CROSS SECTIONS OF BOREL FLOWS WITH RESTRICTIONS ON THE DISTANCE SET

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Abstract. Given a set of positive reals, we provide a necessary and sufficient condition for a free Borel flow to admit a cross section with all distances between adjacent points coming from this set.

1. Introduction

This paper completes the study initiated in [Slu], where a Borel version of D. Rudolph's [Rud76] two-step suspension flow representation is given. The main result of the current work is a criterion for a given set $S \subseteq \mathbb{R}^{>0}$ and a free Borel flow $\mathfrak{F}$ to admit a cross section with distances between adjacent points belonging to S.

A cross section for a flow leads to a representation of the flow as a flow under a function (see Figure 1 and [Nad13, Section 7], [Slu, Section 2]). Properties of the flow are reflected in the properties of the base automorphism, but details of their interplay are obscured by the gap function. To get a more transparent connection between the flow and the base automorphism, it is often desirable to impose restrictions on the distances between adjacent points in the cross section.

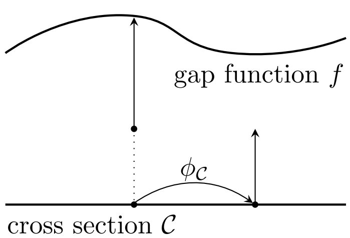

Figure 1.

Of particular importance here are cross sections with only two distinct distances between adjacent points. Their existence in the sense of ergodic theory was proved in [Rud76], and they were used to resolve a problem of Sinai on equivalence of two definitions of K-flows. Further improvement of Rudolph's construction by U. Krengel [Kre76] gave a version of Dye's Theorem for ergodic flows. The Borel version of these results obtained in [Slu], gives a short proof of the analog of the R. Dougherty, S. Jackson, A. S. Kechris [DJK94] classification of Borel flows up to Lebesgue orbit equivalence (see Theorem 10.4 in [Slu] and Theorem 9.1 in [Slu15]). We hope that constructions of cross sections in the present paper will be useful in further explorations of connections between properties of flows and automorphisms they induce on cross sections.

A Borel flow is a Borel measurable action of $\mathbb{R}$ on a standard Borel space Ω. Actions are denoted additively: $\omega + r$ denotes the action of $r \in \mathbb{R}$ upon $\omega \in \Omega$. A cross section for a flow $\mathbb{R} \curvearrowright \Omega$ is a Borel set $\mathcal{C} \subseteq \Omega$ that intersects every orbit in a non-empty lacunary set ("lacunarity" means existence of $c \in \mathbb{R}^{>0}$ such that for any $x \in \mathcal{C}$ and $r \in \mathbb{R}^{>0}$ inclusion $x + r \in \mathcal{C} \setminus \{x\}$ implies $r \geqslant c$). Existence of cross sections was first shown by V. M. Wagh [Wag88], improving upon earlier works of W. Ambrose and S. Kakutani [Amb41], [AK42]. When the flow is free, every orbit becomes an affine copy of $\mathbb{R}$, and any translation invariant notion can therefore be transferred from $\mathbb{R}$ onto orbits of the flow. In particular, given two points $\omega_1, \omega_2 \in \Omega$ within the same orbit one may naturally define the distance $\mathrm{dist}(\omega_1, \omega_2)$ between them.

We always assume that our flows are free and cross sections are "bi-infinite" on each orbit — if $\mathcal{C} \subseteq \Omega$ is a cross section, then every $x \in \mathcal{C}$ has a successor and a predecessor among elements of $\mathcal{C}$ from the same orbit. This allows us to endow $\mathcal{C}$ with an induced automorphism $\phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ which sends a point to the next one. We also let $\vec{\mathrm{gap}}_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}^{>0}$ to denote the gap function which measures distance to the next point: $\vec{\mathrm{gap}}_{\mathcal{C}}(x) = \mathrm{dist}\big(x, \phi_{\mathcal{C}}(x)\big)$.

Given a non-empty set $S \subseteq \mathbb{R}^{>0}$, we say that a cross section $\mathcal{C}$ is S-regular if $\vec{\mathrm{gap}}_{\mathcal{C}}(x) \in S$ for all $x \in \mathcal{C}$, i.e., if the distances between adjacent points in $\mathcal{C}$ belong to S. The following was proved in [Slu]: Given any two positive rationally independent reals $\alpha, \beta \in \mathbb{R}^{>0}$, any free Borel flow admits an $\{\alpha, \beta\}$-regular cross section. In this paper we push the methods of [Slu] a little further and for a given $S \subseteq \mathbb{R}^{>0}$ give a criterion

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for a flow to admit an S-regular cross section. Recall that a flow is said to be sparse if it admits a cross section with gaps "bi-infinitely" unbounded on each orbit. A subgroup of $\mathbb{R}$ generated by S is denoted by $\langle S\rangle$.

Theorem (see Theorem 4.1)**.** *Let $\mathfrak{F}$ be a free Borel flow on a standard Borel space X and let $S \subseteq \mathbb{R}^{>0}$ be a set bounded away from zero.*

(I) Assume $\langle S\rangle = \lambda\mathbb{Z}$, $\lambda > 0$. The flow $\mathfrak{F}$ admits an S-regular cross section if and only if it admits a $\{\lambda\}$-regular cross section.
(II) Assume $\langle S\rangle$ is dense in $\mathbb{R}$, but $\langle S \cap [0, n]\rangle = \lambda_n\mathbb{Z}$, $\lambda_n \geqslant 0$, for all natural $n \in \mathbb{N}$ (we take $\lambda_n = 0$ if $S \cap [0, n]$ is empty). The flow $\mathfrak{F}$ admits an S-regular cross section if and only if the phase space X can be partitioned into $\mathfrak{F}$-invariant Borel pieces (some of which may be empty)

$$X = \Big(\bigsqcup_{i=0}^{\infty} X_i\Big) \sqcup X_\infty$$

such that $\mathfrak{F}|_{X_\infty}$ is sparse and $\mathfrak{F}|_{X_i}$ admits a $\{\lambda_i\}$-regular cross section.
(III) Assume there $n \in \mathbb{N}$ such that $\langle S \cap [0, n]\rangle$ is dense in $\mathbb{R}$. Any flow admits an S-regular cross section.

To further explore item (I), it is, perhaps, helpful to recall a criterion of W. Ambrose [Amb41] (see also [Slu, Proposition 2.5]) for a flow to admit a $\{\lambda\}$-regular cross section.

Proposition 1.1. *A free Borel flow $\mathfrak{F}$ on X admits a cross section with all gaps of size $\lambda > 0$ if and only if there is a Borel function $f : X \to \mathbb{C} \setminus \{0\}$ such that*

$$f(x + r) = e^{\frac{2\pi i r}{\lambda}} f(x) \quad \text{for all } x \in X \text{ and } r \in \mathbb{R}.$$

The paper is concluded with an example of a flow which shows that the condition in item (II) of the main theorem is not vacuous.

1.1. **Notations.** The following notations are used throughout the paper. For a set $S \subseteq \mathbb{R}^{>0}$ and a cross section $\mathcal{C}$, $\mathsf{E}^S_{\mathcal{C}}$ denotes the equivalence relations defined by

$$x \,\mathsf{E}^S_{\mathcal{C}}\, y \iff \exists n \in \mathbb{N}\ \phi^n_{\mathcal{C}}(x) = y \text{ and } \vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi_{\mathcal{C}}(x)^k\big) \in S \text{ for all } 0 \leqslant k < n,$$

or the same condition with roles of x and y switched. In plain words, $x \,\mathsf{E}^S_{\mathcal{C}}\, y$ if all the gaps, when going from x to y in $\mathcal{C}$, belong to S. To say that $\mathcal{C}$ is S-regular is the same as to say that $\mathsf{E}^S_{\mathcal{C}}$ coincides with the orbit equivalence relation induced on $\mathcal{C}$. We also let $\mathsf{E}^{\leqslant K}_{\mathcal{C}}$ denote the relation $\mathsf{E}^{[0,K]}_{\mathcal{C}}$.

A set $S \subseteq \mathbb{R}$ is said to be ϵ-dense in an interval $I \subseteq \mathbb{R}$ if for every open sub-interval $J \subseteq I$ of length ϵ the intersection $J \cap S$ is non-empty. An ϵ-neighborhood $(x - \epsilon, x + \epsilon)$ of $x \in \mathbb{R}$ is denoted by $\mathcal{U}_\epsilon(x)$. For a set $S \subseteq \mathbb{R}^{>0}$, the semigroup generated by S is denoted by $\mathcal{T}(S)$:

$$\mathcal{T}(S) = \Big\{ \sum_{k=1}^{n} s_k \ \Big|\ n \geqslant 1,\ s_k \in S \Big\}.$$

The group generated by S is, as usually, denoted by $\langle S\rangle$. We say that a set $S \subseteq \mathbb{R}^{\geqslant 0}$ is asymptotically dense in $\mathbb{R}$ if for every $\epsilon > 0$ there is $K \geqslant 0$ such that S is ϵ-dense in $[K, \infty)$.

2. Regular cross sections of sparse flows

Lemma 2.1. *Let $S \subseteq \mathbb{R}^{>0}$ be a non-empty subset. The following are equivalent.*

(i) $\langle S\rangle$ is dense in $\mathbb{R}$.
(ii) For every $\epsilon > 0$ there exists a finite $F \subseteq S$ and $K \in \mathbb{R}^{\geqslant 0}$ such that $\mathcal{T}(F)$ is ϵ-dense in $[K, \infty)$.
(iii) $\mathcal{T}(S)$ is asymptotically dense in $\mathbb{R}$.

Proof. Implications (iii) $\implies$ (i) and (ii) $\implies$ (iii) are obvious. We prove (i) $\implies$ (ii).

Suppose S generates a dense subgroup. Pick an element $\tilde{s} \in S$ and an $0 < \epsilon < \tilde{s}/2$. Select a finite set $\tilde{F} \subseteq \langle S\rangle$ such that $\tilde{F} \subseteq [0, \tilde{s}]$ and $\tilde{F}$ is $\epsilon/2$-dense in $[0, \tilde{s}]$. Let $F = \{s_k\}_{k=0}^{n} \subseteq S$ be a finite set such that $s_0 = \tilde{s}$ and any $f \in \tilde{F}$ is of the form $f = \sum_{k=0}^{n} a_{f,k} s_k$ for some $a_{f,k} \in \mathbb{Z}$, $k \leqslant n$. Such coefficients $a_{f,k}$ may not be unique, for each $f \in \tilde{F}$ we fix one such decomposition. Let $M = \max_{f,k} |a_{f,k}|$ and take

$K = M \cdot \sum_{k=0}^{n} s_k$. We claim that $\mathcal{T}(F)$ is ϵ-dense in $[K,\infty)$. Indeed, take $\mathcal{U}_{\epsilon/2}(y) \subseteq [K,\infty]$, and pick $r \in \mathbb{N}$ such that $y - r\tilde{s} \in [K, K+\tilde{s})$. Since

$$\text{either } (y, y+\epsilon/2) - r\tilde{s} \subseteq [K, K+\tilde{s}) \text{ or } (y-\epsilon/2, y) - r\tilde{s} \subseteq [K, K+\tilde{s}),$$

one may find $f \in \tilde{F}$ such that $f + r\tilde{s} + K \in \mathcal{U}_{\epsilon/2}(y)$ Since $f + K \in \mathcal{T}(F)$ and $r\tilde{s} \in \mathcal{T}(F)$, we get $f + r\tilde{s} + K \in \mathcal{T}(S)$, and so $\mathcal{T}(F)$ is ϵ-dense in $[K,\infty)$. □

Theorem 2.2. *Let $S \subseteq \mathbb{R}^{>0}$ be a non-empty set bounded away from zero. If $\langle S\rangle$ is dense in $\mathbb{R}$, then any sparse flow admits an S-regular cross section.*

Proof. Let $\mathfrak{F}$ be a free sparse Borel flow on a standard Borel space Ω, and let $S \subseteq \mathbb{R}^{>0}$ be such that $\langle S\rangle$ is dense in $\mathbb{R}$. It is easy to see that if $\langle S\rangle$ is dense in $\mathbb{R}$, then there is a countable (possibly finite) subset $S' \subseteq S$ which also generates a dense subgroup of $\mathbb{R}$, and we may therefore assume without loss of generality that S is countable.

By Lemma 2.1, the semigroup $\mathcal{T}(S)$ is asymptotically dense in $\mathbb{R}^{>0}$, and so there exists a function $\xi : \mathbb{R}^{>0} \to \mathbb{R}$ such that $x + \xi(x) \in \mathcal{T}(S)$ and $\xi(x) \to 0$ as $x \to +\infty$. Such a function can be picked Borel. Set $\epsilon_n = 2^{-n-1}/3$, and let $(K_n)_{n=0}^{\infty}$ be an increasing sequence, $K_{n+1} > K_n + 1$, such that $|\xi(x)| < \epsilon_n$ for all $x \geqslant K_n - 2$.

We construct cross sections $\mathcal{C}_n$, Borel functions $h_{n+1} : \mathcal{C}_n \to (-\epsilon_n, \epsilon_n)$, and finite Borel equivalence relations E_n on $\mathcal{C}_n$ which will satisfy the following list of properties.

(1) The relation E_0 on $\mathcal{C}_0$ is the trivial equivalence relation: $x \,\mathsf{E}_0\, y$ if and only if $x = y$.
(2) $\mathcal{C}_n$ is a sparse cross section for every n and $\vec{\mathrm{gap}}_{\mathcal{C}_n}(x) \geqslant 1$ for all $x \in \mathcal{C}_n$.
(3) $\mathcal{C}_{n+1} = \mathcal{C}_n + h_{n+1}$, i.e.,
$$\mathcal{C}_{n+1} = \bigl\{x + h_{n+1}(x) \bigm| x \in \mathcal{C}_n\bigr\}.$$
(4) h_{n+1} is constant on E_n-classes: $x \,\mathsf{E}_n\, y \implies h_{n+1}(x) = h_{n+1}(y)$.
(5) E_n-classes are $\mathcal{T}(S)$-regular: $\mathsf{E}_n \subseteq \mathsf{E}_{\mathcal{C}_n}^{\mathcal{T}(S)}$.
(6) E_{n+1} is coarser than E_n:
$$x \,\mathsf{E}_n\, y \implies \bigl(x + h_{n+1}(x)\bigr) \,\mathsf{E}_{n+1} \bigl(y + h_{n+1}(y)\bigr).$$
(7) Distinct E_n-classes are far from each other: if $x, y \in \mathcal{C}_n$ belong to the same orbit and are not E_n-equivalent, then $\mathrm{dist}(x,y) > K_n - 1$.
(8) If $x \,\mathsf{E}_n\, \phi_{\mathcal{C}_n}(x)$, then $\mathrm{dist}\bigl(x, \phi_{\mathcal{C}_n}(x)\bigr) \leqslant K_n + 1$.

Let us first finish the proof under the assumption that such cross sections have been manufactured. Set $f_{n,n+1} : \mathcal{C}_n \to \mathcal{C}_{n+1}$ to be the map $f_{n,n+1}(x) = x + h_{n+1}(x)$ and define $f_{m,n} : \mathcal{C}_m \to \mathcal{C}_n$ for $m \leqslant n$ to be

$$f_{m,n} = f_{n-1,n} \circ f_{n-2,n-1} \circ \cdots \circ f_{m,m+1}$$

with the agreement that $f_{m,m} : \mathcal{C}_m \to \mathcal{C}_m$ is the identity map. Since $|h_n(x)| < \epsilon_{n-1}$, and since $\vec{\mathrm{gap}}_{\mathcal{C}_n}(x) \geqslant 1$ by (2), it follows that maps $f_{n,n+1}$ are injective, and thus so are all the maps $f_{m,n}$, $m \leqslant n$. Since they are also surjective by (3), the maps $f_{m,n}$ are Borel isomorphisms between $\mathcal{C}_m$ and $\mathcal{C}_n$. In simple words, $\mathcal{C}_n$ is obtained from $\mathcal{C}_m$ by moving each point of $\mathcal{C}_m$ by at most $\sum_{i=m}^{n-1} \epsilon_i$ as prescribed by functions h_i, $m < i \leqslant n$. Let

$$H_m : \mathcal{C}_m \to \Bigl(-\sum_{i=m} \epsilon_i, \sum_{i=m} \epsilon_i\Bigr) \qquad H_m(x) = \sum_{n=m}^{\infty} h_{n+1}\bigl(f_{m,n}(x)\bigr).$$

be the "total shift" function. Note that $H_m(x) = H_n\bigl(f_{m,n}(x)\bigr)$ for any $x \in \mathcal{C}_m$ and $m \leqslant n$. The limit cross section $\mathcal{C}_\infty$ is defined by $\mathcal{C}_\infty = \mathcal{C}_0 + H_0$, i.e.,

$$\mathcal{C}_\infty = \{x + H_0(x) \mid x \in \mathcal{C}_0\}.$$

Note also that $\mathcal{C}_\infty = \{x + H_m(x) \mid x \in \mathcal{C}_m\}$ for any $m \in \mathbb{N}$, and the map $x \mapsto x + H_m(x)$ is a bijection between $\mathcal{C}_m$ and $\mathcal{C}_\infty$.

We claim that $\mathcal{C}_\infty$ is a $\mathcal{T}(S)$-regular cross section. It is clear that $\mathcal{C}_\infty$ is a cross section. Let $y_1, y_2 \in \mathcal{C}_\infty$, $y_1 \neq y_2$, be given and let m be so large that $K_m > \mathrm{dist}(y_1, y_2) + 2$.

Pick $z_1, z_2 \in \mathcal{C}_m$ such that $y_i = z_i + H_m(z_i)$. Since $H_m(z_i) \leqslant 1/3$,

$$\mathrm{dist}(z_1, z_2) \leqslant \mathrm{dist}(y_1, y_2) + 2/3 < K_m - 1,$$

hence $z_1 \mathsf{E}_m z_2$ by (7), whence (5) implies that $\mathrm{dist}(z_1, z_2) \in \mathcal{T}(S)$, but by (4) and (6) we get $H_m(z_1) = H_m(z_2)$. Therefore, $\mathrm{dist}(y_1, y_2) = \mathrm{dist}(z_1, z_2) \in \mathcal{T}(S)$. Thus, $\mathcal{C}_\infty$ is a $\mathcal{T}(S)$-regular cross section.

We now add some points to $\mathcal{C}_\infty$ to make it S-regular. Let $S_\uparrow^{<\omega}$ be the set of all tuples $(0, t_1, \ldots, t_m)$, $t_k \in \mathbb{R}$, such that $0 < t_1 < \cdots < t_m$, and $t_{k+1} - t_k \in S$, for all $k < m$, $m \in \mathbb{N}$. Fix a map $\zeta : \mathcal{T}(S) \to S_\uparrow^{<\omega}$ such that for any $t \in \mathcal{T}(S)$ one has $t = t_m$, where $\zeta(t) = (t_k)_{k=1}^m$. In other words, $\zeta(t)$ is a way to decompose an interval of length t into intervals of lengths in S. Let $\mathcal{C}$ be given by

$$\mathcal{C} = \bigl\{ x + t \bigm| x \in \mathcal{C}_\infty,\ t \text{ is one of the coordinates in } \zeta\bigl(\vec{\mathrm{gap}}_{\mathcal{C}_\infty}(x)\bigr) \bigr\}.$$

Since S is bounded away from zero, $\mathcal{C}$ is a lacunary S-regular cross section.

It remains to show how such $\mathcal{C}_n$, E_n, and h_n can be constructed. Let $\mathcal{C}_0$ be a sparse cross section; by passing to a sub cross section we may assume that $\vec{\mathrm{gap}}_{\mathcal{C}_0}(x) > K_0$ for all $x \in \mathcal{C}_0$. We take E_0 to be the trivial equivalence relation.

Suppose we have constructed $\mathcal{C}_n$, E_n, and $h_n : \mathcal{C}_{n-1} \to (-\epsilon_{n-1}, \epsilon_{n-1})$. Consider the relation $\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}$ on $\mathcal{C}_n$. By item (8) $\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}$ is coarser than E_n (recall that $K_{n+1} \geqslant K_n + 1$). Since $\mathcal{C}_n$ is sparse by (2), each $\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}$-class is finite and constitutes an interval in $\mathcal{C}_n$. Any $\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}$-class consists of finitely many E_n-classes. Consider one such class and let $x_1, \ldots, x_m \in \mathcal{C}_n$ be representatives of E_n-classes in the $\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}$-class:

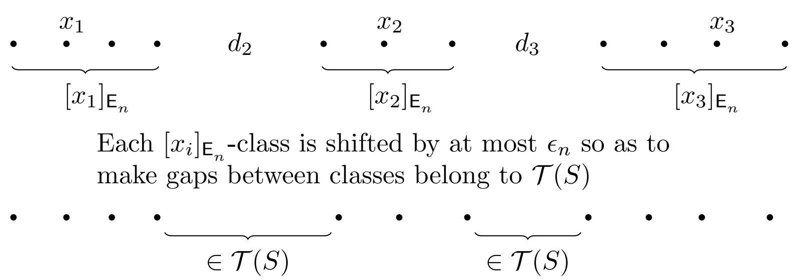

FIGURE 2. Constructing $\mathcal{C}_{n+1}$ from $\mathcal{C}_n$.

- $x_1 < x_2 < \cdots < x_m$;
- $x_i \mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}} x_j$;
- $[x_1]_{\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}} = \bigsqcup_{k=1}^m [x_k]_{\mathsf{E}_n}$.

Let d_k, $2 \leqslant k \leqslant m$, be the gap between the k^{th} and $k-1^{\text{st}}$ E_n-classes:

$$d_k = \mathrm{dist}\bigl(\max[x_{k-1}]_{\mathsf{E}_n}, \min[x_k]_{\mathsf{E}_n}\bigr).$$

By (7), $d_k \geqslant K_n - 1$, and therefore $|\xi(d_2)| < \epsilon_n$. We let $h_{n+1}(x) = 0$ for $x \in [x_1]_{\mathsf{E}_n}$ and $h_{n+1}(x) = \xi(d_2)$ for $x \in [x_2]_{\mathsf{E}_n}$. By induction on k we set

$$h_{n+1}(x) = \xi\bigl(d_k - h_{n+1}(x_{k-1})\bigr) \text{ for } x \in [x_k]_{\mathsf{E}_n}.$$

In words, we shift E_n-classes one by one by at most ϵ_n to make distances between them elements of $\mathcal{T}(S)$. This can be done within each $\mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}}$-class in a Borel way, thus defining a Borel map $h_{n+1} : \mathcal{C}_n \to (-\epsilon_n, \epsilon_n)$. Finally, we let $\mathcal{C}_{n+1} = \mathcal{C}_n + h_{n+1}$, and $\mathsf{E}_{n+1} = \mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}} + h_{n+1}$, i.e.,

$$\bigl(x + h_{n+1}(x)\bigr) \mathsf{E}_{n+1} \bigl(y + h_{n+1}(y)\bigr) \quad \text{if and only if} \quad x \mathsf{E}_{\mathcal{C}_n}^{\leqslant K_{n+1}} y.$$

All the items (1-8) are now easily verified. □

3. LARGE REGULAR BLOCKS

In this section we fix a positive real $\upsilon \in \mathbb{R}^{>0}$ and a strictly monotone sequence $(t_m)_{m=0}^\infty$ that converges to 0 and is such that $\upsilon + t_0 > 0$. We set

$$\mathcal{T}_m = \mathcal{T}(\{\upsilon + t_0, \ldots, \upsilon + t_m\})$$

to denote the semigroup generate by $\{v + t_i : i \leqslant m\}$, and also $\mathcal{T}_m^* = v + \mathcal{T}_m$. In this section we do the necessary preparation to show that every flow admits a cross section with arbitrarily large $\bigcup_m \mathcal{T}_m$-regular blocks.

Let $d_1, \ldots, d_n$ be a family of positive reals and let $R_i \subseteq \mathcal{U}_\epsilon(d_i)$ be non-empty subsets of the ϵ-neighborhoods of d_i. We let $\mathcal{A}_n = \mathcal{A}\big(\epsilon, (d_i)_{i=1}^n, (R_i)_{i=1}^n\big)$ to denote the set of all $z \in \mathcal{U}_\epsilon\big(\sum_{i=1}^n d_i\big)$ for which there exist $x_i \in R_i$ such that $z = \sum_{i=1}^n x_i$ and

$$\Big|\sum_{i=1}^r (d_i - x_i)\Big| < \epsilon \text{ for all } r \leqslant n.$$

When sequences (d_i) and (R_i) are constant, $d := d_i$ and $R := R_i$, we use the notation $\mathcal{A}_n(\epsilon, d, R)$. For the geometric explanation of sets $\mathcal{A}_n$ we refer the reader to Subsection 6.2 of [Slu].

For two non-zero reals $a, b \in \mathbb{R}$ we let $\gcd(a, b)$ to denote the largest positive real c such that both a and b are integer multiples of c. If no such real exists, i.e., if a and b are rationally independent, we set $\gcd(a, b) = 0$.

We need two lemmas from [Slu], which we state below.

Lemma 3.1 (see Lemma 6.7 in [Slu])**.** *Sets $\mathcal{A}_n(\epsilon, d, R)$ have the following additivity properties.*

(i) If $y_i \in R$, $1 \leqslant i \leqslant n$, are such that

$$\Big|nd - \sum_{i=1}^n y_i\Big| < \epsilon,$$

then $\sum_{i=1}^n y_i \in \mathcal{A}_n(\epsilon, d, R)$.

(ii) If $x_i \in \mathcal{A}_{n_i}(\epsilon, d, R)$, $1 \leqslant i \leqslant k$, are such that

$$\Big|\sum_{i=1}^k (x_i - n_i d)\Big| < \epsilon,$$

then $\sum_{i=1}^k x_i \in \mathcal{A}_{\sum_{i=1}^k n_i}(\epsilon, d, R)$.

(iii) If $d \in R$ and $m \leqslant n$, then $\mathcal{A}_m(\epsilon, d, R) + (n - m)d \subseteq \mathcal{A}_n(\epsilon, d, R)$.

Lemma 3.2 (see Lemma 6.8 in [Slu])**.** *Let $\epsilon > 0$, let $0 < \delta \leqslant \epsilon$, and let $x, y \in \mathcal{A}_m(\epsilon, d, R)$, $m \geqslant 1$, be given. Set $a = x - md$ and $b = y - md$. Suppose that $d \in R$, and $a < 0 < b$. There exists $N = N_{\mathrm{Lem\,3.2}}(R, m, \epsilon, \delta, d, x, y)$ such that for all $n \geqslant N$*

- *if $\delta > \gcd(a, b)$, then the set $\mathcal{A}_n(\epsilon, d, R)$ is δ-dense in $\mathcal{U}_\epsilon(nd)$;*
- *if $\delta \leqslant \gcd(a, b)$, then the set $\mathcal{A}_n(\epsilon, d, R)$ is κ-dense in $\mathcal{U}_\epsilon(nd)$ for any $\kappa > \gcd(a, b)$ and moreover*

$$nd + k\gcd(a, b) \in \mathcal{A}_n(\epsilon, d, R) \quad \text{for all integers } k \text{ such that } nd + k\gcd(a, b) \in \mathcal{U}_\epsilon(nd).$$

Let us now explain the meaning of sets $\mathcal{T}_m$ and $\mathcal{T}_m^*$ defined above. We work with sets R_i that are subsets of

$$\mathcal{T}_\infty = \mathcal{T}(\{v + t_i : i \in \mathbb{N}\}).$$

The problem is that there are too many possibilities for the sets R_i, while the argument for Lemma 3.4 below relies upon having only finitely many possibilities for R_i. So, we stratify $\mathcal{T}_\infty$ into sets $\mathcal{T}_m$ and note that for any $D > 0$ the set $\mathcal{T}_m \cap [0, D]$ is finite. While sets R_i will be infinite, each of them will be determined by a finite subset of $\mathcal{T}_m$ and a natural parameter $r \in \mathbb{N}$. This will let us reduce the amount of possibilities for R_i to a finite number. The exact definition is as follows. We say that $R \subseteq \mathcal{U}_\epsilon(d)$ is **r-tamely δ-dense in** $\mathcal{U}_\epsilon(d)$ if there exists a finite set $L \subseteq \mathcal{U}_\epsilon(d)$ satisfying

- $R = \big(L + (t_m)_{m=r}^\infty\big) \cap \mathcal{U}_\epsilon(d)$;
- $L \subseteq \mathcal{T}_r^*$;
- L is δ-dense in $\mathcal{U}_\epsilon(d)$.

We say that R is **tamely δ-dense in** $\mathcal{U}_\epsilon(d)$ if it is r-tamely δ-dense in $\mathcal{U}_\epsilon(d)$ for some $r \in \mathbb{N}$. Note that if a finite $L \subseteq \mathcal{T}_r^*$ is δ-dense in $\mathcal{U}_\epsilon(d)$, then there exists $m_0 \in \mathbb{N}$ so large that for any $m \geqslant m_0$ the set $L + t_m$ is also a subset of $\mathcal{U}_\epsilon(d)$ and is δ-dense in $\mathcal{U}_\epsilon(d)$.

Lemma 3.3 (cf. Lemma 6.10 in [Slu])**.** *For any $\epsilon > 0$, any $0 < \delta \leqslant \epsilon$, any d, any $R \subseteq \mathcal{U}_\epsilon(d)$ such that $d \in R$ and R is tamely ϵ-dense in $\mathcal{U}_\epsilon(d)$ there exist $N = N_{\mathrm{Lem\,3.3}}(\epsilon, \delta, d, R)$ and $M = M_{\mathrm{Lem\,3.3}}(\epsilon, \delta, d, R)$ such that for any $n \geqslant N$ the set $\mathcal{A}_n(\epsilon, d, R)$ contains a subset that is M-tamely δ-dense in $\mathcal{U}_\epsilon(nd)$.*

Proof. Let $r \in \mathbb{N}$ be such that R is r-tamely ϵ-dense in $\mathcal{U}_\epsilon(d)$, and pick $L \subseteq \mathcal{U}_\epsilon(d) \cap \mathcal{T}_r^*$ witnessing this; in particular

$$R = \big(L + (t_m)_{m=r}^{\infty}\big) \cap \mathcal{U}_\epsilon(d).$$

Since L is ϵ-dense in $\mathcal{U}_\epsilon(d)$, we may pick two elements $l^-, l^+ \in L$ such that $l^- < d < l^+$. Using $t_m \to 0$, one may find sufficiently large m_1 and m_2 such that setting $x = l^- + t_{m_1}$ and $y = l^+ + t_{m_2}$ one has

- $m_1, m_2 \geqslant r$;
- $x < d$ and $y > d$;
- $x, y \in \mathcal{U}_\epsilon(d)$ (thus $x, y \in R$);
- $\gcd(x - d, y - d) < \delta$.

Set $a = x - d$ and $b = y - d$; we have $a < 0 < b$. Let $\tilde{R} = \{d, x, y\}$, note that $\mathcal{A}_1(\epsilon, d, \tilde{R}) = \tilde{R}$ and Lemma 3.2 when applied to this $\tilde{R}$, $m = 1$, ϵ, $\delta/2$, d x, and y produces

$$\tilde{N} = \tilde{N}_{\text{Lem 3.2}}(\tilde{R}, 1, \epsilon, \delta/2, d, x, y)$$

such that for any $n \geqslant \tilde{N}$ the set $\mathcal{A}_n(\epsilon, d, \tilde{R})$ is $\delta/2$-dense in $\mathcal{U}_\epsilon(nd)$. Since $\tilde{R}$, x, and y are themselves functions[1] of R, ϵ, δ, and d, we have $\tilde{N} = \tilde{N}(\epsilon, \delta, d, R)$. Set $N = \tilde{N} + 1$ and

$$\bar{L} = \big(\{l^-, l^+\} + \mathcal{A}_{\tilde{N}}(\epsilon, d, \tilde{R})\big) \cap \mathcal{U}_\epsilon(Nd).$$

Note that

- $\bar{L}$ is finite;
- $\bar{L}$ is δ-dense in $\mathcal{U}_\epsilon(Nd)$, because for any $\mathcal{U}_{\delta/2}(z) \subseteq \mathcal{U}_\epsilon(Nd)$

$$\text{either } \ (z - \delta/2, z) - l^- \subseteq \mathcal{U}_\epsilon(\tilde{N}d) \ \text{ or } \ (z, z + \delta/2) - l^+ \subseteq \mathcal{U}_\epsilon(\tilde{N}d).$$

For $M \geqslant \max\{r, m_1, m_2\}$ we have $\tilde{R} \subseteq \mathcal{T}_M$. One has

$$\bar{L} \subseteq L + \mathcal{A}_{\tilde{N}}(\epsilon, d, \tilde{R}) \subseteq \mathcal{T}_r^* + \mathcal{T}_M \subseteq \mathcal{T}_M^*.$$

Since $\bar{L}$ is finite, by increasing M if necessary, we may also assume that

$$\bar{R} := \bar{L} + (t_m)_{m=M}^{\infty} \subseteq \mathcal{U}_\epsilon(Nd) \ \text{ and } \ \{l^-, l^+\} + (t_m)_{m=M}^{\infty} \subseteq \mathcal{U}_\epsilon(d).$$

This guarantees that $\bar{R} \subseteq \mathcal{A}_N(d, \epsilon, R)$. Indeed, any $z \in \bar{R}$ is of the form

$$z = l^{\pm} + t_m + x \ \text{ for some } \ x \in \mathcal{A}_{\tilde{N}}(\epsilon, d, \tilde{R}) \text{ and } m \geqslant M.$$

Since $l^{\pm} + t_m \in R$ (because $M \geqslant r$ and R is r-tamely ϵ-dense), and since

$$\mathcal{A}_{\tilde{N}}(\epsilon, d, \tilde{R}) \subseteq \mathcal{A}_{\tilde{N}}(\epsilon, d, R),$$

item (ii) of Lemma 3.1 applies, and we conclude that $\bar{R} \subseteq \mathcal{A}_N(\epsilon, d, R)$.

We claim these M and N satisfy the conclusion of the lemma. The set $\bar{R}$ is an M-tamely δ-dense in $\mathcal{U}_\epsilon(Nd)$ subset of $\mathcal{A}_N(d, \epsilon, R)$. Since $d \in R$, by Lemma 3.1 one has $\mathcal{A}_{n-1}(d, \epsilon, R) + d \subseteq \mathcal{A}_n(\epsilon, d, R)$, and so $\bar{R} + (n - N)d$ is an M-tamely δ-dense in $\mathcal{U}_\epsilon(nd)$ subset of $\mathcal{A}_n(\epsilon, d, R)$ for all $n \geqslant N$. ☐

Lemma 3.4 (cf. Lemma 6.12 in [Slu])**.** *For any $0 < \epsilon \leqslant 1$, any $0 < \delta \leqslant \epsilon$, any $D > 0$, and any $r \in \mathbb{N}$ there exist $N = N_{\text{Lem 3.4}}(\epsilon, \delta, D, r)$ and $M = M_{\text{Lem 3.4}}(\epsilon, \delta, D, r)$ such that for any $n \geqslant N$, any reals d_i and families $R_i \subseteq \mathcal{U}_\epsilon(d_i)$, $1 \leqslant i \leqslant n$, satisfying*

- *$2\epsilon < d_i \leqslant D$;*
- *R_i is r_i-tamely $\epsilon/12$-dense in $\mathcal{U}_\epsilon(d_i)$ for some $r_i \leqslant r$;*

the set $\mathcal{A}_n\big(\epsilon, (d_i)_{i=1}^n, (R_i)_{i=1}^n\big)$ contains a subset that is M-tamely δ-dense in $\mathcal{U}_{\epsilon/2}\big(\sum_{i=1}^n d_i\big)$.

Proof. First of all, without loss of generality we may assume that r is so big that $|t_m| < \epsilon/12$ for all $m \geqslant r$. Note that for any given r' and $D' > 0$ sets

$$\mathcal{T}_{r'}^* \cap [0, D'] \text{ and } \mathcal{T}_{r'} \cap [0, D'] \text{ are finite,}$$

so there are only finitely many possibilities to choose a subset $L \subseteq \mathcal{T}_{r'}^* \cap [0, D']$ and an element $d \in \mathcal{T}_{r'} \cap [0, D']$. This implies that there are only finitely many pairs (d, R) satisfying

- $d \leqslant D + 1$;
- $d \in \mathcal{T}_r$;

[1] Recall that the sequence $(t_m)_{m=0}^{\infty}$ is fixed throughout the section, so dependence upon this sequence is ignored.

- $R \subseteq \mathcal{U}_{3\epsilon/4}(d)$ is r-tamely $\epsilon/12$-dense in $\mathcal{U}_{3\epsilon/4}(d)$;
- $d \in R$;

Let $\mathcal{Q}$ denote the set of all pairs (d, R) satisfying the conditions above. We set

$$M = \max_{(d,R)\in\mathcal{Q}} M_{\mathrm{Lem\,3.3}}(3\epsilon/4, \delta, d, R) \quad \text{and} \quad N = |\mathcal{Q}| \cdot \max_{(d,R)\in\mathcal{Q}} N_{\mathrm{Lem\,3.3}}(3\epsilon/4, \delta, d, R),$$

and claim that these N and M work. Let $n \geqslant N$ and d_i, R_i, $1 \leqslant i \leqslant n$, be given.

Our plan is to alter d_i to $\tilde{d}_i$ and then apply the pigeon-hole principle together with Lemma 3.3. Let $L_i \subseteq \mathcal{U}_\epsilon(d_i) \cap \mathcal{T}^*_{r_i}$ be such that

$$R_i = \big(L_i + (t_m)_{m=r_i}^{\infty}\big) \cap \mathcal{U}_\epsilon(d_i).$$

Note that since L_i is $\epsilon/12$-dense in $\mathcal{U}_\epsilon(d_i)$, for any i we may pick $l_1, l_2 \in L_i$ such that

$$d_i - \epsilon/6 < l_1 < d_i - \epsilon/12 \text{ and } d_i + \epsilon/12 < l_2 < d_i + \epsilon/6.$$

Since $|t_r| < \epsilon/12$, this ensures

$$d_i - \epsilon/4 < l_1 + t_r < d_i \text{ and } d_i < l_2 + t_r < d_i + \epsilon/4.$$

In other words, for any i we may pick elements

$$x_1 = l_1 + t_r \in R_i \cap \mathcal{T}_r \text{ and } x_2 = l_2 + t_r \in R_i \cap \mathcal{T}_r$$

which are $\epsilon/4$-close to d_i and are below d_i and above d_i respectively.

Using this observation, the construction of $\tilde{d}_i$ is simple. For $\tilde{d}_1$ we pick any element of $R_1 \cap \mathcal{T}_r$ which is $\epsilon/4$-close to d_1. If $\tilde{d}_k$ has been chosen, we pick $\tilde{d}_{k+1}$ to satisfy

- $\tilde{d}_{k+1} \in R_{k+1} \cap \mathcal{T}_r$;
- $\big|\tilde{d}_{k+1} - d_{k+1}\big| < \epsilon/4$;
- if $\sum_{i=1}^{k}(\tilde{d}_i - d_i) < 0$ we want $\tilde{d}_{k+1} > d_{k+1}$, and we take $\tilde{d}_{k+1} < d_{k+1}$ otherwise.

The resulting sequence $\tilde{d}_k$ ensures that

$$\Big|\sum_{i=1}^{k}(\tilde{d}_i - d_i)\Big| < \epsilon/4 \quad \text{ holds for all } k \leqslant n.$$

Now set $\tilde{L}_i = \mathcal{U}_{3\epsilon/4}\big(\tilde{d}_i\big) \cap L_i$ and let

$$\tilde{R}_i = \big(\tilde{L}_i + (t_m)_{m=r}^{\infty}\big) \cap \mathcal{U}_{3\epsilon/4}(\tilde{d}) \subseteq \mathcal{U}_\epsilon(d_i).$$

A typical location of $\tilde{d}_i$ relative to d_i is depicted in Figure 3. Note that $(\tilde{d}_i, \tilde{R}_i) \in \mathcal{Q}$ and

$$\mathcal{A}_n\big(3\epsilon/4, (\tilde{d})_{i=1}^n, (\tilde{R}_i)_{i=1}^n\big) \subseteq \mathcal{A}_n\big(\epsilon, (d_i)_{i=1}^n, (R_i)_{i=1}^n\big).$$

By the choice of N and the pigeon-hole principle, there must be indices

$$1 \leqslant k_1 < k_2 < \ldots < k_{\tilde{N}} \leqslant n$$

such that $\tilde{d}_{k_i} = \tilde{d}_{k_j} =: \tilde{d}$ and $\tilde{R}_{k_i} = \tilde{R}_{k_j} =: \tilde{R}$ for all $1 \leqslant i, j \leqslant \tilde{N}$ and $\tilde{N} \geqslant N_{\mathrm{Lem\,3.3}}(\epsilon, \delta, \tilde{d}, \tilde{R})$. Since $\tilde{d}_i \in \tilde{R}_i$, any element of $\mathcal{A}_{\tilde{N}}(3\epsilon/4, \tilde{d}, \tilde{R})$ naturally corresponds to an element of $\mathcal{A}_n\big(\epsilon, (\tilde{d}_i)_{i=1}^n, (\tilde{R}_i)_{i=1}^n\big)$: an element

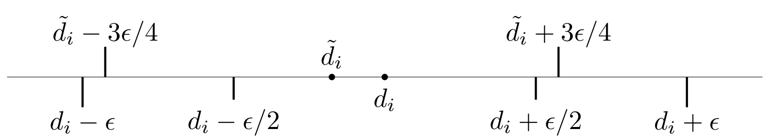

Figure 3. Location of $\tilde{d}_i$ relative to d_i.

$x \in \mathcal{A}_{\tilde{N}}(3\epsilon/4, \tilde{d}, \tilde{R})$ of the form $x = \sum_{i=1}^{\tilde{N}} x_i$, $x_i \in \tilde{R}$, corresponds to $y = \sum_{j=1}^{n} y_j$ given by

$$y_j = \begin{cases} x_i & \text{if } j = k_i; \\ \tilde{d}_j & \text{otherwise.} \end{cases}$$

By the choice of $\tilde{N}$ the set $\mathcal{A}_{\tilde{N}}(3\epsilon/4, \tilde{d}, \tilde{R})$ contains a subset which is M-tamely δ-dense in $\mathcal{U}_{3\epsilon/4}\big(\tilde{N}\tilde{d}\big)$ and therefore $\mathcal{A}_n\big(3\epsilon/4, (\tilde{d}_i)_{i=1}^n, (\tilde{R}_i)_{i=1}^n\big)$ has a subset that is M-tamely δ-dense in $\mathcal{U}_{3\epsilon/4}\big(\sum_{i=1}^n \tilde{d}_i\big)$. Finally,

$$\mathcal{A}_n\big(3\epsilon/4, (\tilde{d}_i)_{i=1}^n, (\tilde{R}_i)_{i=1}^n\big) \subseteq \mathcal{A}_n\big(\epsilon, (d_i)_{i=1}^n, (R_i)_{i=1}^n\big) \text{ and } \mathcal{U}_{\epsilon/2}\Big(\sum_{i=1}^n d_i\Big) \subseteq \mathcal{U}_{3\epsilon/4}\Big(\sum_{i=1}^n \tilde{d}_i\Big),$$

implying that $\mathcal{A}_n\big(\epsilon, (d_i)_{i=1}^n, (R_i)_{i=1}^n\big)$ contains a subset which is M-tamely δ-dense in $\mathcal{U}_{\epsilon/2}\big(\sum_{i=1}^n d_i\big)$ as desired. □

Theorem 3.5. *Let $S = \{\upsilon + t_m : m \in \mathbb{N}\}$. Any free Borel flow admits a cross section $\mathcal{C}$ that has arbitrarily large $\mathsf{E}^S_{\mathcal{C}}$-classes within every orbit.*

Proof. Let $\mathfrak{F}$ be a free Borel flow on a standard Borel space. Set

$$\epsilon_n = 2^{-n-1}\frac{\min\{1, \upsilon + t_0, \upsilon\}}{3}, \quad n \in \mathbb{N}.$$

Since S generates a dense subgroup of $\mathbb{R}$, by Lemma 2.1 we may find

$$K_0 > \max\{\upsilon, \upsilon + t_0\} + 1$$

and M_0 so big that $\mathcal{T}^*_{M_0}$ is $\epsilon_0/12$-dense in $[K_0 - 1, \infty)$.

Let $\mathcal{C}_0$ be a cross section of $\mathfrak{F}$ such that $\vec{\mathrm{gap}}_{\mathcal{C}_0}(x) \in [K_0+1, K_0+2]$ for all $x \in \mathcal{C}_0$ (it exists by [Slu, Corollary 2.3]). Note that $\mathsf{E}^S_{\mathcal{C}_0}$ is the trivial equivalence relation, since $S \subseteq [0, K_0]$. Set $D_0 = K_0 + 3$, $N_0 = 1$, and

$$\begin{aligned}
N_{n+1} &= N_{\mathrm{Lem\,3.4}}(\epsilon_n, \epsilon_{n+1}/12, D_n, M_n)\\
M_{n+1} &= M_{\mathrm{Lem\,3.4}}(\epsilon_n, \epsilon_{n+1}/12, D_n, M_n)\\
D_{n+1} &= (2N_{n+1} + 2)D_n.
\end{aligned}$$

We now construct cross sections $\mathcal{C}_n$ inductively as follows. We begin by selecting a sub cross section of $\mathcal{C}_0$ which consists of pairs of adjacent points in $\mathcal{C}_0$ with at least N_1 at most $2N_1 + 1$ points between any two pairs. By the choice of K_0, within each pair we may move the right point by at most ϵ_0 so as to make the gap an element of $\mathcal{T}_{M_0}$. This means that we can add points into the resulting gap so that the distances between adjacent points will be elements of S. This concludes the construction of $\mathcal{C}_1$. The process is illustrated in Figure 4.

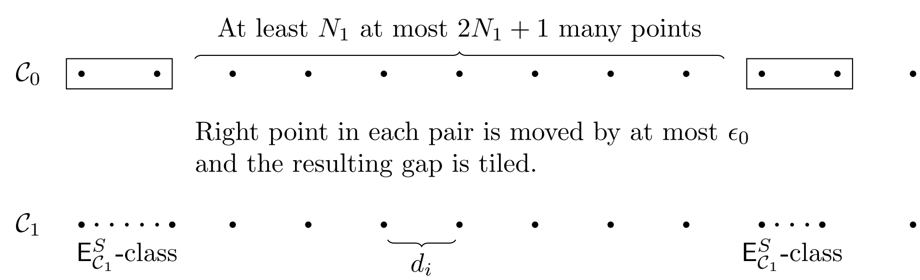

Figure 4. Construction of $\mathcal{C}_1$

We call $\mathsf{E}^S_{\mathcal{C}_1}$-classes, constructed via "tiling the gap" process, rank 1 blocks, and we refer to "isolated points" in $\mathcal{C}_1$ as to rank 0 blocks. It is now a good time to explain the choice of N_1, M_1, and D_1. First of all, D_0 represents an upper bound on the distance between adjacent points in $\mathcal{C}_0$. D_0 was taken with an excess to ensure that it remains a bound even if each point is moved by at most $\sum \epsilon_k$. D_1 respectively represents an upper bound on the distance between adjacent rank 1 blocks in $\mathcal{C}_1$. Between any two adjacent rank 1 blocks there are at least N_1-many rank 0 blocks, and therefore there are at least N_1-many gaps of size at least K_0 each. Let $d_1, \ldots, d_n$ denote the lengths of these gaps (see Figure 4). By the choice of N_1 and Lemma 3.4, each d_i can be distorted by at most ϵ_0 into $\tilde{d}_i$ in such a way that $\tilde{d}_i \in \mathcal{T}_{M_0}$ and the whole sum $\sum_{i=1}^n d_i$ is distorted by at most $\epsilon_1/12$. In fact, we have many ways of doing so. To be more specific, let

$$L_i = \mathcal{U}_\epsilon(d_i) \cap \mathcal{T}^*_{M_0} \text{ and } R_i = \big(L_i + (t_m)_{m=M_0}^\infty\big) \cap \mathcal{U}_\epsilon(d_i).$$

The sets R_i satisfy the assumptions of Lemma 3.4 and the set $\mathcal{A}_n\big(\epsilon, (d_i)_{i=1}^n, (R_i)_{i=1}^n\big)$ corresponds to possible ways of moving the right rank 1 block in Figure 4, when each rank 0 point in the midst is moved according to R_i. By the conclusion of Lemma 3.4, there is a set $\bar{R}$ which is M_1-tamely $\epsilon_1/12$-dense in $\mathcal{U}_{\epsilon_0/2}\big(\sum_{i=1}^n d_i\big) = \mathcal{U}_{\epsilon_1}\big(\sum_{i=1}^n d_i\big)$. To each pair of adjacent rank 1 blocks we associate such a set $\bar{R}$, and during the next step of the construction we shall move rank 1 blocks only as prescribed by $\bar{R}$.

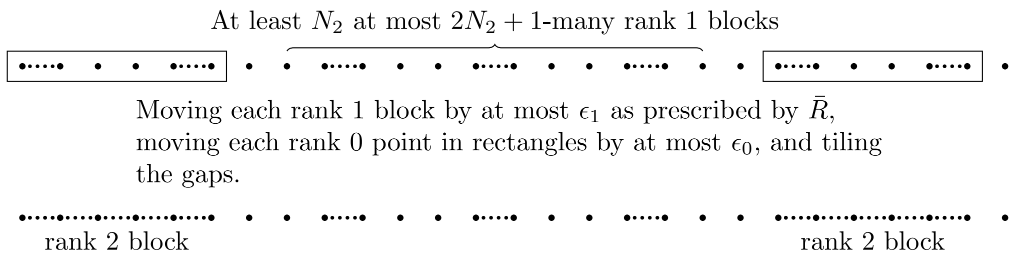

Figure 5. Construction of $\mathcal{C}_2$

The construction of $\mathcal{C}_2$ from $\mathcal{C}_1$ is analogous to the base step. We pick pairs of adjacent rank 1 blocks in $\mathcal{C}_1$ with at least N_2 at most $2N_2+1$ rank 1 blocks in between. Within each pair the right rank 1 block is moved according to (any element of) the corresponding $\bar{R}$, which results in moving each rank 0 point in between according to R_i. All the gaps can now be tiled (i.e., partitioned into segments of lengths in S), resulting in a cross section $\mathcal{C}_2$. Since the argument in Lemma 3.4 provides an algorithm for constructing the required sets, the process can be performed in a Borel way. The $\mathsf{E}^S_{\mathcal{C}_2}$-classes obtained by tiling gaps in $\mathcal{C}_1$ are called rank 2 blocks (see Figure 5). The procedure continues in a similar fashion — to define $\mathcal{C}_3$ we take sufficiently distant pairs of adjacent rank 2 blocks, move the right rank 2 block within each pair by at most ϵ_2 in a way that moves each rank 1 block in between by at most ϵ_1, and each rank 0 block by at most ϵ_0 and turns all the gaps within the pair into elements of $\mathcal{T}(S)$. We add points to tile these gaps, thus creating rank 3 blocks and cross section $\mathcal{C}_3$.

When passing from $\mathcal{C}_n$ to $\mathcal{C}_{n+1}$, each block of rank k is moved by no more than ϵ_k, and if any point is moved, it becomes an element of rank $n+1$ block in $\mathcal{C}_{n+1}$. Since $\sum_{i=0}^{\infty} \epsilon_n$ converges, this ensures that each point "converges to a limit" and the required cross section $\mathcal{C}$ consists of all the "limit points". It is evident from the construction that $\mathcal{C}$ has arbitrarily large $\mathsf{E}^S_{\mathcal{C}}$-blocks within each orbit. The formal details of defining the limit cross section are no different from those of Theorem 9.1 in [Slu] and are similar to those in Theorem 2.2; we therefore omit them. ☐

4. ⚠ Cross sections under construction

Theorem 4.1. *Let $\mathfrak{F}$ be a free Borel flow on a standard Borel space X and let $S \subseteq \mathbb{R}^{>0}$ be a non-empty set bounded away from zero.*

(I) Assume $\langle S \rangle = \lambda\mathbb{Z}$, $\lambda > 0$. The flow $\mathfrak{F}$ admits an S-regular cross section if and only if it admits a $\{\lambda\}$-regular cross section.

(II) Assume $\langle S \rangle$ is dense in $\mathbb{R}$, but $\langle S \cap [0,n] \rangle = \lambda_n\mathbb{Z}$, $\lambda_n \geqslant 0$, for all natural $n \in \mathbb{N}$ (we take $\lambda_n = 0$ if $S \cap [0,n]$ is empty). The flow $\mathfrak{F}$ admits an S-regular cross section if and only if the phase space X can be partitioned into $\mathfrak{F}$-invariant Borel pieces (some of which may be empty)

$$X = \Big(\bigsqcup_{i=0}^{\infty} X_i\Big) \sqcup X_\infty$$

such that $\mathfrak{F}|_{X_\infty}$ is sparse and $\mathfrak{F}|_{X_i}$ admits a $\{\lambda_i\}$-regular cross section.

(III) Assume there is $n \in \mathbb{N}$ such that $\langle S \cap [0,n] \rangle$ is dense in $\mathbb{R}$. Any free flow admits an S-regular cross section.

Proof. (I) Suppose $\mathfrak{F}$ admits an S-regular cross section, say $\mathcal{C}$. Since $\langle S \rangle = \lambda\mathbb{Z}$, every element of S is a multiple of λ, so we may tile all the gaps in $\mathcal{C}$ by intervals of length λ. More precisely,

$$\mathcal{D} = \big\{x + k\lambda : x \in \mathcal{C}, \vec{\mathrm{gap}}_{\mathcal{C}}(x) = n\lambda, 0 \leqslant k < n\big\}$$

is a $\{\lambda\}$-regular cross section.

Suppose now $\mathfrak{F}$ admits a $\{\lambda\}$-regular cross section, say $\mathcal{D}$. It is easy to check that there exists $N \in \mathbb{N}$ such that $n\lambda \in \mathcal{T}(S)$ for all $n \geqslant N$. Let $\mathcal{C}'$ be a sub cross section of $\mathcal{D}$ such that $\vec{\mathrm{gap}}_{\mathcal{C}'}(x) \geqslant N\lambda$ for all $x \in \mathcal{C}'$. We have that $\vec{\mathrm{gap}}_{\mathcal{C}'}(x) \in \mathcal{T}(S)$ for all $x \in \mathcal{C}'$, and so each gap in $\mathcal{C}'$ can be tiled by intervals of lengths in S, which results in an S-regular cross section.

(II) First suppose that X admits a decomposition into invariant pieces of the form

$$X = \Big(\bigsqcup_{i=0}^{\infty} X_i\Big) \sqcup X_\infty.$$

Since $\langle S\rangle$ is dense in $\mathbb{R}$ and $\mathfrak{F}|_{X_\infty}$ is sparse, by Theorem 2.2 $\mathfrak{F}|_{X_\infty}$ admits an S-regular cross section $\mathcal{C}_\infty$. By assumption, $\mathfrak{F}|_{X_i}$ admits a $\{\lambda_i\}$-regular cross section, so by item (I) it also admits an $S \cap [0,i]$-regular cross section $\mathcal{C}_i$. The union

$$\mathcal{C}_\infty \sqcup \bigsqcup_{i\in\mathbb{N}} \mathcal{C}_i$$

of these cross sections is an S-regular cross section on X.

For the other direction suppose $\mathcal{C}$ is an S-regular cross section for $\mathfrak{F}$. Let X_∞ be the set of orbits where the gap function is unbounded:

$$X_\infty = \Big\{x \in \mathcal{C} : \sup\big\{\vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi^k_{\mathcal{C}}(x)\big) : k \in \mathbb{Z}\big\} = \infty\Big\}.$$

Some orbits in X_∞ may not have “bi-infinitely” unbounded gaps, but the restriction of the flow onto the set of such orbits is smooth; we may thus modify $\mathcal{C}$ on this set and assume that $\mathcal{C} \cap X_\infty$ is always “bi-infinitely” unbounded, and is therefore a sparse cross section. Thus $\mathfrak{F}|_{X_\infty}$ is sparse. Let for $i \in \mathbb{N}$

$$X_{i+1} = \Big\{x \in \mathcal{C} : \sup\big\{\vec{\mathrm{gap}}_{\mathcal{C}}\big(\phi^k_{\mathcal{C}}(x)\big) : k \in \mathbb{Z}\big\} \leqslant i+1\Big\} \setminus X_i,$$

where $X_0 = \varnothing$. By assumption there is $\lambda_i \in \mathbb{R}^{\geqslant 0}$ such that $\langle S \cap [0,i]\rangle = \lambda_i\mathbb{Z}$. Since all the gaps in $\mathcal{C} \cap X_i$ belong to $S \cap [0,i]$, item (I) applies, and $\mathfrak{F}|_{X_i}$ admits a $\{\lambda_i\}$-regular cross section.

(III) Suppose $\langle S \cap [0,n]\rangle$ is dense in $\mathbb{R}$. We may assume for notational convenience that S itself is bounded and $\langle S\rangle$ is dense in $\mathbb{R}$. For a bounded subset of $\mathbb{R}$ to generate a dense subgroup, one of two things has to happen. One possibility is that S contains two rationally independent reals $\alpha, \beta \in S$. If this is the case, Theorem 9.1 of [Slu] applies and generates an $\{\alpha,\beta\}$-regular cross section for $\mathfrak{F}$.

The other possibility is that there are infinitely many elements in S. In that case we may select a limit point υ for S. While υ is not necessarily an element of S, there is a sequence $(s_n)_{n=0}^\infty \subseteq S$, which we may assume to be monotone, such that $s_n \to \upsilon$. We therefore find ourselves in the context of Theorem 3.5 for $t_m = s_m - \upsilon$, which ensures existence of a cross section $\mathcal{D}$ with arbitrarily large $\mathsf{E}^S_{\mathcal{D}}$-classes within each orbit. Orbits in $\mathcal{D}$ split into three categories: $\mathcal{D} = \mathcal{D}_r \sqcup \mathcal{D}_0 \sqcup \mathcal{D}_s$, where

- D_r consists from those orbits which constitute a single $\mathsf{E}^S_{\mathcal{D}}$-class;
- $\mathcal{D}_0$ contains orbits which have at least two $\mathsf{E}^S_{\mathcal{D}}$-classes at least one of which is infinite;
- $\mathcal{D}_s$ draws all the orbits with all $\mathsf{E}^S_{\mathcal{D}}$-classes being finite.

More formally, sets D_r, D_0, and D_s are given by

$$\begin{aligned}
\mathcal{D}_r =& \big\{x \in \mathcal{D} : x\, \mathsf{E}^S_{\mathcal{D}} \phi^k_{\mathcal{D}}(x) \text{ for all } k \in \mathbb{Z}\big\},\\
\mathcal{D}_0^* =& \big\{x \in \mathcal{D} : \exists k \in \mathbb{Z}\ \forall n \in \mathbb{N} \quad \phi^k_{\mathcal{D}}(x)\, \mathsf{E}^S_{\mathcal{D}}\, \phi^{k+n}_{\mathcal{D}}(x)\big\} \cup\\
& \big\{x \in \mathcal{D} : \exists k \in \mathbb{Z}\ \forall n \in \mathbb{N} \quad \phi^k_{\mathcal{D}}(x)\, \mathsf{E}^S_{\mathcal{D}}\, \phi^{k-n}_{\mathcal{D}}(x)\big\},\\
\mathcal{D}_0 =& \mathcal{D}_0^* \setminus \mathcal{D}_r,\\
\mathcal{D}_s =& \big\{x \in \mathcal{D} : \forall k \in \mathbb{Z}\ \exists m,n \in \mathbb{Z}\ (m<0) \text{ and } (n>0) \text{ and },\\
& \neg\big(\phi^k_{\mathcal{D}}(x)\, \mathsf{E}^S_{\mathcal{D}} \phi^{k+m}_{\mathcal{D}}(x)\big) \text{ and } \neg\big(\phi^k_{\mathcal{D}}(x)\, \mathsf{E}^S_{\mathcal{D}} \phi^{k+n}_{\mathcal{D}}(x)\big)\big\}.
\end{aligned}$$

Set X_r, X_0, and X_s to be the saturation of sets $\mathcal{D}_r$, $\mathcal{D}_0$, and $\mathcal{D}_s$:

$$X_r = \mathcal{D}_r + \mathbb{R}, \qquad X_0 = \mathcal{D}_0 + \mathbb{R}, \qquad X_s = \mathcal{D}_s + \mathbb{R}.$$

The set $\mathcal{D}_r$ is an S-regular cross section for $\mathfrak{F}|_{X_r}$. The restriction of $\mathfrak{F}$ onto X_0 is smooth, as taking finite endpoints of infinite $\mathsf{E}^S_{\mathcal{D}_0}$-classes picks at most two points from each orbit. The flow $\mathfrak{F}|_{X_0}$ therefore admits any kind of cross section. It remains to deal with the restriction of $\mathfrak{F}$ on X_s. Let $\mathcal{C} \subseteq \mathcal{D}_s$ to consist of endpoints of $\mathsf{E}^S_{\mathcal{D}_s}$-classes:

$$\mathcal{C} = \left\{x \in \mathcal{D}_s : \neg\big(x\,\mathsf{E}^S_{\mathcal{D}_s}\,\phi_{\mathcal{D}_s}(x)\big) \text{ or } \neg\big(x\,\mathsf{E}^S_{\mathcal{D}_s}\,\phi^{-1}_{\mathcal{D}_s}(x)\big)\right\}.$$

The condition of having arbitrarily large $\mathsf{E}^S_{\mathcal{D}_s}$-classes ensures that $\mathcal{C}$ is a *sparse* cross section for $\mathfrak{F}|_{X_s}$. Theorem 2.2 applies and finishes the proof. □

Remark 4.2. In the ergodic theoretical framework, when two flows that differ on a set of measure zero are identified, items (II) and (III) collapse. This is because every flow that preserves a finite measure is sparse on an invariant set of full measure (see [Slu, Theorem 3.3]). For an ergodic theorist any free flow admits an S-regular cross section whenever S generates a dense subgroup of $\mathbb{R}$.

In conclusion we would like to give an example of a flow which illustrates the difference between items (II) and (III) above in the Borel setting. Let $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the odometer map: if $x \in 2^{\mathbb{N}}$ is such that $x = 1^n 0*$, then $\sigma(x) = 0^n 1*$; also $\sigma(1^\infty) = 0^\infty$. This is a free Borel automorphisms on the Cantor space.

Proposition 4.3. *Let $S \subseteq \mathbb{R}^{>0}$ be a non-empty set of positive reals bounded away from zero such that*

- *S generates a dense subgroup of $\mathbb{R}$.*
- *All elements in $\mathbb{R}$ are pairwise rationally dependent, i.e., $\langle S \rangle = \beta\mathbb{Q}$ for some $\beta \in \mathbb{R}^{>0}$.*
- *$S \cap [0, n]$ is finite for every $n \in \mathbb{N}$.*

A typical example is the set of partial sums of the harmonic series:

$$S = \left\{\sum_{i=1}^{n} \frac{1}{i} : n \geqslant 1\right\}.$$

Let $\alpha \in \mathbb{R}^{>0}$ be any real that is rationally independent form β. Let $\mathfrak{F}$ be the flow under the constant function α over the Cantor space $2^{\mathbb{N}}$ with odometer σ as the base automorphism. Such a flow does not admit an S-regular cross section.

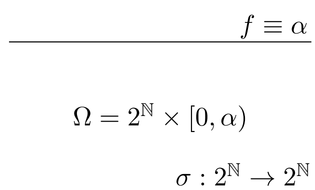

Figure 6. The flow $\mathfrak{F}$ under the function $f \equiv \alpha$ with the base automorphism σ.

Proof. The proof is by contradiction. Set $\Omega = 2^{\mathbb{N}} \times [0, \alpha)$ and suppose that $\mathcal{C} \subseteq \Omega$ is an S-regular cross section for $\mathfrak{F}$. The space Ω can naturally be endowed with a compact topology which turns $\mathfrak{F}$ into a continuous flow on a compact metric space. Indeed, $\Omega = 2^{\mathbb{N}} \times [0, \alpha]/\!\sim$, where $\sim$ identifies (x, α) with $\big(\sigma(x), 0\big)$. Since $\sim$ is closed, the factor topology turns Ω into a compact metric space, and the flow $\mathfrak{F}$ is seen to be continuous. Moreover, since σ is a minimal[2] homeomorphism of the Cantor space, one easily checks that $\mathfrak{F}$ on Ω is also minimal. By Proposition 3.2 in [Slu], there is a Borel invariant comeager subset $Z \subseteq \Omega$ such that $Z \cap \mathcal{C}$ has all gaps bounded by some $n_0 \in \mathbb{N}$. Since $S \cap [0, n_0]$ is finite, and since all elements in $S \cap [0, n_0]$ are rationally dependent, there is $\lambda \in \mathbb{R}^{>0}$ such that $\langle S \cap [0, n_0]\rangle = \lambda\mathbb{Z}$. By item (I) of Theorem 4.1 this means that there is a $\{\lambda\}$-regular cross section $\mathcal{D} \subseteq Z$ for the flow $\mathfrak{F}|_Z$. By Ambrose's criterion for existence of a $\{\lambda\}$-regular cross section (see Proposition 1.1) there exists a Borel function $f : Z \to \mathbb{C} \setminus \{0\}$ such that

$$f(x + r) = e^{\frac{2\pi i r}{\lambda}} f(x) \text{ for all } x \in Z \text{ and } r \in \mathbb{R}.$$

Let $X = \mathrm{proj}_{2^{\mathbb{N}}}(Z)$. Since Z is $\mathfrak{F}$-invariant,

$$X = Z \cap \left\{(x, 0) : x \in 2^{\mathbb{N}}\right\}.$$

[2]A homeomorphism is minimal if every its orbit is dense.

Note that X must be Borel, σ-invariant, and comeager in $2^{\mathbb{N}}$. We restrict the function f to the base $2^{\mathbb{N}} \times \{0\}$. Since f is Borel, there is a comeager subset $\tilde{X}$ of $2^{\mathbb{N}} \times \{0\}$ such that $f|_{\tilde{X}}$ is continuous (see [Kec95, 8.38]). Without loss of generality we may assume that $\tilde{X} \subseteq X$ and that $\tilde{X}$ is σ-invariant. Pick $x_0 \in \tilde{X}$. Since $x + \alpha = \sigma(x)$ for all $x \in 2^{\mathbb{N}}$, for any $k \in \mathbb{N}$ we have

$$f\big(\sigma^k(x_0)\big) = f(x_0 + k\alpha) = e^{\frac{2\pi i k\alpha}{\lambda}} f(x_0).$$

We take $k = 2^m$ in the above. Since $\sigma^{2^m}(x) \to x$ for all $x \in 2^{\mathbb{N}}$, and since f is continuous on $\tilde{X}$ which is σ-invariant, we get

$$e^{\frac{2\pi i 2^m \alpha}{\lambda}} f(x_0) \to f(x_0) \text{ as } m \to \infty,$$

which is equivalent to $2^m\alpha/\lambda \to 0 \mod \mathbb{Z}$, because $f(x_0) \neq 0$. By assumption on S, α/λ is an irrational number, thus to finish the proof it remains to show that $2^m\gamma \not\to 0 \mod \mathbb{Z}$ for any irrational $\gamma \in \mathbb{R}^{>0}$.

Suppose towards a contradiction that $2^m\gamma \to 0 \mod \mathbb{Z}$ for an irrational γ. Pick m_0 so big that for every $m \geqslant m_0$ there is $k_m \in \mathbb{Z}$ such that $|2^m\gamma - k_m| < 1/4$. Let $a = 2^{m_0}\gamma - k_{m_0}$, and let $p \in \mathbb{N}$ be the smallest natural such that $|2^p a| \geqslant 1/4$. It is easy to see that $|2^p a| < 1/2$. Therefore,

$$1/4 \leqslant \big|2^{p+m_0}\gamma - 2^p k_{m_0}\big| < 1/2.$$

Thus $2^m \not\to 0 \mod \mathbb{Z}$ as claimed. □

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