}_k^-)$, where

$$\widetilde{T}_k^0 = \Big\{f \in \mathcal{E}_{1,1} : \forall r \in [0,1] \cap \mathbb{Q} \quad |f(r)| \leq k\Big\},$$

$$\widetilde{T}_k^+ = \Big\{f \in \mathcal{E}_{1,1} : \forall n\ \exists m\ \forall s \in \mathbb{Q}^{>1/n}\ \exists r \in [0,1] \cap \mathbb{Q} \quad |f(\mathrm{i}s + r)| > k + \frac{1}{m}\Big\},$$

$$\widetilde{T}_k^- = \Big\{f \in \mathcal{E}_{1,1} : \forall n\ \exists m\ \forall s \in \mathbb{Q}^{<-1/n}\ \exists r \in [0,1] \cap \mathbb{Q} \quad |f(\mathrm{i}s + r)| > k + \frac{1}{m}\Big\}.$$

The set $\widetilde{T}_k$ is invariant under the real shifts: if $f \in \widetilde{T}_k$ then $z \mapsto f(z+r) \in \widetilde{T}_k$, $r \in \mathbb{R}$. By 1-periodicity, we have the action of $\mathbb{T} = [0,1]$ on $\widetilde{T}_k$. Since Borel actions of compact groups admit Borel transversals, we can pick a Borel transversal $S_k \subseteq \widetilde{T}_k$ for the action of $\mathbb{T}$. Note that S_k intersects each $\mathbb{C}$-orbit of $f \in W_{f,k}$ in one or two points, depending on whether $W_{f,k}$ is bounded from only one side or from both. Choosing in a Borel way either of the two points on those orbits where $|S_k \cap (\mathbb{C} \cdot f)| = 2$, gives a Borel transversal $T_k \subseteq S_k$ for the restriction of the action onto $Z_k = \{f \in \mathcal{E}_{1,1} : W_{f,k} \neq \varnothing\}$.

It only remains to glue the sets T_k together into a single Borel transversal by setting $T = \bigcup_k (T_k \setminus \bigcup_{\ell<k} Z_\ell)$. □

Remark 7.5. In fact, the action of $\mathbb{C} \curvearrowright \mathcal{E}_1$ on all of $\mathcal{E}_1$ admits a Borel transversal. Let $\alpha : \mathcal{E}_{1,1} \to Y$ be a Borel reduction of $E_{\mathbb{C} \curvearrowright \mathcal{E}_{1,1}}$ to the equality on some standard Borel space Y. There is a Borel $E_{\mathbb{C} \curvearrowright \mathcal{E}_1}$-invariant function $\pi : \mathcal{E}_1 \to \mathbb{C} \setminus \{0\}$ such that $\pi(f)$ is a generator for the group of periods of f (see, for instance, [48, Cor. 5.3]). Note that the function $\widetilde{f}(z) = f(\pi(f)z)$ belongs to $\mathcal{E}_{1,1}$ and therefore

$$\widetilde{\alpha} : \mathcal{E}_1 \to Y \times (\mathbb{C} \setminus \{0\}),$$

given by $\widetilde{\alpha}(f) = (\alpha(\widetilde{f}), \pi(f))$, is a reduction from $E_{\mathbb{C} \curvearrowright \mathcal{E}_1}$to the equality on $Y \times (\mathbb{C} \setminus \{0\})$. Thus, $E_{\mathbb{C} \curvearrowright \mathcal{E}_1}$ is concretely classifiable and therefore admits a Borel transversal.

The action $\mathbb{C} \curvearrowright \mathcal{D}^+_{1,1}$, on the other hand, does not admit Borel transversals. Indeed, one can embed the free part of the Bernoulli shift on $2^{\mathbb{Z}}$ into $E_{\mathbb{C} \curvearrowright \mathcal{D}^+_{1,1}}$ by viewing a sequence $x \in 2^{\mathbb{Z}}$ as a $\{0,1\}$-valued divisor along $\mathrm{i}\mathbb{Z}$. For the benefit of the readers unfamiliar with this type of argument, we carefully spell out the details of the following standard ideas.

Lemma 7.6. *Consider the action $\mathbb{Z} \curvearrowright \mathcal{D}^+_{1,1}$, where the generator τ acts by $d(z) \mapsto d(z - \mathrm{i})$. The action $\mathbb{Z} \curvearrowright \mathcal{D}^+_{1,1}$ does not admit a Borel transversal.*

Proof. Let $\mathrm{Free}(2^{\mathbb{Z}})$ denote the set of non-periodic binary sequences indexed by $\mathbb{Z}$, and denote by s the (left) shift map $(s \cdot x)(i) = x(i-1)$, $x \in \mathrm{Free}(2^{\mathbb{Z}})$. The following two items ensure the lack of Borel transversals.

1. The action $s \curvearrowright \mathrm{Free}(2^{\mathbb{Z}})$ does not admit a Borel transversal.

2. There exists a continuous equivariant injection $\varphi : \mathrm{Free}(2^{\mathbb{Z}}) \to \mathcal{D}^+_{1,1}$.

Any Borel transversal T for $\mathbb{Z} \curvearrowright \mathcal{D}_{1,1}^{+}$, had it existed, would then pull back to a Borel transversal $\varphi^{-1}(T)$ for $s \curvearrowright \mathrm{Free}(2^{\mathbb{Z}})$, violating (1).

It thus remains to establish Properties (1) and (2) above. Suppose towards a contradiction that there exists a Borel transversal B for $s \curvearrowright \mathrm{Free}(2^{\mathbb{Z}})$. The sets $B_n = s^n \cdot B$ are pairwise disjoint (by the freeness of the action) and form a partition of $\mathrm{Free}(2^{\mathbb{Z}})$. However, the shift map has invariant probability measures. For instance, the product measure $\prod_{k\in\mathbb{Z}} \mu_0$, where μ_0 is the Bernoulli measure on $\{0,1\}$, $\mu_0(0) = 1/2 = \mu_0(1)$. Sets B_n must therefore all have the same measure, which implies $\mu(B_n) = 0$ by finiteness of μ. The conclusion $\mu(X) = 0$ leads to a contradiction. The proof of Property (1) is complete.

It remains to establish the existence of the equivariant Borel injection φ. Given $x \in 2^{\mathbb{Z}}$, let d_x be the divisor given by

$$d_x(z) = \begin{cases} x(k) & \text{if } z = k\mathrm{i} + m \text{ for some } k, m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $d_x \in \mathcal{D}_{1,1}^{+} \sqcup \mathcal{D}_2^{+}$ for all $x \in 2^{\mathbb{Z}}$ and $d_x \in \mathcal{D}_{1,1}^{+}$ precisely when $x \in \mathrm{Free}(2^{\mathbb{Z}})$. The map φ is then given by

$$\mathrm{Free}(2^{\mathbb{Z}}) \ni x \mapsto \varphi(x) = d_x \in \mathcal{D}_{1,1}^{+}.$$

Continuity and injectivity of φ are both clear, and a direct computation verifies that $d_{s\cdot x} = \tau d_x$, so φ is equivariant. □

Proposition 7.7. *The action $\mathbb{C} \curvearrowright \mathcal{D}_{1,1}^{+}$ does not have Borel transversals.*

Proof. Let $K = \{s + \mathrm{i}t : 0 \le s, t < 1\} \subset \mathbb{C}$. If $T \subseteq \mathcal{D}_{1,1}^{+}$ were a Borel transversal for $\mathbb{C} \curvearrowright \mathcal{D}_{1,1}^{+}$, then $K \cdot T$ would be a Borel transversal for $\mathbb{Z} \curvearrowright \mathcal{D}_{1,1}^{+}$, which contradicts Lemma 7.6. □

Proof of Theorem 7.1. Suppose towards a contradiction that there existed a Borel equivariant map $\xi : \mathcal{D}_1^{+} \to \mathcal{E}_1$ such that $\mathrm{div}(\xi(d)) = d$. Since div maps $\mathcal{E}_{1,1}$ onto $\mathcal{D}_{1,1}^{+}$ and ξ is equivariant, ξ maps $\mathcal{D}_{1,1}^{+}$ to $\mathcal{E}_{1,1}$.

Let T be a Borel transversal for $\mathcal{E}_{1,1}$, whose existence is ensured by Proposition 7.4. Then the pre-image $\xi^{-1}(T)$ is a Borel transversal for $\mathcal{D}_{1,1}^{+}$, contradicting Lemma 7.6. □

Remark 7.8. We note that, in contrast to Proposition 7.4, the action $\mathbb{C} \curvearrowright \mathcal{MR}_1$ on the space of 1-periodic meromorphic functions is not concretely classifiable. In fact, there are compact invariant subsets of $\mathbb{C} \curvearrowright \mathcal{MR}_1$.

Consider a 1-periodic Yosida-type normal meromorphic function[8], which is not doubly-periodic. For instance, let

$$S_{a,b}(z) = e^{\pi \mathrm{i}(a-b)} \times \frac{\sin \pi(z-a)}{\sin \pi(z-b)},$$

where $a, b \in [0,1]$. This function is meromorphic, 1-periodic, and satisfies

$$S_{a,b}(z) = 1 + O(1)e^{-2\pi|y|}$$

for $|y| \geq c > 0$. Hence the infinite product

$$F(z) = \prod_{k \in \mathbb{Z}} S_{a_k,b_k}(z - \mathrm{i}k)$$

converges very fast to a 1-periodic meromorphic function. If the points a_k, b_k are uniformly separated, $|a_k - b_k| \geq c > 0$ for all $k \in \mathbb{Z}$, then the function F is normal, i.e., it's orbit is a compact subset of $\mathcal{MR}$, and each limiting meromorphic function is not constant. The freedom in the choice of a_k, b_k allows us to avoid doubly-periodic signed divisors in the limit, and hence, we can ensure that the closure of the orbit of F is a subset of $\mathcal{MR}_1$.

The existence of compact invariant subsets of $\mathcal{MR}_1$ guarantees that $\mathbb{C} \curvearrowright \mathcal{MR}_1$ is not concretely classifiable. Moreover, the action admits invariant probability measures, which can be obtained via the classical Krylov–Bogolyubov construction.

7.2 Subharmonic functions and Riesz measures

As shown in Section 5.11, if Γ is a closed subgroup of $\mathbb{R}^d$ with dimension $\dim \Gamma \leq d-2$, then there exists a Borel equivariant $\xi : \mathcal{M}_\Gamma^+(\mathbb{R}^d) \to \mathcal{SH}_\Gamma(\mathbb{R}^d)$ that is a right-inverse to Δ. We now show that such equivariant inverses do not exist when $\dim \Gamma = d-1$.

The group Γ is isomorphic to a group of the form $\mathbb{R}^q \times \mathbb{Z}^p$ for some unique choice of p and q satisfying $p+q \leq d$, where $p+q$ is $\dim \Gamma$. Moreover, there exists a matrix $B \in \mathrm{GL}_d(\mathbb{R})$ such that $\Gamma = B(\mathbb{R}^q \times \mathbb{Z}^p \times \{0\}^{d-p-q})$.

We aim to prove the following theorem.

[8] This is the class of meromorphic functions f on $\mathbb{C}$ such that the family of translates $\{f(z+w) : w \in \mathbb{C}\}$ is normal with respect to locally uniform convergence and such that no limit function is constant. This class was thoroughly studied by Favorov in [16].

Theorem 7.9. *Suppose that Γ is a closed subgroup of $\mathbb{R}^d$ of dimension $\dim \Gamma \geq d-1$. There are no Borel equivariant maps $\xi : \mathcal{M}_\Gamma^+(\mathbb{R}^d) \to \mathcal{SH}_\Gamma(\mathbb{R}^d)$ satisfying $\Delta\xi(\mu) = \mu$ for all $\mu \in \mathcal{M}_\Gamma^+(\mathbb{R}^d)$. If $\dim \Gamma = d-1$, then the argument shift action $\mathbb{R}^d \curvearrowright \mathcal{SH}_\Gamma(\mathbb{R}^d)$ admits a Borel transversal. Furthermore, the only subharmonic functions whose stabilizer has dimension d are the constant functions.*

7.2.1 The case $\dim \Gamma = d-1$.

Choose a $(d-1)$-dimensional vector subspace V of $\mathbb{R}^d$ and a $(d-1)$-dimensional lattice Λ such that $\Lambda \subseteq \Gamma \subseteq V$. For instance, if $\Gamma = B(\mathbb{R}^q \times \mathbb{Z}^p \times \{0\})$, we can take

$$V = B(\mathbb{R}^{d-1} \times \{0\}) \quad \text{and} \quad \Lambda = B(\mathbb{Z}^{d-1} \times \{0\}).$$

We denote by $D_0 = D_0(\Lambda)$ a fundamental domain of the lattice Λ.

Let e be one of the two (parallel) unit vectors in $V^\perp$. Any element in $\mathbb{R}^d$ is therefore uniquely represented as $x + se$, with $x \in V$ and $s \in \mathbb{R}$. Given a function $u \in \mathcal{SH}_\Gamma(\mathbb{R}^d)$ and $s \in \mathbb{R}$, define

$$M_u(s) = \sup\Big\{u(x+se) : x \in V\Big\} = \sup\Big\{u(x+se) : x \in V \cap D_0\Big\}, \tag{7.1}$$

where the last equality follows from the Γ-periodicity of u. Since the closure $\overline{V \cap D_0}$ is compact and u is upper semi-continuous, the supremum is finite for all s. The following lemma is analogous to Lemma 7.3 and is the key to establishing the existence of Borel transversals of $\mathbb{R}^d \curvearrowright \mathcal{SH}_\Gamma(\mathbb{R}^d)$.

Lemma 7.10. *Let $\dim(\Gamma) = d-1$. For any $u \in \mathcal{SH}_\Gamma(\mathbb{R}^d)$ one has*

$$\lim_{s\to+\infty} M_u(s) = +\infty \quad \textit{or} \quad \lim_{s\to-\infty} M_u(s) = +\infty, \tag{7.2}$$

not excluding the case that both limits hold simultaneously.

Remark 7.11. Note that, in contrast to the planar case, there are plenty of bounded subharmonic functions in $\mathbb{R}^d$ for $d \geq 3$. However, the lemma asserts that boundedness, even along sequences of affine hyperplanes of the form

$$\{x + t_j e : x \in V\}, \quad \{x - s_j e : x \in V\},$$

is incompatible with Γ-periodicity.

Before we proceed with the proof, we recall a subharmonic counterpart of Lemma 7.2.

Proposition 7.12. *Let u be an upper bounded subharmonic function in the domain $D = \{x \in \mathbb{R}^d : |x_d| < 1\}$, with $\limsup_{y \to x} u(y) \le 0$ for $x \in \partial D$. Then $u(x) \le 0$ everywhere on D.*

The proof is a reduction to the classical maximum principle for subharmonic function on bounded domains (see, for example, [23, Thm. 2.4]).

Proof. For $\varepsilon > 0$, let

$$u_\varepsilon(x) = u(x) - \varepsilon\left(x_1^2 + \cdots + x_{d-1}^2\right) + \varepsilon(d-1)x_d^2.$$

Then $\Delta u_\varepsilon = \Delta u$, so the function u_ε is also subharmonic. On the boundary ∂D, we have the upper bound $u_\varepsilon \le (d-1)\varepsilon$ (understood in the sense of upper limits), and by taking R large enough we can ensure that the same upper bound holds on the whole of $\partial(B(0,R) \cap D)$. Hence, by the classical maximum principle we have $u_\varepsilon(x) \le (d-1)\varepsilon$ for all $x \in B(0,R) \cap D$. Since R can be chosen arbitrarily large, we get $u_\varepsilon(x) \le (d-1)\varepsilon$ throughout D. In terms of the original subharmonic function u, this means that

$$u(x) \le (d-1)\varepsilon + \varepsilon\left(x_1^2 + \ldots + x_{d-1}^2\right) - \varepsilon(d-1)x_n^2$$

on D, which in particular implies that

$$u(x) = \mathrm{O}(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

the error term estimate being uniform on compact subsets. Letting $\varepsilon \to 0$ we obtain the desired conclusion. □

We turn to the proof of Lemma 7.10.

Proof of Lemma 7.10. For the sake of contradiction, assume that (7.2) fails. Then, there exists a constant M such that for some sequences $(s_j)_j$ and $(t_j)_j$ tending to $+\infty$, both $M_u(-s_j)$ and $M_u(t_j)$ are bounded from above by M. That is to say, the upper bound $u \le M$ holds on the parallel affine hyperplanes

$$\{x - s_j e : x \in V\} \quad \text{and} \quad \{x + t_j e : x \in V\}, \qquad \text{for } j \in \mathbb{N}.$$

Moreover, for each j individually, u is bounded from above in the domain

$$F_j = \{x + se : x \in V, -s_j < s < t_j\}.$$

But then $u \leq M$ on F_j by Proposition 7.12, and, since F_j exhaust $\mathbb{R}^d$, u is upper bounded by M globally. Let $\mu = \Delta u$ be the Riesz mass of u. By [23, Thm. 3.20], we have that

$$\int_0^\infty \frac{\mu(B(0,t))}{t^{d-1}} dt < +\infty. \tag{7.3}$$

We conclude the proof by showing that (7.3) is incompatible with $(d-1)$-periodicity. Let $u_1, \dots, u_{d-1}$ be a linearly independent set of elements of Γ, and consider the truncated rectangular cylinder

$$Q_R = \left\{ se + \sum_{j=1}^{d-1} x_j u_j : 0 \leq x_j \leq 1,\ |s| \leq R \right\}.$$

For large enough R, $\mu(Q_R) \geq c_0 > 0$, and by the Γ-invariance of μ, we have $\mu(Q_R + x) = \mu(Q_R)$ for any x in the lattice generated by $\Lambda = \{u_1, \dots, u_{d-1}\}$. Moreover, there exists a constant $d_0 > 0$, depending only on R and Λ, such that for t large enough the ball $B(0,t)$ contains at least $d_0 t^{d-1}$ disjoint translates of Q_R by elements of Λ. Hence, $\mu(B(0,t)) \geq d_0 t^{d-1} \mu(Q_R) \geq c_0 d_0 t^{d-1}$. But this contradicts the integrability (7.3), and we have reached a contradiction. □

Empowered by Lemma 7.10, the action $\mathbb{R}^d \curvearrowright \mathcal{SH}_\Gamma(\mathbb{R}^d)$ can be seen to admit a Borel transversal, by an argument that parallels Proposition 7.4. On the other hand, a construction similar to the one given in Lemma 7.6 for divisors, shows that $\mathbb{R}^d \curvearrowright \mathcal{M}_\Gamma^+(\mathbb{R}^d)$ does not. These two facts let us conclude that there are no $\mathbb{R}^d$-equivariant Borel right-inverses to Δ on $\mathcal{SH}_\Gamma(\mathbb{R}^d)$.

7.2.2 The case $\dim \Gamma = d$

It is even simpler to exclude existence of equivariant inverses to Δ on $\mathcal{M}_\Gamma^+(\mathbb{R}^d)$, when stabilizers have full dimension. The key observation is the following.

Lemma 7.13. *Let Γ be a closed subgroup of $\mathbb{R}^d$ of dimension d. Then any Γ-invariant subharmonic function is constant.*

In contrast, $\mathcal{M}_\Gamma^+(\mathbb{R}^d)$ contains non-zero elements whenever Γ is a proper subgroup of $\mathbb{R}^d$. The proof of Lemma 7.13 therefore concludes the proof of Theorem 7.9.

Proof. Since any subharmonic function can be approximated by smooth subharmonic functions, it is enough to show that any smooth Γ-invariant subharmonic function is constant. If $u \in C^2(\mathbb{R}^d)$, then for any bounded domain D with regular boundary, Green's formula gives

$$\int_D \Delta u \, dm = \int_{\partial D} \nabla u \cdot \mathrm{n} \, dS, \tag{7.4}$$

where dS denotes the surface element and n is the outward unit normal to D. We apply this with D being a box

$$D = \left\{ \sum_{j=1}^{d} x_j u_j : x \in (0,1)^d \right\}$$

with $\{u_j : 1 \le j \le d\}$ a set of linearly independent elements of Γ. Then, by periodicity, the right-hand side of (7.4) vanishes (for any k, the integrand coincides on the two faces $\{\sum_j x_j u_j : x_k = 0\}$ and $\{\sum_j x_j u_j : x_k = 1\}$, but the direction of the normal is reversed). Hence u is harmonic, and, since it is Γ-periodic, it is bounded and thus constant by the Liouville's theorem for harmonic functions [43]. □

7.3 Non-existence for the $\bar{\partial}$-equation

Let $C_1^\infty(\mathbb{C}, \mathbb{C})$ denote the space of smooth complex-valued functions on $\mathbb{C}$ with a discrete 1-dimensional stabilizer. The goal of this section is to prove the following result.

Theorem 7.14. *There does not exist any equivariant Borel right-inverse to $\bar{\partial}$ on $C_1^\infty(\mathbb{C}, \mathbb{C})$.*

Proof. Informally speaking, the idea is to start with an appropriate *non-stationary* random $C_1^\infty(\mathbb{C}, \mathbb{C})$-function f such that $g = \bar{\partial} f$ is stationary. If an equivariant Borel right-inverse Ψ to $\bar{\partial}$ existed, then $F = \Psi(g)$ would be a *stationary* random function such that $F - f$ is entire. But by the ergodic theorem combined with 1-periodicity, $F - f$ can be shown to be a constant, contradicting the non-stationarity of f.

Let us fill in the missing details. We let ξ be a stationary random element of $C_1^\infty(\mathbb{C}, \mathbb{C})$. For definiteness, we may take a stationary process ν on $\mathbb{R}$ with smooth and non-periodic trajectories, and put

$$\xi(x + \mathrm{i}y) = \sin(2\pi x)\nu(y).$$

Then let $f(x + \mathrm{i}y) = -2\mathrm{i}y + \xi(x + \mathrm{i}y)$. Since $\bar{\partial} f = 1 + \bar{\partial}\xi$, the function f satisfies the desired properties. Next, note that $\partial(F - f)$ is a 1-periodic stationary random entire function, and hence constant. The argument is similar to the one used in Section 7.1. Indeed, by the ergodic theorem, there is a constant M and sequences $s_j, t_j \to +\infty$ such that $|\partial(F-f)|$ is upper bounded by M on

$$\{x - \mathrm{i}s_j : x \in [0,1]\} \cup \{x + \mathrm{i}t_j : x \in [0,1]\}\,.$$

By Lindelöf's maximum principle we have that $|\partial(F-f)(x+\mathrm{i}y)| \le M$ for $y \in [-s_j, t_j]$. But then $|\partial(F-f)|$ is bounded on $\mathbb{C}$ and hence constant. It thus follows that $F(z) = f(z) + az + b$, and by 1-periodicity we can further claim that $F(z) = f(z) + b$.

To finish the proof, we claim that this is a contradiction. Note that there are non-stationary random functions which become stationary after adding a constant—take e.g., $\nu(t) - \nu(0)$ with ν a stationary process on $\mathbb{R}$ as above. However, this is not the case for our random function f. By the Birkhoff ergodic theorem, we get that for fixed $M > 0$,

$$\lim_{R\to\infty} \frac{\left|\left\{y \in [-R,R] : \max_{x\in[0,1]} |\xi(x+\mathrm{i}y)| \le M\right\}\right|}{2R} = \mathbb{E}\left(\mathbb{1}_{\{|\xi(x)|\le M \text{ for } x\in[0,1]\}} | \mathcal{I}\right),$$

where $\mathcal{I}$ is the σ-algebra of invariant events. In particular, as $M \to \infty$ the right-hand side tends to 1 in L^1. Looking at the corresponding limit for F, we have in the same way that

$$\lim_{M\to\infty} \lim_{R\to\infty} \frac{\left|\left\{y \in [-R,R] : \max_{x\in[0,1]} |F(x+\mathrm{i}y)| \le M\right\}\right|}{2R} = 1 \tag{7.5}$$

in L^1. Now, if $|\xi(x+\mathrm{i}y)| \le M$ and $|y| > M$, then the reverse triangle inequality gives the lower bound $|f(x+\mathrm{i}y)| \ge |2y| - |\xi| > M$ so that

$$\left\{\max_{x\in[0,1]} |f(x+\mathrm{i}y)| > M\right\} \supseteq \left\{\max_{x\in[0,1]} |\xi(x+\mathrm{i}y)| \le M\right\} \cap \left\{|y| > M\right\}.$$

As a consequence, the ergodic averages pertaining to f satisfy

$$\limsup_{R\to\infty} \frac{\left|\left\{y \in [-R,R] : \max_{x\in[0,1]} |f(x+\mathrm{i}y)| \le M\right\}\right|}{2R} \le 1 - \mathbb{E}\left(\mathbb{1}_{|\xi(x)|\le M \text{ for } x\in[0,1]} | \mathcal{I}\right).$$

Taking finally the limit as $M \to \infty$, we have the L^1-convergence

$$\lim_{M\to\infty} \lim_{R\to\infty} \frac{\left|\left\{y \in [-R,R] : \max_{x\in[0,1]} |f(x+\mathrm{i}y)| \le M\right\}\right|}{2R} = 0. \tag{7.6}$$

The conclusions (7.5) and (7.6) are clearly incompatible with $F - f$ being constant, and hence we have reached a contradiction. □

8 Non-existence condition for equivariant inverses

In this section, we continue our investigation of the lifting problem. While Theorem 4.9 gives sufficient conditions for the existence of Borel liftings, here we give several examples where liftings do not exist.

The prototype example is the Borel entire function F generated by the action $\mathbb{C} \curvearrowright \mathcal{E}$ by translation of the argument $f \mapsto f(\cdot + z)$, and the map $\pi\colon \mathcal{E} \to \mathcal{E}$, $f \mapsto f'$. As it turns out, this map π does not have a Borel equivariant (right) inverse, i.e., F does not have a Borel primitive. There are several other natural maps π without Borel equivariant right inverses. The general framework, that covers the derivative, as well as other maps π, is given in Theorem 8.1 and uses the classical Birkhoff observation that the action $\mathbb{C} \curvearrowright \mathcal{E}$ by translation of the argument has a dense orbit, (i.e., is topologically transitive) combined with the Baire category argument.

8.1 Prerequisites

8.1.1 The Baire property

Recall that a subset $A \subseteq Z$ of a Polish space Z is *meager* (or of the first category) if there are nowhere dense sets $F_n \subseteq Z$, $n \in \mathbb{N}$, such that $A \subseteq \bigcup_n F_n$. The set A is *comeager* (or residual) if its complement $Z \setminus A$ is meager, that is, there exist dense open sets U_n, $n \in \mathbb{N}$, such that $A \supseteq \bigcap_n U_n$. By the Baire category theorem, every comeager set is dense in Z.

The set $A \subseteq Z$ has the *Baire property* if there exists an open set U such that the set $A \mathbin{\triangle} U$ is meager, i.e., A can be represented in the form $A = U \mathbin{\triangle} M$, where U is open, and M is meager. Note that all open sets and all meager sets have the Baire property. We recall several basic facts about the sets with the Baire properties, which we will be using.

1. The class of sets having the Baire property is a σ-algebra containing all open sets. In particular, all Borel sets have the Baire property.

2. A somewhat deeper fact is that all analytic sets have the Baire property [32, 21.6], [51, Thm. 4.3.2]. To see this, one needs to observe that closed sets possess the Baire property, and then to use the fact that the class of sets having the Baire

property is closed under the Suslin scheme, and that every analytic set can be obtained by applying the Suslin scheme to closed sets.

3. For a set $A \subseteq Z$ with the Baire property, put

$$U(A) = \bigcup \Big\{ U : U \text{ open}, U \setminus A \text{ is meager} \Big\}.$$

Then,

(a) $A \bigtriangleup U(A)$ is meager;

(b) the open set $U(A)$ is *regular*, i.e., it equals the interior of its closure;

(c) $U(A)$ is the unique regular open set U such that $A \bigtriangleup U$ is meager.

For the details, see [32, 8G].

8.1.2 Generic ergodicity

Let $G \curvearrowright Z$ be a continuous action of a Polish group G on a Polish space Z. The action is called *generically ergodic* if every G-invariant set in Z with the Baire property is either meager, or comeager.

Note that if $A \subseteq Z$ is a G-invariant set that has the Baire property, and $A = U \bigtriangleup M$ for a regular open U and a meager M, then U is G-invariant. Indeed, for $g \in G$,

$$U \bigtriangleup M = A = gA = gU \bigtriangleup gM$$

implies $U = gU$ since both U and gU are regular open, while M and gM are both meager.

Furthermore, generic ergodicity is equivalent to the existence of a dense orbit (aka *topological transitivity*). Indeed, if $G \curvearrowright Z$ is generically ergodic and $(U_n)_n$ is a countable basis of non-empty open sets for the topology of Z, then the G-invariant set $\bigcap_n GU_n$ is comeager and therefore dense, and, in fact, the orbit of each $z \in \bigcap_n GU_n$ is dense. Conversely, let $G \curvearrowright Z$ have a dense orbit. Denote by A a G-invariant set with the Baire property. Then $A = U \bigtriangleup M$, where U is regular open and M is meager. Then U is also G-invariant, so it contains a dense orbit and is therefore dense (assuming U is non-empty). Consequently, either $U = \varnothing$ and A is meager, or U is a dense open set and A is comeager; in other words, the action is generically ergodic.

8.2 Non-existence condition

We are now ready to formulate our *non-existence condition* for Borel equivariant liftings. Here is the setting of our result:

1. Let $G \curvearrowright Z$ be a continuous action by a Polish group G on a Polish space Z, and $G \curvearrowright Y$ be a Borel action on a standard Borel space Y.

2. Let $\pi : Z \to Y$ a Borel equivariant map. Our aim is to give conditions under which π does not have a Borel equivariant right inverse.

3. Suppose that $H \curvearrowright Z$ is a free continuous action by a Polish group H, whose orbits are classified by π:

$$\pi(z_1) = \pi(z_2) \iff z_1 E_H z_2. \tag{8.1}$$

4. Let $\tau : G \curvearrowright H$ be a continuous action by automorphisms, and let $H \rtimes_\tau G$ be the corresponding semidirect product. We assume that $H \rtimes_\tau G$ acts continuously on Z, and that this action is compatible with the actions $G \curvearrowright Z$ and $H \curvearrowright Z$.

There are some differences with the setting of the main result, cf. Section 2.1.1. Most notably, Z is assumed to be a Polish space and $H \rtimes_\tau G \curvearrowright Z$ is assumed to be continuous.

Theorem 8.1. *If $G \curvearrowright Z$ is generically ergodic while $\tau : G \curvearrowright H$ is not, then there are no Borel G-equivariant inverses to $Z \to \pi(Z)$. Moreover, if $Y' \subseteq Y \cap \pi(Z)$ is a Borel G-invariant set such that $Z' = \pi^{-1}(Y')$ is comeager in Z, then there are no Borel G-equivariant inverses to $\pi|_{Z'} : Z' \to Y'$ either.*

Proof. Suppose that $\xi : Y' \to Z'$ is Borel G-equivariant, satisfies $\pi(\xi(y)) = y$ for all $y \in Y'$, and that Z' is comeager in Z. Such a map ξ is necessarily injective. Consider the set $T = \xi(Y')$. It is Borel by the Luzin–Suslin theorem ([32, Thm. 15.1], [51, Prop. 4.5.1]), it is G-invariant (by the equivariance of ξ), intersects each H-orbit at most once (by Eq. (8.1)), and its saturation $H \cdot T = \pi^{-1}(Y') = Z'$ is comeager in Z.

Suppose that the action τ is not generically ergodic. Then there exists a G-invariant Borel set $A \subset H$ such that both A and $H \setminus A$ are non-meager subsets of H. Consider the disjoint subsets $A \cdot T$ and $(H \setminus A) \cdot T$ of Z. We claim that these sets are (i) G-invariant, (ii) Borel, (iii) non-meager. This contradicts the generic ergodicity of the action $G \curvearrowright Z$. It suffices to check these properties for the set $A \cdot T$.

The G-invariance of $A \cdot T$ follows from the relation $g(hz) = (\tau^g h)(gz)$ combined with the G-invariance of A and T. To check this relation, we note that, by assumption, G and H act on Z as follows: $gz = (e_H, g)z$, $hz = (h, e_G)z$, where e_H and e_G are the units in H and G. Thus,

$$g(hz) = g((h, e_G)z) = (e_H, g)(h, e_G)z = (e_H \tau^g h, g)z = (\tau^g h, g)z,$$

and therefore,

$$(\tau^g h)(gz) = (\tau^g h, e_G)(e_H, g)z = (\tau^g h, g)z = g(hz).$$

Borelness of $A \cdot T$ follows from applying the Luzin–Suslin theorem to the continuous injective map $\psi \colon (h, t) \mapsto ht$.

To see that the set $A \cdot T$ is non-meager, first note that since A is non-meager in H and H is Polish, there exists a non-empty open set $O \subseteq H$ such that $A \cap O$ is comeager in O. Then $(A \cap O) \times T$ is comeager in $O \times T$, and therefore, $A \times T$ cannot be meager in $H \times T$. It remains to note that $A \cdot T = \psi(A \times T)$, where the map $\psi \colon A \times T \to Z'$ is continuous and injective, so it cannot map non-meager sets onto meager ones. Thus $A \cdot T$ is non-meager in $Z' = H \cdot T$, and since Z' is comeager in Z, $A \cdot T$ remains non-meager in Z. □

8.3 Applications

Theorem 8.1 has multiple applications when $G = \mathbb{C}$ acts on the space of entire functions $Z = \mathcal{E}$ via the argument shift. Note that, by Birkhoff's theorem, $\mathbb{C} \curvearrowright \mathcal{E}$ has dense orbits and is thus generically ergodic. In all of the following examples τ is the trivial action (that is, $\tau^g h = h$ for all g and h), and the semidirect product is really just the direct product $\mathbb{C} \times H$. In particular, $\tau : \mathbb{C} \curvearrowright H$ is automatically not generically ergodic provided that H is non-trivial. In the following examples, elements of Y are functions, and $\mathbb{C}$ acts on Y via the argument shift. For each of the following maps π, there are no Borel equivariant inverses $\xi : \pi(Z) \to Z$.

1. The derivative $\frac{d}{dz} : \mathcal{E} \to \mathcal{E}$. Here $H = \mathbb{C}$ is viewed as the space of constant functions and $H \curvearrowright \mathcal{E}$ by $(w \cdot f)(z) = f(z) + w$.

2. The logarithmic derivative $L : \mathcal{E}_{\neq 0} \to \mathcal{MR}$, $f \mapsto f'/f$. Here H is the multiplicative group $\mathbb{C}^\times$, and $H \curvearrowright \mathcal{E}_{\neq 0}$ by multiplication, $(w \cdot f)(z) = wf(z)$.

3. The exponentiation $\exp : \mathcal{E} \to \mathcal{E}^\times$. Here $H = \mathbb{Z}$ is identified with constant functions of the form $2\pi i n$, $n \in \mathbb{Z}$ and $H \curvearrowright \mathcal{E}$ by $(n \cdot f)(z) = f(z) + 2\pi i n$.

4. The absolute value map $|\cdot| : \mathcal{E} \to C(\mathbb{C})$. Here H is the circle $\mathbb{T}$, identified with the multiplicative group of complex numbers of absolute value 1. $H \curvearrowright \mathcal{E}$ via $(\alpha \cdot f)(z) = e^{i\alpha} f(z)$ for $\alpha \in [0, 2\pi)$.

5. The Schwarzian derivative $S : \mathcal{E} \setminus \{\text{constants}\} \to \mathcal{MR}$, where
$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$
Here H is the group of Möbius transformations $\mathrm{PGL}(2, \mathbb{C})$ acting by post-composition:
$$f \mapsto \frac{af + b}{cf + d}.$$

6. The spherical derivative $(\,\cdot\,)^\# : \mathcal{E} \setminus \{\text{constants}\} \to C(\mathbb{C})$ where
$$f^\# = \frac{2|f'|}{1 + |f|^2}.$$
Here H is the special unitary group $\mathrm{SU}(2)$ acting on $\mathcal{E}$ by post-composition:
$$f \mapsto \frac{\alpha f - \bar{\beta}}{\beta f + \bar{\alpha}} \qquad \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1.$$

An application of Theorem 8.1 with a non-trivial action τ is given by the difference operator $D : \mathcal{E} \to \mathcal{E}$ defined by $D(f)(z) = f(z+1) - f(z)$. Here $H = \mathcal{E}_1$ is the space of 1-periodic entire functions, $H \curvearrowright \mathcal{E}$ via $(h \cdot f)(z) = f(z) + h(z)$, and τ is the argument shift. The action $\tau : \mathbb{C} \curvearrowright \mathcal{E}_1$ has a Borel transversal as was argued in Remark 7.5. Hence, it cannot be generically ergodic by the following standard fact, whose proof is included for the reader's convenience.

Lemma 8.2. *Consider a continuous action $G \curvearrowright Z$ of a Polish group G on a Polish space Z. Assume that no orbit of the action is comeager. If the action has a Borel transversal, then it cannot be generically ergodic.*

Proof. First, consider a simpler case with countably many orbits. Note that every orbit is an analytic set (as the image of $G \times \{z\}$ under the continuous map $(g, z) \mapsto gz$, $g \in G$, $z \in Z$, and therefore, has the Baire property). All the orbits cannot be meager (otherwise, Z becomes meager in itself, which contradicts the Baire category theorem). There could not be a unique non-meager orbit—in that case, its complement is meager as a countable union of meager sets, and the orbit becomes comeager, which contradicts the assumption. Hence, there are at least two non-meager orbits, and we can decompose $Z = Z_1 \sqcup Z_2$, where the sets Z_1 and Z_2 are G-invariant and have the Baire property. Each of them contains a non-meager subsets, hence, the action cannot be generically ergodic.

Now, we assume that the action has uncountably many orbits, and let $T \subseteq Z$ be the Borel transversal of the action. By the assumption, the set T is uncountable. Fix a Borel bijection $\alpha : 2^{\mathbb{N}} \to T$ between T and the Cantor set $2^{\mathbb{N}}$. For each finite string $s \in 2^{<\mathbb{N}}$, define $Z_s = G \cdot \alpha(N_s)$, where $N_s \subseteq 2^{\mathbb{N}}$ consists of all infinite binary extensions of s. Each Z_s is G-invariant and satisfies $Z_s = Z_{s^\frown 0} \sqcup Z_{s^\frown 1}$ (the sign "$^\frown$" denotes concatenation of finite strings). Furthermore, the sets Z_s are analytic (as the images of Borel sets $G \times N_s$ by the Borel map $(g, x) \mapsto g\alpha(x)$, $g \in G$, $x \in N_s$), and therefore possess the Baire property.

Assume towards a contradiction that $G \curvearrowright Z$ is generically ergodic. Since $Z = Z_0 \sqcup Z_1$ and both Z_0 and Z_1 are G-invariant with the Baire property, one of them must be comeager. Let $i_1 \in \{0, 1\}$ be such that Z_{i_1} is comeager. Proceeding inductively, at each step n we have $Z_{i_1 \cdots i_n} = Z_{i_1 \cdots i_n 0} \sqcup Z_{i_1 \cdots i_n 1}$, and we choose i_{n+1} such that $Z_{i_1 \cdots i_{n+1}}$ is comeager.

This yields a sequence $a = (i_n)_n \in 2^{\mathbb{N}}$ for which each $Z_{i_1 \cdots i_n}$ is G-invariant and comeager. Consequently, $\bigcap_n Z_{i_1 \cdots i_n}$ is comeager as well. However, this intersection equals the single orbit $G\alpha(a)$, contradicting the assumption that no orbit is comeager. $\square$

8.4 Two remarks

1. All our non-existence applications pertained to the action $\mathbb{C} \curvearrowright \mathcal{E}$ by the argument shift, i.e., to the Borel entire function $\mathbb{C} \curvearrowright \mathrm{Free}(\mathcal{E}) \xrightarrow{\mathrm{id}} \mathbb{C} \curvearrowright \mathcal{E}$. This does not exclude the possibility that for other Borel measurable entire functions $\mathbb{C} \curvearrowright X \xrightarrow{\varphi} \mathbb{C} \curvearrowright \mathcal{E}$, the same map π might have a Borel equivariant inverse. For one, we have the tautological example: If $F : X \to \mathcal{E}$ is a Borel entire function,

then the Borel entire function F' given by $x \mapsto F'_x$ evidently admits the Borel entire primitive F. Moreover, a modification of the original Weiss' construction [55] can be used to build Borel entire functions that admit primitives of all orders.

2. It is also worth mentioning that, for some of the maps π discussed above, it is possible to formulate a cohomological criterion for the existence of the Borel equivariant inverse to π. For instance, this can be done when $\pi = \mathrm{d}/\mathrm{d}z$ is the derivative map. With any Borel entire function $\varphi\colon X \to \mathcal{E}$, we associate the *cocycle* $\alpha\colon X \times \mathbb{C} \to \mathbb{C}$,

$$\alpha_\varphi(x, z) = \int_0^z \varphi_x(w)\,\mathrm{d}w.$$

Then, *a Borel entire function φ admits a Borel primitive if and only if α_φ is a coboundary*, that is, there is a Borel function $g\colon X \to \mathbb{C}$ such that $\alpha(x, z) = g(z \cdot x) - g(x)$ for all $x \in X$ and $z \in \mathbb{C}$. If such g exists, then $g(x) = \varphi_x(0)$, the evaluation of φ at the origin.

Appendices

A Growth of Borel continuous functions

The purpose of this appendix is to highlight the differences in the growth rates between measurable and Borel entire functions by showing that for some free Borel $\mathbb{C}$-actions, any nowhere constant Borel entire function will have arbitrarily fast growth. In fact, this phenomenon has little to do with the structure of entire functions, and applies to arbitrary continuous functions.

The main result, namely Theorem A.2, is derived from the following theorem from [19, Thm. 1.1].

Theorem A.1. *Let $\mathbb{Z}^d \curvearrowright X'$ be a continuous action on a compact Polish space. Let also $(S'_n)_n$, $S'_n \subseteq X'$, be a sequence of Borel complete sections* [9] *and $(A_n)_n$, $A_n \subseteq \mathbb{Z}^d$, be an increasing sequence of finite subset of $\mathbb{Z}^d$ such that $\bigcup_n A_n = \mathbb{Z}^d$. Then there is $x \in X'$ such that $(A_n \cdot x) \cap S'_n \neq \varnothing$ for infinitely many n.*

Let $(\Omega, ||\cdot||)$ be a separable Banach space and $C(\mathbb{R}^d, \Omega)$ be the space of Ω-valued continuous functions on $\mathbb{R}^d$ endowed with the topology of uniform convergence on

[9] A section is complete if it intersects each orbit of the action.

compact subsets. Naturally, we have the argument shift action $\mathbb{R}^d \curvearrowright C(\mathbb{R}^d, \Omega)$. Let $\mathbb{R}^d \curvearrowright X$ be a free Borel action. Given a Borel $\mathbb{R}^d$-equivariant map $\varphi : X \to C(\mathbb{R}^d, \Omega)$ let $M_{\varphi,x} : \mathbb{R}^{>0} \to \mathbb{R}^{\geq 0}$ be given by

$$M_{\varphi,x}(R) = \max_{|r| \leq R} ||\varphi(x)(r)||.$$

We say that φ is *everywhere unbounded* if for all $x \in X$ one has $M_{\varphi,x}(R) \to +\infty$ as $R \to +\infty$. By a *rate function* we mean a non-decreasing $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ satisfying $\lim_{R\to\infty} f(R) = +\infty$.

Theorem A.2. *For each d, there is a free Borel action $\mathbb{R}^d \curvearrowright X$ such that for any everywhere unbounded Borel equivariant $\varphi : X \to C(\mathbb{R}^d, \Omega)$ and any rate function f there is a point $x \in X$ such that*

$$\limsup_{R\to\infty} \frac{M_{\varphi,x}(R)}{f(R)} = +\infty.$$

Proof. Let $K \subseteq \mathbb{R}^d$ be a compact symmetric subset so large that $\mathbb{Z}^d + K = \mathbb{R}^d$, and let $\operatorname{diam} K$ denote its diameter. For instance, $K = [-1/2, 1/2]^d$ with $\operatorname{diam} K = \sqrt{d}$ will do. Note that $(r + K) \cap \mathbb{Z}^d \neq \varnothing$ for any $r \in \mathbb{R}^d$. It is notationally more convenient to prove the weaker inequality

$$\limsup_{R\to\infty} \frac{M_{\varphi,x}(R + \operatorname{diam} K)}{f(R)} \geq 1. \tag{A.1}$$

The latter, however, is equivalent to the conclusion of Theorem A.2, for Eq. (A.1) applied to $f_1(R) = Rf(R + \operatorname{diam} K)$ yields the conclusion of Theorem A.2 for f.

Let $\mathbb{Z}^d \curvearrowright X'$ be a free continuous action of $\mathbb{Z}^d$ on some compact Polish space X'. We turn it into a free action of $\mathbb{R}^d$ on $X = X' \times [0, 1)^d$ in the natural way:

$$r \cdot (x, s) = (\lfloor s + r \rfloor \cdot x, s + r - \lfloor s + r \rfloor),$$

where $\lfloor r \rfloor \in \mathbb{Z}^d$ is the coordinate-wise integer part of the vector $r \in \mathbb{R}^d$. Note that X' can be identified with the $\mathbb{Z}^d$-invariant subset $X' \times \{0\} \subseteq X$.

Fix an everywhere unbounded Borel equivariant $\varphi : X \to C(\mathbb{R}^d, \Omega)$ and a growth rate function f. Our goal is to find an $x \in X$ such that Eq. (A.1) holds. Let $\Phi : X \to \Omega$ be given by $\Phi(x) = \varphi(x)(0)$ and let

$$B(R) = \{a \in \Omega : ||a|| < R\} \subseteq \Omega$$

denote the open ball of radius R around the origin. Given $n \in \mathbb{N}$, set

$$S_n = \Phi^{-1}(\Omega \setminus B(f(n))) = \{x \in X : ||\Phi(x)|| \geq f(n)\}.$$

Everywhere unboundedness of φ is equivalent to the assertion that each S_n is complete, i.e., it intersects every orbit of the action. Furthermore, the sets S_n are nested, $S_n \supseteq S_{n+1}$, and vanish, $\bigcap_n S_n = \varnothing$. In fact, the latter holds in the following stronger sense: for any compact $L \subseteq \mathbb{R}^d$ and any $x \in X$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ one has $(L \cdot x) \cap S_n = \varnothing$.

Set $S'_n = (K \cdot S_n) \cap X'$ and note that each S'_n is a Borel complete section for the action $\mathbb{Z}^d \curvearrowright X'$. One may now apply Theorem A.1 to sets S'_n and $A_n = \{a \in \mathbb{Z}^d : ||a|| \leq n\}$, $n \in \mathbb{N}$. It yields an $x \in X'$ such that $(A_n \cdot x) \cap S'_n \neq \varnothing$ for infinitely many n. In other words, there are n_k, $a_k \in A_{n_k}$, $y_k \in S_{n_k}$ and $s_k \in K$ such that $||a_k|| \leq n_k$ and $a_k x = s_k y_k$. Note that

$$\begin{aligned} ||\rho(x, y_k)|| &\leq ||\rho(x, a_k x)|| + ||\rho(a_k x, y_k)|| \\ &= ||\rho(x, a_k x)|| + ||\rho(s_k y_k, y_k)|| \\ &= ||a_k|| + ||s_k|| \leq n_k + \operatorname{diam} K. \end{aligned}$$

In particular,

$$\begin{aligned} M_{\varphi,x}(n_k + \operatorname{diam} K) \geq ||\varphi(x)(\rho(x, y_k))|| &= ||\Phi(\rho(x, y_k)x)|| \\ &= ||\Phi(y_k)|| \geq f(n_k), \end{aligned}$$

and therefore $R = n_k$, $k \in \mathbb{N}$, witness that Eq. (A.1) does indeed hold. $\square$

Specializing to entire functions, we get the following corollary.

Corollary A.3. *There is a free Borel action $\mathbb{C} \curvearrowright X$ such that for any rate function f and any Borel equivariant $\varphi : X \to \mathcal{E} \setminus \{constants\}$ there is a point $x \in X$ such that*

$$\limsup_{R \to \infty} \frac{M_{\varphi,x}(R)}{f(R)} = +\infty.$$

Proof. Apply Theorem A.2 to $\mathbb{R}^2 = \mathbb{C}$, $\Omega = \mathbb{C}$ so that $\mathcal{E} \subseteq C(\mathbb{R}^2, \Omega)$, and note that any $\varphi : X \to \mathcal{E} \setminus \{\text{constants}\}$ is necessarily everywhere unbounded by Liouville's theorem. $\square$

B Topologies on meromorphic functions

As discussed in Section 5.1, the space of meromorphic functions $\mathcal{MR}$ is typically endowed with the topology $\tau_{\mathcal{MR}}$ of uniform convergence on compact subsets of $\mathbb{C}$ with respect to, say, the spherical metric on the Riemann sphere $\overline{\mathbb{C}}$. However, the algebraic operations on $\mathcal{MR}$ are not continuous with respect to $\tau_{\mathcal{MR}}$. This appendix aims to demonstrate that there is no Polish group topology on the multiplicative group $\mathcal{MR}^\times$ of non-zero meromorphic functions. This result is established in Corollary B.8, which proves a slightly more general statement.

B.1 Dudley's theorem

Our argument relies on a theorem by Dudley [14] concerning the automatic continuity of homomorphisms into groups equipped with norms having linear growth. We begin by recalling the relevant definitions. (Our terminology differs from that used in [14].) In contrast to Section 4, all seminorms in this appendix take values in $\mathbb{N}$.

Definition B.1. A *seminorm* on a group H is a map $||\cdot|| : H \to \mathbb{N}$ such that for all $h, h_1, h_2 \in H$ one has

1. $||e_H|| = 0$;
2. $||h|| = ||h^{-1}||$;
3. $||h_1h_2|| \leq ||h_1|| + ||h_2||$.

Given a seminorm $||\cdot||$, its *kernel* is $\ker(||\cdot||) = \{h \in H : ||h|| = 0\}$. Note that $\ker(||\cdot||)$ is necessarily a subgroup of H. A seminorm is a *norm* if its kernel is trivial. We say that $||\cdot||$ has *linear growth* if the inequality $||h^n|| \geq \max\{n, ||h||\}$ holds for all $n \geq 1$ and all $h \in H \setminus \ker(||\cdot||)$. Since $||h^n|| \leq n||h||$ holds for all seminorms, no seminorm can grow faster than linearly in this sense. All seminorms considered in this appendix are, in fact, *additive*, meaning they satisfy the stronger condition $||h^n|| = n||h||$ for all $h \in H$ and $n \in \mathbb{N}$, and thus have linear growth.

Given a group G and a topology τ_G on it, we say that τ_G is a *semigroup topology* if the multiplication is τ_G-continuous (with no assumption on the continuity of the inverse map).

The following theorem is stated in [14, Thm. 1] for group topologies on G. However, continuity of the inverse map is not used in the proof, and we will make use of the more general statement later on. For the reader's convenience, the proof of Dudley's theorem is reproduced below.

Theorem B.2 (Dudley [14, Thm. 1]). *Suppose a group H admits a norm with linear growth. Let G be a group equipped with a completely metrizable semigroup* *topology. Any homomorphism from G to H is continuous with respect to the discrete topology on H.*

Proof. Let $\varphi : G \to H$ be a homomorphism, d be a complete metric on G that generates a semigroup topology, and $||\cdot||$ be a norm on H with linear growth. We claim that $\ker\varphi = \{g \in G : \varphi(g) = e_H\}$ contains a neighborhood of the identity.

Suppose towards a contradiction that this is not the case. We construct a sequence $(g_n)_n \subseteq G \setminus \ker\varphi$ such that for $r_n = n + \sum_{i=0}^{n} ||\varphi(g_i)||$ and $h_{m,n} \in G$ given by

$$h_{m,n} = g_m(g_{m+1}(\cdots(g_{n-1}(g_n)^{r_{n-1}})^{r_{n-2}}\cdots)^{r_{m+1}})^{r_m},$$

the sequence $(h_{m,n})_{n=m}^{\infty}$ is d-Cauchy for each m. Note that

$$h_{m,n} = g_m(h_{m+1,n})^{r_m}. \tag{B.1}$$

Take g_0 to be any element of $G \setminus \ker\varphi$. Suppose that g_m, $m \leq n$, and therefore also r_m, $h_{m,n}$, have been constructed for all $m \leq n$. Consider the function $\xi_m : G \to G$ given by

$$\xi_m(x) = g_m(g_{m+1}(\cdots(g_{n-1}(g_n \cdot x^{r_n})^{r_{n-1}})^{r_{n-2}}\cdots)^{r_{m+1}})^{r_m}.$$

This map is continuous by the continuity of the multiplication and $\xi_m(e) = h_{m,n}$ for all $m \leq n$. Therefore, for a small enough neighborhood of the identity $U \subseteq G$ the inequality $d(\xi_m(g), h_{m,n}) < 2^{-n}$ holds for all $m \leq n$ and all $g \in U$. Since $\ker\varphi$ does not contain any neighborhoods of the identity by our assumption, we may pick for g_{n+1} an element of $U \setminus \ker\varphi$. This ensures that $d(h_{m,n+1}, h_{m,n}) < 2^{-n}$ and $(h_{m,n})_{n=m}^{\infty}$ is thus d-Cauchy.

Set $h_m = \lim_{n\to\infty} h_{n,m}$ and note that $h_m = g_m h_{m+1}^{r_m}$ holds for all m by Eq. (B.1) and the continuity of the multiplication. A priori, we have little control as to whether h_m's are elements of $\ker\varphi$ or not. Nonetheless, we claim that

$$||\varphi(h_m^{r_{m-1}})|| \geq r_{m-1} \tag{B.2}$$

holds for all m. Indeed, if $||\varphi(h_{m+1})|| \neq 0$, then $||\varphi(h_{m+1}^{r_m})|| \geq r_m$ by the linearity of the length function and

$$\begin{aligned} ||\varphi(h_m^{r_{m-1}})|| &\geq ||\varphi(h_m)|| = ||\varphi(g_m h_{m+1}^{r_m})|| \\ &\geq ||\varphi(h_{m+1}^{r_m})|| - ||\varphi(g_m)|| \geq r_m - ||\varphi(g_m)|| = r_{m-1} + 1 > r_{m-1}. \end{aligned}$$

If on the other hand $||\varphi(h_{m+1})|| = 0$, then $\varphi(h_{m+1}) = e_H$ and so

$$||\varphi(h_m^{r_{m-1}})|| = ||\varphi((g_m h_{m+1}^{r_m})^{r_{m-1}})|| = ||\varphi(g_m^{r_{m-1}})|| \geq r_{m-1},$$

where the last inequality relies on the choice of $g_m \in G \setminus \ker\varphi$. This justifies Eq. (B.2), which, in particular, implies that h_m must be an element of $G \setminus \ker\varphi$.

Iterating Eq. (B.2) yields a contradiction. Indeed

$$\begin{aligned}
||\varphi(h_0)|| &= ||\varphi(g_0 h_1^{r_0})|| \geq ||\varphi(h_1^{r_0})|| - ||\varphi(g_0)|| \geq ||\varphi(h_1)|| - ||\varphi(g_0)|| \\
&= ||\varphi(g_1 h_2^{r_1})|| - ||\varphi(g_0)|| \geq ||(h_2^{r_1})|| - ||\varphi(g_1)|| - ||\varphi(g_0)|| \\
&\geq ||\varphi(h_2)|| - ||\varphi(g_1)|| - ||\varphi(g_0)|| \\
&= \cdots \\
&= ||\varphi(g_m h_{m+1}^{r_m})|| - \sum_{i=0}^{m-1} ||\varphi(g_i)|| \geq ||\varphi(h_{m+1}^{r_m})|| - \sum_{i=0}^{m} ||\varphi(g_i)|| \\
\text{Eq. (B.2)} &\geq r_m - \sum_{i=0}^{m} ||\varphi(g_i)|| = m
\end{aligned}$$

is true for all m, i.e., $||\varphi(h_0)|| = \infty$, which is impossible.

We conclude that $\ker\varphi$ contains a neighborhood of the identity. From this it is easy to see that $\ker\varphi$ must be open. Recall that $\ker\varphi$ is a subgroup of G. If $U \subseteq \ker\varphi$ is open, then we can write $\ker\varphi = \bigcup_g gU$, where the union is taken over all $g \in \ker\varphi$. The topology of the metric d is only a semigroup topology and in a general topological semigroup the translation map $x \mapsto gx$ is continuous but not necessarily open. However, in our setup G is a group, so $x \mapsto g^{-1}x$ is a continuous inverse to the former map, hence all translations are homeomorphisms and gU is thus open for all $g \in G$. This shows that $\ker\varphi$ is open.

Finally, given any $A \subseteq H$, $\varphi^{-1}(A) = \bigcup_g g\ker\varphi$, where the union is over $g \in \varphi^{-1}(A)$, which shows that $\varphi^{-1}(A)$ is open. We conclude that φ is continuous with respect to the discrete topology on H. □

Corollary B.3. *Suppose* $||\cdot||$ *is a seminorm of linear growth on a group* G*, and* $\ker(||\cdot||)$ *is a normal subgroup of* G*. Then* $\ker(||\cdot||)$ *is open in any completely metrizable semigroup topology on* G*.*

Proof. Let $N = \ker(||\cdot||)$ be normal in G and consider the factor group $H = G/N$. Let $\pi : G \to H$ be the quotient map. It is straightforward to verify that $||g_1|| = ||g_2||$ whenever $g_1 N = g_2 N$, allowing us to define a function $||\cdot||_H : H \to \mathbb{N}$ via $||gN||_H = ||g||$. Furthermore, $||\cdot||_H$ is a norm with linear growth, $\ker(||\cdot||_H) = \{e_H\}$.

Let τ_G be a completely metrizable semigroup topology on G. By Theorem B.2, π is τ_G-continuous with respect to the discrete topology on H. It follows that $N = \pi^{-1}(e_H)$ must be τ_G-open, as claimed. $\square$

B.2 Applications to meromorphic functions

Here is an example of a linearly growing seminorm relevant to our work.

Example B.4. Let $\mathcal{D}$ be the Abelian group of divisors on $\mathbb{C}$. For a compact $K \subseteq \mathbb{C}$, define $||\cdot||_K : \mathcal{D} \to \mathbb{N}$ by $||d||_K = \sum_{z\in K} |d(z)|$. By the definition of a divisor, $d(z)$ is non-zero for only finitely many $z \in K$, so the sum is well-defined. One can easily verify that $||\cdot||_K$ is an additive, and thus linearly growing, seminorm. The group $\ker(||\cdot||_K)$ consists of divisors d satisfying $d(z) = 0$ for all $z \in K$. Since $\mathcal{D}$ is Abelian, all its subgroups are normal. Corollary B.3 therefore implies that $\ker(||\cdot||_K)$ is open in any completely metrizable semigroup topology on $\mathcal{D}$.

Example B.5. A seminorm on a group H can be pulled back to a group G through any homomorphism $G \to H$. For instance, consider the multiplicative group $\mathcal{MR}^\times$ of non-zero meromorphic functions and the homomorphism $\mathrm{div} : \mathcal{MR}^\times \to \mathcal{D}$ given by the divisor map. Given a compact set $K \subseteq \mathbb{C}$, define a seminorm $||\cdot||'_K$ on $\mathcal{MR}^\times$ by $||f||'_K = ||\mathrm{div}(f)||_K$ for $f \in \mathcal{MR}^\times$, where $||\cdot||_K$ is the seminorm from Example B.4. Note that $\ker(||\cdot||'_K)$ consists of meromorphic functions with no poles or zeros in K. Corollary B.3 shows that $\ker(||\cdot||'_K)$ is open with respect to any completely metrizable semigroup topology on $\mathcal{MR}^\times$.

Given a set K, let $\mathcal{A}(K)$ denote the free Abelian group with generators K. If $K \subseteq K'$, there is a natural surjective homomorphism $\pi : \mathcal{A}(K') \to \mathcal{A}(K)$ defined on the generators by

$$\pi(x) = \begin{cases} x & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

The family of compact subsets of $\mathbb{C}$ is directed under inclusion, allowing us to form the inverse limit $\varprojlim \mathcal{A}(K)$ of the groups $\mathcal{A}(K)$ as K ranges over compact subsets of $\mathbb{C}$. Observe that $\varprojlim \mathcal{A}(K)$ is naturally isomorphic to the group of divisors $\mathcal{D}$. We endow each $\mathcal{A}(K)$ with the discrete topology and let $\tau_{\mathcal{D}}$ be the inverse limit topology on $\mathcal{D}$. For $K \subseteq \mathbb{C}$, let $\pi_K : \mathcal{D} \to \mathcal{A}(K)$ be the corresponding projection.

Proposition B.6. *Let G be a group, τ_G a completely metrizable semigroup topology on G, and $\alpha : G \to \mathcal{D}$ a homomorphism.*

1. *The homomorphism α is $(\tau_G, \tau_{\mathcal{D}})$-continuous.*

2. *If the range of $\pi_L \circ \alpha : G \to \mathcal{A}(L)$ is uncountable for some compact $L \subseteq \mathbb{C}$, then τ_G is not separable.*

Proof. (1) By the universal property of the inverse limit topology, it suffices to show that for each compact $K \subseteq \mathbb{C}$, the homomorphism

$$\alpha_K = \pi_K \circ \alpha : G \to \mathcal{A}(K)$$

is continuous with respect to the discrete topology on $\mathcal{A}(K)$. Define a seminorm $||\cdot||_K : G \to \mathbb{N}$ by $||g||_K = \sum_{z \in K} |\alpha(g)(z)|$. This seminorm is additive, and its kernel $\ker(||\cdot||_K)$ is normal in G. By Corollary B.3, $\ker(||\cdot||_K)$ is τ_G-open, and thus α_K is continuous.

(2) Let L be such that the range of $\pi_L \circ \alpha$ is uncountable. By item (1), $\pi_L \circ \alpha : G \to \mathcal{A}(L)$ is continuous, and the sets $(\pi_L \circ \alpha)^{-1}(z)$ for $z \in \operatorname{ran}(\pi_L \circ \alpha)$ are τ_G-open and pairwise disjoint. Since $\operatorname{ran}(\pi_L \circ \alpha)$ is uncountable, τ_G cannot be separable and hence is not Polish. □

Applying this proposition to the identity map on $\mathcal{D}$ and the homomorphism $\operatorname{div} : \mathcal{MR}^\times \to \mathcal{D}$, we obtain the following corollaries.

Corollary B.7. *Any completely metrizable semigroup topology on $\mathcal{D}$ is finer than $\tau_{\mathcal{D}}$. The homomorphism $\operatorname{div} : \mathcal{MR}^\times \to \mathcal{D}$ is $\tau_{\mathcal{D}}$-continuous with respect to any completely metrizable semigroup topology τ on $\mathcal{MR}^\times$.*

Corollary B.8. *There is no Polish semigroup topology on $\mathcal{D}$. There is no Polish semigroup topology on $\mathcal{MR}^\times$.*

Proof. Since $\operatorname{div} : \mathcal{MR}^\times \to \mathcal{D}$ is surjective, Proposition B.6 implies that no semigroup topology on $\mathcal{D}$ or $\mathcal{MR}^\times$ is separable. □

Recall that a topological algebra is an algebra A equipped with a topology τ that turns A into a topological vector space and with respect to which the multiplication on A is continuous. A complete topological algebra is a topological algebra that is complete as a topological vector space.

Corollary B.8 rules out the existence of Polish topologies on $\mathcal{MR}$ that turn it into a topological algebra. However, it is now easy to show that $\mathcal{MR}$ is not a complete topological algebra under any (not necessarily separable) completely metrizable topology. Recall that a subset Z of a vector space X is *absorbing* if for any $x \in X$ there is $r > 0$ such that for all scalars $|c| \leq r$ one has $cx \in Z$. In a topological vector space, all neighborhoods of the origin are absorbing.

Theorem B.9. *There is no metrizable topology on $\mathcal{MR}$ that turns it into a complete topological algebra.*

Proof. Suppose, towards a contradiction, that τ is a completely metrizable topology with respect to which $\mathcal{MR}$ is a topological algebra. The space $\mathcal{MR}^\times = \mathcal{MR} \setminus \{0\}$ is an open subset of $\mathcal{MR}$ and is therefore completely metrizable in the induced topology. Let $|| \cdot ||' = || \cdot ||'_{\{0\}}$ be the seminorm on $\mathcal{MR}^\times$ from Example B.5 corresponding to $K = \{0\}$. Since τ is an algebra topology, multiplication is continuous, and Corollary B.3 implies that the set $\ker(|| \cdot ||')$ is τ-open in $\mathcal{MR}^\times$ and hence in $\mathcal{MR}$.

By construction, $\ker(|| \cdot ||')$ is a subset of $\mathcal{MR}^\times$, so $0 \notin \ker(|| \cdot ||')$. However, the constant function $1 \in \ker(|| \cdot ||')$, and thus $0 \in \ker(|| \cdot ||') - 1$. Since τ is a vector space topology, translations are τ-open, and $\ker(|| \cdot ||') - 1$ is therefore an open neighborhood of the origin, hence absorbing. However, if $h(z) = 1/z$ and $c \neq 0$, then $ch + 1$ has a pole at the origin, so $ch \notin \ker(|| \cdot ||') - 1$. This is a contradiction. $\square$

Grosse-Erdmann [22] noted that, as a consequence of Arens's extension [1] of the Gelfand–Mazur theorem, there is no Hausdorff locally convex topology on $\mathcal{MR}$ that is a field topology. It was also asked in [22] whether such a topology exists if we drop either the local convexity or the continuity of the inverse requirement, and, furthermore, whether such a topology can be metrizable. Theorem B.9 answers the latter part of this question in the negative.

Corollary B.10. (cf. [22, p. 302])

1. *There is no completely metrizable vector space and field topology on $\mathcal{MR}$.*
2. *There is no completely metrizable locally convex algebra topology on $\mathcal{MR}$.*

Finally, we consider vector space topologies on $\mathcal{MR}$. Of course, it would be too much to expect $\mathcal{MR}$ not to have any Polish vector space topologies. Being a vector space with a Hamel basis of size continuum, $\mathcal{MR}$ is (abstractly) isomorphic to an infinite-dimensional separable Banach space and thus admits many distinct Polish topologies. None of these, however, generate the Borel σ-algebra of $\tau_{\mathcal{MR}}$. We recall that all the field operations on $\mathcal{MR}$ are Borel.

Theorem B.11. *There is no Polish vector space topology on $\mathcal{MR}$ whose Borel σ-algebra equals $\mathcal{B}\tau_{\mathcal{MR}}$.*

Proof. Suppose, towards a contradiction, that such a topology τ exists. For $n \in \mathbb{N}$, define

$$\begin{aligned} M_n &= \{f \in \mathcal{MR} : 0 \text{ is not a pole of } z^n f\} \\ &= \{0\} \cup \{f \in \mathcal{MR}^\times : \operatorname{div}(f)(0) \geq -n\}. \end{aligned}$$

Note that M_n is Borel and thus has the Baire property with respect to τ. Since $\mathcal{MR} = \bigcup_n M_n$, the Baire category theorem ensures that M_{n_0} is non-meager for some n_0. However, each M_n is a vector subspace of $\mathcal{MR}$, which forces $\mathcal{MR} = M_{n_0}$. This is absurd, as $1/z^{n_0+1} \notin M_{n_0}$. □

C Runge's theorem for periodic harmonic functions

Let Γ be a discrete subgroup of $\mathbb{R}^d$, $d \geq 3$, of dimension at most $d-2$. We are given a closed set $K \subset \mathbb{R}^d$ which is invariant with respect to the action of Γ, such that $\mathbb{R}^d \setminus K$ is connected. We assume throughout that $K \cap D_0$ is compact, where $D_0 = D_0(\Gamma)$ is a (closed) fundamental domain of Γ, and fix a Γ-invariant neighborhood U of K.

Theorem C.1. *Given a harmonic function h on U with* $\operatorname{Stab}(h) = \Gamma$ *and a number $\varepsilon > 0$, there exists a harmonic function H on $\mathbb{R}^d$ with* $\operatorname{stab}(H) \supseteq \Gamma$, *such that* $\sup_{x\in K} |H(x) - h(x)| \leq \varepsilon$.

Remark C.2. The above result is true whenever $\mathbb{R}^d/\Gamma$ is a non-compact manifold, which also holds in the case $\dim(\Gamma) = d-1$. However, the following simple proof does not apply (the periodization of the Green function in Lemma C.3 diverges). The more general theorem is a special case of the Lax–Malgrange approximation theorem, see Section 3.10 in [41].

The first step towards Theorem C.1 is the following lemma.

Lemma C.3. *There exists a function G_Γ on $\mathbb{R}^d$ with* $\operatorname{stab}(G_\Gamma) = \Gamma$, *such that*

$$-\Delta G_\Gamma = \sum_{x\in\Gamma} \delta_x.$$

Proof. We obtain a fundamental solution $G(x)$ by $G(x) = \frac{1}{(d-2)\omega_d} f(x)$ where ω_d is the volume of the unit ball in $\mathbb{R}^d$ and where f is the function

$$f(x) = \frac{1}{|x|^{d-2}} + \sum_{y\in\Gamma\setminus\{0\}} \left(|x-y|^{-(d-2)} - |y|^{-(d-2)}\right).$$

Indeed, in an annular shell of bounded width and inner radius r, there are about r^{d-3} lattice points. Moreover, for fixed x and r large, the terms are of order $r^{-(d-1)}$ and hence the series converges absolutely. Thus, $f(x)$ defines a superharmonic function on $\mathbb{R}^d$ with Newtonian singularities at the lattice points, with

$$-\Delta f = (d-2)\omega_d \sum_{y\in\Gamma} \delta_y.$$

It is not immediately clear, however, that f and G are Γ-periodic. To investigate this, let $a \in \Gamma$ and consider the difference $f(x+a) - f(x)$. A direct computation yields, for $x \notin \Gamma$,

$$\begin{aligned} f(x+a) - f(x) &= \frac{1}{|x+a|^{d-2}} - \frac{1}{|x|^{d-2}} \\ &\quad + \sum_{y\in\Gamma\setminus\{0\}} \left(\frac{1}{|x-(y-a)|^{d-2}} - \frac{1}{|x-y|^{d-2}} \right) \\ &= \sum_{y\in\Gamma} \left(\frac{1}{|x-(y-a)|^{d-2}} - \frac{1}{|x-y|^{d-2}} \right). \end{aligned}$$

We claim that the right-hand side vanishes identically. Indeed, it holds that

$$f(x+a) - f(x) = \sum_{y\in\Gamma\cap B(0,R)} \left(\frac{1}{|x-(y-a)|^{d-2}} - \frac{1}{|x-y|^{d-2}} \right) + \mathrm{o}(1)$$

as $R \to +\infty$, and in the sum on the right-hand side all but $\mathrm{O}(R^{(d-3)})$ terms cancel exactly. As the remaining terms are of order $R^{-(d-1)}$, the right-hand side tends to 0 as $R \to +\infty$. This shows that $f(x+a) = f(x)$, so $G_\Gamma(x) = G(x)$ is Γ-invariant.

It only remains to show that G_Γ is not invariant under any larger subgroup $G' \subset \mathbb{R}^d$. This, however, is immediate from the fact that $\mathrm{stab}(\Delta G) = \Gamma$. This completes the proof. □

With the periodic fundamental solution G_Γ at hand, the necessary periodic approximation property is standard. First, we establish a representation of smooth functions g with $\mathrm{stab}(g) = \Gamma$ and suitable conditions on the support. We denote by $D_0 = D_0(\Gamma)$ the closed fundamental domain of Γ.

Lemma C.4. *Assume that g is smooth and Γ-invariant and that* $\mathrm{supp}(g) \cap D_0$ *is compact. Then we have*

$$g(x) = -\int_{D_0} G_\Gamma(x-y)\Delta g(y)dm(y). \tag{C.1}$$

Proof. If we denote the right-hand side by $g_0(x)$, then g_0 is a well-defined function with $\mathrm{stab}(g_0) = \Gamma$, and $g_0(x) \to 0$ as $x \to \infty$, $x \in D_0$. It follows that $(g - g_0)(x)$ is a Γ-periodic harmonic function which tends to zero as $x \to \infty$ along D_0. Any such function is bounded, and hence constant by Liouville's theorem. Due to the limiting behavior at infinity, we must have $g = g_0$. $\square$

We turn to the proof of the periodic Runge theorem stated at the beginning of this appendix.

Proof of Γ*-periodic Runge (Theorem C.1).* We follow one of the standard text-book proofs of Runge's theorem, which is based on the Hahn-Banach theorem, and can be found for instance in the paper [3].

Denote by $C_\Gamma(\mathbb{R}^d)$ the space of continuous functions with stabilizer Γ, and endow it with the topology of locally uniform convergence in D_0 (that is, the topology corresponding to locally uniform convergence on the quotient manifold $\mathbb{R}^d/\Gamma$ where the Γ-invariant functions naturally live). We use the notation $\mathcal{H}_\Gamma(K)$ for the subspace of $C_\Gamma(\mathbb{R}^d)$ consisting of functions harmonic on some neighborhood of the closed set K. If $\mathcal{O}$ is open and Γ-invariant, we write $\mathcal{H}_\Gamma(\mathcal{O})$ for those functions in $C_\Gamma(\mathbb{R}^d)$ harmonic on $\mathcal{O}$.

It is sufficient to show that for an arbitrarily large open Γ-invariant set $\mathcal{O}$, $\mathcal{H}_\Gamma(\mathcal{O})$ is dense in $\mathcal{H}_\Gamma(K)$. This, in turn, will follow if we show that whenever μ is a finite signed Borel measure on $K \cap D_0$ whose associated linear functional

$$g \mapsto \int_{D_0} g \, d\mu$$

annihilates $\mathcal{H}_\Gamma(\mathcal{O})$, then $\mathcal{H}_\Gamma(K)$ is annihilated as well.

To that end, let μ be such a measure and let $f \in \mathcal{H}_\Gamma(K)$. Form the function

$$\nu(x) = \int_{D_0} G_\Gamma(x - y) \, d\mu(y). \tag{C.2}$$

The function ν is harmonic outside K (since for each y, $x \mapsto G_\Gamma(x-y)$ is harmonic outside Γ and since μ is supported on $K \cap D_0$). Moreover, we claim that ν vanishes outside $\mathcal{O}$, and since $\mathbb{R}^d \setminus K$ is connected the unique continuation property of harmonic functions implies that ν is in fact supported on K. To see why ν vanishes outside $\mathcal{O}$, note that $x \in \mathcal{O}^c$, $x - y \in \Gamma$ implies that $y \in \mathcal{O}^c$ as well, so $y \mapsto G_\Gamma(x - y)$ is an element of $\mathcal{H}_\Gamma(\mathcal{O})$. But μ annihilates $\mathcal{H}_\Gamma(\mathcal{O})$, so we conclude from the definition (C.2) that $\nu = 0$ on $\mathcal{O}^c$.

Now, for the trick. We let N be a neighborhood of K such that f is smooth and harmonic on N, and let φ be a smooth Γ-invariant cut-off function supported in N which equals 1 on a neighborhood of K. We have that

$$\int f(x)d\mu(x) = \int f(x)\varphi(x)d\mu(x) = -\int\int G_\Gamma(x-y)\Delta(f\varphi)(y)dm(y)d\mu(x)$$

by Lemma C.4 applied to $f\varphi$. Applying Fubini's theorem and Green's formula, we get

$$\int f(x)d\mu(x) = -\int \Delta(f\varphi)(y)\nu(y)dm(y).$$

But $\Delta(f\varphi) \equiv 0$ on a neighborhood of K, while $\nu = 0$ outside K. Hence, $\int f(x)d\mu(x)$ vanishes, and the proof that $\mathcal{H}_\Gamma(\mathcal{O})$ is dense in $\mathcal{H}_\Gamma(K)$ is complete.

A standard iterative application of the above version of Runge with respect to an exhaustive family K_j such that $K_j \cap D_0$ is compact completes the proof (cf. the proof in [3]). □

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