SMOOTH ORBIT EQUIVALENCE OF MULTIDIMENSIONAL BOREL FLOWS

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ABSTRACT. Free Borel $\mathbb{R}^d$-flows are smoothly equivalent if there is a Borel bijection between the phase spaces that maps orbits onto orbits and is a C^∞-smooth orientation preserving diffeomorphism between orbits. We show that all free non-tame Borel $\mathbb{R}^d$-flows are smoothly equivalent in every dimension $d \geq 2$. This answers a question of B. Miller and C. Rosendal.

1. INTRODUCTION

Let us begin by defining the notions mentioned in the title as well as the related concepts that are needed to state the main results of our work. A **Borel flow** is a Borel action $\mathbb{R}^d \curvearrowright \Omega$ of the Euclidean group on a standard Borel space Ω. An action of $\vec{r} \in \mathbb{R}^d$ upon $x \in \Omega$ is denoted by $x + \vec{r}$. An **orbit equivalence** between two flows $\mathbb{R}^d \curvearrowright \Omega$ and $\mathbb{R}^d \curvearrowright \Omega'$ is a Borel bijection $\xi : \Omega \to \Omega'$ that sends orbits onto orbits: $\xi\big(x + \mathbb{R}^d\big) = \xi(x) + \mathbb{R}^d$ for all $x \in \Omega$; when such a map exists, we say that the flows are **orbit equivalent**. For an action $\mathbb{R}^d \curvearrowright \Omega$ we let E denote the corresponding orbit equivalence relation: $xEy \iff x + \mathbb{R}^d = y + \mathbb{R}^d$. When the action is moreover free, $\rho : E \to \mathbb{R}^d$ will stand for the associated **cocycle**, determined uniquely by the condition $x + \rho(x, y) = y$. Given free Borel flows on phase spaces Ω and Ω', any orbit equivalence $\xi : \Omega \to \Omega'$ gives rise to a map $\alpha_\xi : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ defined by $\alpha_\xi(x, \vec{r}) = \rho\big(\xi(x), \xi(x + \vec{r})\big)$. A Borel orbit equivalence ξ is said to be a **smooth equivalence** if $\alpha_\xi(x, \cdot\,) : \mathbb{R}^d \to \mathbb{R}^d$ is a C^∞-smooth orientation preserving diffeomorphism for all points $x \in \Omega$.

1.1. Prior work.

The concept of orbit equivalence originated in ergodic theory, where the set-up differs in two essential aspects. First, one endows phase spaces of flows with probability measures. The flows are then assumed to preserve (or to quasi-preserve) these measures. Likewise, orbit equivalence maps are required to be at least quasi-measure-preserving. Second, all the properties of interest are expected to hold up to a null set. For instance, an orbit equivalence map may mix elements between orbits as long as this behavior is confined to a set of measure zero. The latter is a notable relaxation of the Borel definition.

Smooth equivalence of one-dimensional flows, better known under the name of **time-change equivalence**, is closely connected to the notion of Kakutani equivalence of automorphisms [Kak43], and has been studied extensively since the pioneering works of J. Feldman [Fel76] and A. Katok [Kat75, Kat77]. An important milestone was the monograph of D. Ornstein, D. Rudolph, and B. Weiss [ORW82], which showed, in particular, that there is a continuum of pairwise time-change inequivalent ergodic measure-preserving flows. The higher-dimensional case was considered by D. Rudolph [Rud79], where he found a striking difference with the one-dimensional situation—all ergodic measure-preserving $\mathbb{R}^d$-flows, $d \geq 2$, are smoothly equivalent. J. Feldman obtained a similar result for quasi-measure-preserving flows in [Fel91, Fel92].

Note that in the definition of time-change equivalence, it is essential to allow for the orbit equivalence maps to be quasi-measure-preserving even if all the flows are assumed to be measure-preserving (see [Nak88, Remark 4.5] regarding the connection between the integrability of the cocycle as required in [Kat75] and measure class preservation of the orbit equivalence). This underlines the strength of Rudolph's result, as it is shown in [Rud79, Proposition 1.1] that in the dimensions $d \geq 2$ there is a single class of ergodic measure-preserving flows under *measure-preserving* smooth equivalence relation. Further discussion on what restrictions on the orbit equivalence maps may produce finer equivalence relations among higher-dimensional flows can also be found in [Rud79].

In this paper, we are interested in the descriptive set-theoretical viewpoint. This means that neither flows are assumed to preserve any measures (thus increasing the pool of flows to consider), nor orbit equivalence

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maps have to be quasi-measure-preserving (which may potentially collapse previously inequivalent flows into the same class). On the other hand, the necessity to run constructions on all orbits may, in principle, increase the number of equivalence classes, as complicated dynamics of a flow can be contained in a null set. All in all, this framework is in a general position to the one of ergodic theory, and ahead of time, it is not clear how versatile smooth equivalence will turn out to be. The key work that investigated the subject from this purely Borel vantage point is the article by B. Miller and C. Rosendal [MR10], where they studied Kakutani equivalence of Borel automorphisms and classified one-dimensional flows up to descriptive time-change equivalence.

Theorem 1 (Miller–Rosendal [MR10, Theorem B])**.** *All non-tame[1] free Borel $\mathbb{R}$-flows are smoothly equivalent.*

As the one-dimensional case has been settled, they posed [MR10, Problem C] the following problem: "Classify free Borel $\mathbb{R}^d$-actions on Polish spaces up to (C^∞-)time-change isomorphism." In other words, does the analog of D. Rudolph and J. Feldman theorems hold? Are there two (non-tame) inequivalent free Borel $\mathbb{R}^d$-flows for any $d \geq 2$?

These and related topics were studied in [Slu19], where we showed that any two non-tame free $\mathbb{R}^d$-flows, $d \geq 2$, are smoothly equivalent *up to a compressible set*. The method to prove this result was an expansion of the one used in [Fel91], and such a statement is about as far as ergodic-theoretical methods can go, since a compressible set has measure zero relative to any probability measure invariant under the flow.

1.2. **Main results.** In the present work, we give a complete answer to Problem C of [MR10] by showing that all non-tame free $\mathbb{R}^d$-flows, $d \geq 2$, are smoothly equivalent (Theorem 21). Table 1 provides a concise summary and compares the number of classes up to smooth equivalence in ergodic theory and Borel dynamics.

TABLE 1. Number of classes of smooth orbit equivalence.

	Ergodic Theory	Borel Dynamics
$d = 1$	$\mathfrak{c}$-many [ORW82]	one [MR10]
$d \geq 2$	one [Rud79, Fel91]	one

Many results in ergodic theory and Borel dynamics of $\mathbb{Z}^d$ and $\mathbb{R}^d$ actions are based on the fact that such actions are (essentially) hyperfinite. In ergodic theory, this is manifested by a group of related theorems that usually go under the name of "Rokhlin Lemma". The key idea here is that one can find a measurable set that intersects every orbit of the flow in a set of pairwise disjoint rectangles (more precisely, d-dimensional parallelepipeds). Moreover, one often takes a sequence of such sets, where rectangles cohere and eventually cover all the orbits (at least, up to a null set). The details of the assumptions on such regions vary, but a construction of this form is present in many arguments, including the references above. The direct analog of such a tower of coherent rectangular regions is not possible in Borel dynamics. One, therefore, has to rely on more complicated geometric shapes (see, for instance, [JKL02, Theorem 1.16] and [GJ15]).

Our argument also requires regions witnessing hyperfiniteness. The key property we need is for them to be smooth disks. S. Gao, S. Jackson, E. Krohne, and B. Seward [GJKS] have shown the possibility to construct such regions for low-dimensional flows. Their argument is an elaboration of the orthogonal marker regions technique developed in [GJ15]. We take a different path and build upon the approach presented by A. Marks and S. Unger in [MU17, Appendix A]. Section 2 is devoted to these topics and it leads to Theorem 5 that shows the existence of such disk-shaped regions in all dimensions.

In order to prove that all non-tame free $\mathbb{R}^d$-flows are smoothly equivalent, we leverage the work of Miller–Rosendal that handles the case of $d = 1$. To this end Section 3 introduces the concept of a special flow, which is a type of an $\mathbb{R}^d$-flow that is build over a one-dimensional flow in a very primitive way. We show in Theorem 20 that all $\mathbb{R}^d$-flows are smoothly equivalent to a special flow. This piece is the technical core of this paper.

Finally, in Section 4 we prove the main result on smooth equivalence of free $\mathbb{R}^d$-flows (Theorem 21) and conclude with some remarks on its potential strengthening.

[1]A flow $\mathbb{R}^d \curvearrowright \Omega$ is **tame** if there is a Borel set $S \subseteq \Omega$ that intersects every orbit of the flow in a single point. The term **smooth** is often used in the literature instead, but since we also work with diffeomorphisms, this word will be used in the traditional sense of differential geometry. Tame flows should be considered trivial in the context of the questions we are interested in this paper.

1.3. **Notations.** The following notations are used throughout the paper: $B(R) \subseteq \mathbb{R}^d$ denotes a ball of radius R centered at the origin; $|| \cdot ||$ stands for the ℓ^2- norm in $\mathbb{R}^d$; and $\mathrm{dist}(\vec{r}, \vec{r}\,')$ refers to the Euclidean distance in $\mathbb{R}^d$. By a diffeomorphism we always mean a C^∞-smooth orientation preserving diffeomorphism. A smooth disk therefore refers to any compact region in $\mathbb{R}^d$ that is diffeomorphic to a ball. Interior of a set $F \subseteq \mathbb{R}^d$ is denoted by $\mathrm{int}\, F$, and ∂F stands for the boundary of F. Given a Cartesian product $X_1 \times X_2 \times \cdots \times X_m$, $\mathrm{proj}_k : \prod_{i=1}^m X_i \to X_k$ denotes the projection onto the k^{th} coordinate, and, more generally, $\mathrm{proj}_{[k,l]}$ will denote the projection onto $X_k \times \cdots \times X_l$, for $k \leq l$.

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2. Disk-Shaped Coherent Regions

We begin by stating the following classical fact from differential topology (see, for instance, [Fel91, Proposition 2.6]), which will be used throughout the paper to justify the existence of diffeomorphisms moving disks in a prescribed fashion.

Lemma 2 (Extension Lemma). *Let F and F' be smooth disks in $\mathbb{R}^d$, $d \geq 2$, each containing m smooth disks in its interior: $D_1, \ldots, D_m \subset \mathrm{int}\, F$ and $D'_1, \ldots, D'_m \subset \mathrm{int}\, F'$. Suppose that disks D_i are pairwise disjoint and so are the disks D'_i. Any collection of orientation preserving diffeomorphisms $\phi_i : D_i \to D'_i$ can be extended to an orientation preserving diffeomorphism $\psi : F \to F'$.*

Theorems in Borel dynamics of $\mathbb{R}^d$ and $\mathbb{Z}^d$ actions often rely on variants of the hyperfiniteness construction. Our argument is no exception, and this section gives the specific version to be used later in Section 3. The cases of $d = 2$ and $d = 3$ of Theorem 5 are due to S. Gao, S. Jackson, E. Krohne, and B. Seward; they are announced to appear in [GJKS]. We borrow the structure of our argument from A. Marks and S. Unger [MU17, Appendix A] and supplement it with Lemma 3 to get the desired shape of the regions for all dimensions $d \geq 2$.

Lemma 3 (Separation Lemma). *Let $0 < R_1 < R_2$ be positive reals and let $D_1, \ldots, D_n \subset \mathbb{R}^d$, $d \geq 2$, be smooth pairwise disjoint disks of diameter $\mathrm{diam}(D_i) < (R_2 - R_1)/2$. There exists a smooth disk F wedged between the two balls, $B(R_1) \subseteq F \subseteq B(R_2)$, such that for each i either $D_i \subseteq \mathrm{int}\, F$ or $F \cap D_i = \varnothing$.*

Figure 1 illustrates the statement. Disks D_i are marked in gray and the required disk F is dashed.

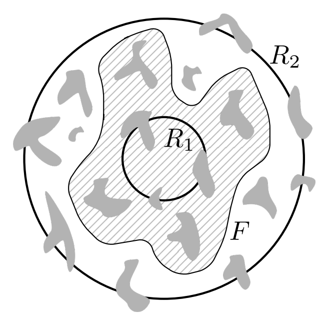

FIGURE 1. Separation Lemma.

Proof. The proof is by induction on the number of disks n. The base case $n = 0$ is trivial, we argue the step from $n - 1$ to n. If none of the disks D_i lie inside the open annulus $A = \mathrm{int}\, B(R_2) \setminus B(R_1)$, then the ball $F = B((R_2 + R_1)/2)$ works. Otherwise, select a ball $D_{i_0} \subset A$. By the inductive assumption there is a disk F' that fulfills the conclusions of the lemma for all disks D_i, $i \neq i_0$. We are done if also $D_{i_0} \cap F' = \varnothing$ or $D_{i_0} \subset \mathrm{int}\, F'$, so assume otherwise (Figure 2a).

Find a smooth disk $G \subset A$ that contains $D_{i_0} \subset \mathrm{int}\, G$ in its interior and does not intersect any other disk D_i. Pick a disk $Z \subset \mathrm{int}\, G$ that is disjoint from $\partial F'$ (Figure 2b). Such a disk can be found, since the boundary $\partial F'$ is nowhere dense. Choose a diffeomorphism ψ supported on G such that $\psi(D_{i_0}) = Z$. Lemma 2 may be used to justify the existence of such a diffeomorphism. We have either $\psi(D_{i_0}) \subset \mathrm{int}\, F'$ or $\psi(D_{i_0}) \cap F' = \varnothing$. Set $F = \psi^{-1}(F')$ (Figure 2c).

Since ψ is supported on G, both conditions $F \cap D_i = \varnothing$ and $D_i \subset \mathrm{int}\, F$, $i \neq i_0$, continue to hold whenever they did so for F' instead of F. By construction we now also have either $F \cap D_{i_0} = \varnothing$ or $D_{i_0} \subset \mathrm{int}\, F$. $\square$

Let $\mathbb{R}^d \curvearrowright \Omega$ be a free Borel flow, and let E be its orbit equivalence relation. A set $\mathcal{C} \subset \Omega$ is said to be

- R-**discrete**, where R is a positive real, if $(c + B(R)) \cap (c' + B(R)) = \varnothing$ for all distinct $c, c' \in \mathcal{C}$;
- **discrete** if it is R-discrete for some $R > 0$;
- **cocompact** if there exists $R > 0$ such that $\mathcal{C} + B(R) = \Omega$;
- **complete** if it intersects every orbit of the action;

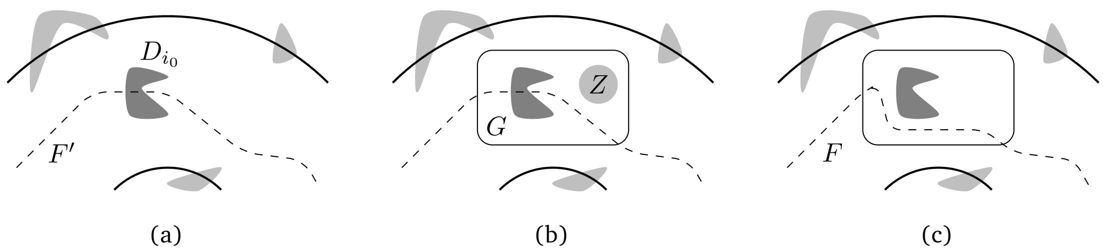

FIGURE 2. Construction of a disk F that separates disks D_i.

- a **cross section** if it is discrete and complete;
- **on a rational grid** (or simply **rational**, for short) if $\rho(c, c') \in \mathbb{Q}^d$ for all $c, c' \in \mathcal{C}$ such that cEc'.

We make use of the following result due to C. M. Boykin and S. Jackson.

Lemma 4 (Boykin–Jackson [BJ07], cf. Lemma A.2 of [MU17]). *Let $a_1 < a_2 < \cdots$ be an increasing sequence of natural numbers. For any free Borel flow $\mathbb{R}^d \curvearrowright \Omega$ there exists a sequence of a_i-discrete cocompact cross sections $\mathcal{C}_i$ such that $\bigcup_i \mathcal{C}_i$ is rational and for all $\epsilon > 0$, for every $x \in \Omega$, there are infinitely many i such that $||\rho(x, c)|| < \epsilon a_i$ for some $c \in \mathcal{C}_i$.*

Proof. The direct adaptation to $\mathbb{R}^d$-flows of the argument [MU17, Lemma A.2] (presented therein for $\mathbb{Z}^d$ actions) produces cross sections $\mathcal{C}_i'$ that satisfy all the conclusions except possibly for $\bigcup_i \mathcal{C}_i'$ being rational. As shown in [Slu19, Lemma 2.3], there is a *rational grid* for the flow, i.e., there is a complete rational set $Q \subset \Omega$ invariant under the action of $\mathbb{Q}^d$. Using Luzin-Novikov Theorem (see [Kec95, 18.14]), one can find cross sections $\mathcal{C}_i \subset Q$ and Borel bijections $\zeta_i : \mathcal{C}_i' \to \mathcal{C}_i$ such that $\bigcup_i \mathcal{C}_i$ is rational and for all $c \in \mathcal{C}_i'$ one has $cE\zeta_i(c)$ and $||\rho(c, \zeta_i(c))|| < 1$. In other words, every element in $\mathcal{C}_i'$ can be shifted by distance < 1 to ensure that all the cross sections are on the same rational grid. This argument is the content of [Slu19, Lemma 2.4].

The cross sections $\mathcal{C}_i$ continue to be cocompact and still satisfy the key property that for every $x \in \Omega$ and $\epsilon > 0$ there are infinitely many i with $||\rho(c, x)|| < \epsilon a_i$ for some $c \in \mathcal{C}_i$. The only minor issue is that this modification reduces the discreteness parameter by 1. Therefore if the original cross sections $\mathcal{C}_i'$ were chosen to be $(a_i + 1)$-discrete, then each of $\mathcal{C}_i$ is guaranteed to be a_i-discrete. □

To formulate the next theorem we need an extra bit of notation. Let $\mathbb{R}^d \curvearrowright \Omega$ be a free Borel flow. For a set $W \subseteq \Omega \times \Omega$ and $c \in \Omega$ we let $W(c)$ denote the slice over c, i.e., $W(c) = \{x \in \Omega : (c, x) \in W\}$. We also denote by $\widetilde{W}$ the set $\{(c, \vec{r}) \in \Omega \times \mathbb{R}^d : c + \vec{r} \in W(c)\}$. Note that $\widetilde{W}(c)$ is the region of $\mathbb{R}^d$ described by $W(c)$, when c is taken to be the origin of the coordinate system.

Theorem 5. *Let $\mathbb{R}^d \curvearrowright \Omega$ be a free Borel flow and let E denote its orbit equivalence relation. There exist cross sections $\mathcal{C}_n$ and Borel sets $W_n \subseteq (\mathcal{C}_n \times \Omega) \cap E$ such that $\bigcup_n \mathcal{C}_n$ is rational and for all $n \in \mathbb{N}$:*

(i) $\widetilde{W}_n(c)$ is a smooth disk for every $c \in \mathcal{C}_n$.
(ii) Sets $W_n(c)$, $c \in \mathcal{C}_n$, are pairwise disjoint.
(iii) For every $c' \in \mathcal{C}_m$, $m < n$, and every $c \in \mathcal{C}_n$, either $W_m(c') \cap W_n(c) = \varnothing$ or $W_m(c') \subseteq W_n(c)$. Moreover, in the latter case $\rho(c, c') + \widetilde{W}_m(c')$ is contained in the interior of $\widetilde{W}_n(c)$.
(iv) For all $x \in \Omega$ and all compact $K \subset \mathbb{R}^d$ there are m and $c \in \mathcal{C}_m$ such that $x + K \subseteq W_m(c)$.
(v) There are smooth disks $A_{n,k} \subseteq \mathbb{R}^d$, $k \in \mathbb{N}$, and a Borel partition $\mathcal{C}_n = \bigsqcup_{k \in \mathbb{N}} \mathcal{C}_{n,k}$ such that

$$W_n = \bigsqcup_k \{(c, c + \vec{r}) : c \in \mathcal{C}_{n,k}, \vec{r} \in A_{n,k}\} \quad \text{and} \quad \widetilde{W}_n = \bigsqcup_k \{(c, \vec{r}) : c \in \mathcal{C}_{n,k}, \vec{r} \in A_{n,k}\}.$$

Proof. Set $a_n = 5^n$, and let $\mathcal{C}_n$ be a sequence of cross sections produced by Lemma 4. Note that $\bigcup_n \mathcal{C}_n$ is guaranteed to be rational. We construct a sequence of Borel sets $W_n \subseteq (\mathcal{C}_n \times \Omega) \cap E$, which will also satisfy

$$B(a_n/2) \subseteq \widetilde{W}_n(c) \subseteq B(a_n) \quad \text{for all } c \in \mathcal{C}_n. \tag{1}$$

This property will later be helpful in establishing item (iv).

For the base of the argument set $W_1 = \{(c, c + \vec{r}) : c \in \mathcal{C}_1, \vec{r} \in B(a_1)\}$. Note that item (v) holds with a trivial partition $\mathcal{C}_{1,1} = \mathcal{C}_1$, $\mathcal{C}_{1,j} = \varnothing$ for $j \geq 2$, and $A_{1,1} = B(a_1)$. Suppose now that W_i have been constructed for $i < n$ and satisfy all the items of the theorem. Cross section $\mathcal{C}_n$ is a_n-discrete, so regions $c + B(a_n)$ are pairwise disjoint as c ranges over $\mathcal{C}_n$.

For a given $c \in \mathcal{C}_n$ we consider regions $W_i(c')$, $i < n$, that intersect $c + B(a_n)$ and that are not contained in a bigger such region. More formally, begin by choosing all the elements $c_1^{n-1}, \ldots, c_{l_{n-1}}^{n-1} \in \mathcal{C}_{n-1}$ such that $W_{n-1}(c_j^{n-1}) \cap (c + B(a_n)) \neq \varnothing$; next, pick all $c_1^{n-2}, \ldots, c_{l_{n-2}}^{n-2} \in \mathcal{C}_{n-2}$ such that $W_{n-2}(c_j^{n-2}) \cap (c + B(a_n)) \neq \varnothing$ and $W_{n-2}(c_j^{n-2}) \cap W_{n-1}(c_i^{n-1}) = \varnothing$ for all $1 \leq i \leq l_{n-1}$; continue in the same fashion, terminating in a collection $c_1^1, \ldots, c_{l_1}^1 \in \mathcal{C}_1$ such that $W_1(c_j^1) \cap (c + B(a_n)) \neq \varnothing$ and $W_1(c_j^1) \cap W_k(c_i^k) = \varnothing$ for all $2 \leq k < n$, and all $1 \leq i \leq l_k$. Note that in view of Eq. (1), there can only be finitely many points c_i^k at each step. Let $c_1, \ldots, c_l \in \bigcup_{i<n} \mathcal{C}_i$ be an enumeration of the elements c_i^k, $1 \leq k < n$, $1 \leq i \leq l_k$, and let for $1 \leq j \leq l$, the number $i(j)$ be such that $c_j \in \mathcal{C}_{i(j)}$.

Sets $W_{i(j)}(c_j)$ are pairwise disjoint, and we therefore find ourselves in the set up of Lemma 3, where the ball $B(a_n)$ interacts with a number of pairwise disjoint smooth disks $\rho(c, c_j) + \widetilde{W}_{i(j)}(c_j)$, each having diameter $\leq 2 * a_{n-1} < a_n/2$. Lemma 3 claims that we can find a smooth disk F squeezed according to $B(a_n/2) \subseteq F \subseteq B(a_n)$, and such that every region $\rho(c, c_j) + \widetilde{W}_{i(j)}(c_j)$ is either contained in the interior of F or is disjoint from it. Set $W_n(c) = \{c + \vec{r} : \vec{r} \in F\}$ and note that $\widetilde{W}_n(c)$ fulfills Eq. (1).

We claim that this construction can be done in such a way that only countably many distinct shapes for F are used. Indeed, the input to Lemma 3, which produced F, is determined by the number l of regions $W_{i(j)}(c_j)$ intersecting $c + B(a_n)$, by the shape of these regions, and by their location relative to c. Since the union $\bigcup_k \mathcal{C}_k$ is rational, the vector $(\rho(c, c_1), \ldots, \rho(c, c_l))$ is in $\mathbb{Q}^l$. By inductive assumption, for each $c_j \in \mathcal{C}_{i(j)} = \bigsqcup_k \mathcal{C}_{i(j),k}$, there is some $k(j) \in \mathbb{N}$ such that $W_{i(j)}(c_j) = c_j + A_{i(j),k(j)}$ for a smooth disk $A_{i(j),k(j)} \subseteq \mathbb{R}^d$. Thus, the input to Lemma 3 is uniquely determined by the tuple

$$\Big(l,\ \rho(c, c_1), \ldots, \rho(c, c_l),\ i(1), k(1), \ldots, i(l), k(l)\Big).$$

There are only countably many such tuples and we can assume that the same disk F is used whenever the input tuple is the same. This guarantees compliance with item (v). Note also that such regions W_n are automatically Borel.

It remains to verify that sets W_n satisfy the rest of the conclusions of the theorem. Item (i) is fulfilled by the choice of F. Item (ii) holds since $\mathcal{C}_n$ is a_n-discrete and $F \subseteq B(a_n)$. Compliance with item (iii) is the key property of the disk F produced by Lemma 3.

We argue that item (iv) holds. Pick a point $x \in \Omega$ and a compact $K \subset \mathbb{R}^d$. Let n_0 be so large that $K \subseteq B(a_{n_0}/4)$. According to the property of cross sections $\mathcal{C}_n$ guaranteed by Lemma 4, for $\epsilon = 1/4$ there exists $n_1 \geq n_0$ such that $||\rho(x, c)|| < a_{n_1}/4$ for some $c \in \mathcal{C}_{n_1}$, i.e., $x \in c + B(a_{n_1}/4)$. We therefore have

$$x + K \subseteq x + B(a_{n_0}/4) \subseteq c + B(a_{n_1}/4) + B(a_{n_0}/4) \subseteq c + B(a_{n_1}/2) \subseteq W_{n_1}(c),$$

where the last inclusion follows from Eq. (1). $\square$

In the proof above we chose a family of pairwise disjoint regions $W_{i(j)}(c_j)$ that intersect $c + B(a_n)$. In the sequel, we will need a similar family of subregions of a region $W_n(c)$. The following lemma and definition isolate the relevant notion.

Lemma 6. *Let $\mathcal{C}_n$ and W_n, $n \in \mathbb{N}$, be as in Theorem 5. For each n and each $c \in \mathcal{C}_n$ there exists a family $c_1, \ldots, c_l \in \bigcup_{i<n} \mathcal{C}_i$ such that for $i(j)$ given by the condition $c_j \in \mathcal{C}_{i(j)}$ one has*

(i) $W_{i(j)}(c_j) \subseteq W_n(c)$ for all $1 \leq j \leq l$;
(ii) sets $W_{i(j)}(c_j)$ are pairwise disjoint for $1 \leq j \leq l$;
(iii) for any $m < n$ and $c' \in \mathcal{C}_m$ such that $W_m(c') \subseteq W_n(c)$ there exists $1 \leq j \leq l$ such that $W_m(c') \subseteq W_{i(j)}(c_j)$.

Proof. Just like in the proof of Theorem 5, let $c_1^{n-1}, \ldots, c_{l_{n-1}}^{n-1} \in \mathcal{C}_{n-1}$ be all the elements (if any) such that $W_{n-1}(c_j^{n-1}) \subseteq W_n(c)$. In view of 5(ii), sets $W_{n-1}(c_j^{n-1})$ are pairwise disjoint. Pick all the elements $c_1^{n-2}, \ldots, c_{l_{n-2}}^{n-2} \in \mathcal{C}_{n-2}$ satisfying $W_{n-2}(c_j^{n-2}) \subseteq W_n(c)$, but $W_{n-2}(c_j^{n-2})$ is disjoint from all $W_{n-1}(c_i^{n-1})$, $1 \leq i \leq l_{n-1}$. Note that by 5(iii) the latter is equivalent to saying that $W_{n-2}(c_j^{n-2})$ is not contained in any of $W_{n-1}(c_k^{n-1})$, $1 \leq k \leq l_{n-1}$.

One continues in the same fashion. At step k we pick elements $c_1^{n-k}, \ldots, c_{l_{n-k}}^{n-k} \in \mathcal{C}_{n-k}$ that are contained in $W_n(c)$ and are disjoint from all the sets $W_{n-j}(c_i^{n-j})$, $1 \le j < k$, $1 \le i \le l_{n-j}$, constructed at the previous steps. The process terminates with the selection of elements $c_1^1, \ldots, c_{l_1}^1 \in \mathcal{C}_1$.

The points c_j^k, $1 \le k < n$, $1 \le j \le l_k$, satisfy the conditions of this lemma. Items (i) and (ii) are evident, and (iii) follows from the observation that if $W_m(c') \subseteq W_n(c)$ was not picked during the construction, then it had to intersect some set $W_k(c_j^k)$ for an element c_j^k, $k > m$, picked earlier. By the condition 5(iii) this means $W_m(c') \subseteq W_k(c_j^k)$ as desired. □

Definition 7. A family of regions $W_{i(j)}(c_j)$ satisfying the conclusions of Lemma 6 is called **a maximal family of subregions** of $W_n(c)$.

Remark 8. It is easy to check that the maximal family of subregions of any $W_n(c)$ is necessarily unique, but this will not play a role in our arguments.

Lemma 9. *Let $\mathcal{C}_n$ and W_n, $n \in \mathbb{N}$, be as in Theorem 5. For every $m_1 \in \mathbb{N}$ and $c_1 \in \mathcal{C}_{m_1}$ there exist a sequence of integers $m_1 < m_2 < m_3 < \cdots$ and elements $c_j \in \mathcal{C}_{m_j}$ such that the regions $W_{m_j}(c_j)$ satisfy the inclusions $W_{m_j}(c_j) \subseteq W_{m_{j+1}}(c_{j+1})$ for all $1 \le j < \infty$.*

Proof. The set $\widetilde{W}_{m_1}(c_1)$ is a disk by 5(i), and in particular it is a compact region in $\mathbb{R}^d$. We may therefore pick a compact $K \subseteq \mathbb{R}^d$ such that the inclusion $\widetilde{W}_{m_1}(c_1) \subset K$ is proper. By 5(iv) there exists some m_2 and $c_2 \in \mathcal{C}_{m_2}$ such that $W_{m_1}(c_1) \subset c_1 + K \subseteq W_{m_2}(c_2)$. Items 5(iii) and 5(ii) guarantee that $m_2 > m_1$. The same choice can now be iterated to construct the desired sequence $m_1 < m_2 < m_3 < \cdots$ and elements $c_j \in \mathcal{C}_{m_j}$. □

3. Equivalence to Special Flows

One of the simplest ways to construct an $\mathbb{R}^d$-flow is to start with an $\mathbb{R}$-flow on some standard Borel space Ω_1 and define the action $\mathbb{R}^d \curvearrowright \Omega_1 \times \mathbb{R}^{d-1}$ by

$$\Omega_1 \times \mathbb{R}^{d-1} \ni (y, \vec{q}) + (r_1, \ldots, r_d) = (y + r_1, \vec{q} + (r_2, \ldots, r_d)).$$

We say that a flow $\mathbb{R}^d \curvearrowright \Omega$ is **special** if it is isomorphic to a flow of the form above. This is an ad hoc notion, which we use to reduce smooth equivalence of multidimensional flows to the one dimensional situation. Our goal in this section is to show that every free Borel $\mathbb{R}^d$-flow is smoothly equivalent to a special one. The argument goes through a sequence of lemmas, and we begin by establishing some common notation.

Throughout the section we fix a free Borel $\mathbb{R}^d$-flow $\mathfrak{F}$, $d \ge 2$, let $\mathcal{C}_n$ be the cross sections and $W_n \subseteq \mathcal{C}_n \times \Omega$ be the corresponding regions produced by Theorem 5. Let $V_n \subseteq \Omega$ denote the projection of W_n onto the second coordinate, and let $\pi_n : V_n \to \mathcal{C}_n$ be defined by the condition $(\pi_n(x), x) \in W_n$ for all $x \in V_n$. Note that V_n is Borel as $\mathrm{proj}_2 : W_n \to V_n$ is injective by 5(ii) and π_n is Borel since its graph is the flip of W_n. Define for $m < n$ sets

$$P_{m,n} = \big\{(c', c) \in \mathcal{C}_m \times \mathcal{C}_n : W_m(c') \subseteq W_n(c)\big\},$$

which encode regions of the level m inside a given region of the level n. Sets $\widetilde{W}_n(c)$ are smooth disks, and our first lemma shows that specific diffeomorphisms onto balls can be chosen to cohere across levels.

Lemma 10. *There exist radius maps $t_n : \mathcal{C}_n \to \mathbb{R}^{>0}$, "diffeomorphism" functions $\phi_n : V_n \to \mathbb{R}^d$, and shift maps $s_{m,n} : P_{m,n} \to \mathbb{R}^{\ge 0}$ subject to the following conditions to be valid for all $m < n$, and all $(c', c) \in P_{m,n}$:*

- *(i)* $t_m(c') \ge m$;
- *(ii)* $\widetilde{W}_m(c') \ni \vec{r} \mapsto \phi_m(c' + \vec{r}) \in B(t_m(c'))$ *is a C^∞ orientation preserving diffeomorphism onto the ball* $B(t_m(c'))$;
- *(iii)* $\phi_n(x) = \phi_m(x) + \vec{s}_{m,n}(c', c)$ *for all $x \in W_m(c')$, where* $\vec{s}_{m,n}(c', c) = s_{m,n}(c', c) \times \vec{0}^{d-1}$;
- *(iv)* $t_m(c') + s_{m,n}(c', c) \le t_n(c) - 1$;
- *(v) there is a Borel partition $\mathcal{C}_m = \bigsqcup_k \mathcal{C}_{m,k}$ such that $\widetilde{W}_m(c_1) = \widetilde{W}_m(c_2)$, $t_m(c_1) = t_m(c_2)$, and $\phi_m(c_1 + \vec{r}) = \phi_m(c_2 + \vec{r})$ for all $c_1, c_2 \in \mathcal{C}_{m,k}$ and all $\vec{r} \in \widetilde{W}_m(c_1)$.*

The meaning of these conditions is as follows. Item (i) ensures that radii go to infinity as $m \to \infty$. According to item (ii), each map ϕ_m encodes a family of diffeomorphisms, one for each $c' \in \mathcal{C}_m$. Formally speaking, these diffeomorphisms are maps from $\widetilde{W}_m(c')$ onto $B(t_m(c'))$. However, we will occasionally abuse the language by calling the map $\phi_m|_{W_m(c')} : W_m(c') \to B(t_m(c'))$ a diffeomorphism.

Item (iii) postulates that ϕ_n extends ϕ_m up to a translation of the range along the x-axis, where the translation value is constant over each $W_m(c')$ region and is equal to $s_{m,n}(c', c)$. Condition (iv) is a reformulation of the inequality

$$\operatorname{dist}\big(\phi_m(W_m(c')) + \vec{s}_{m,n}(c', c),\ \partial B(t_n(c))\big) \geq 1.$$

It means that disks $\phi_m(W_m(c')) + \vec{s}_{m,n}(c', c)$ are at least one unit of distance away from the boundary of $B(t_n(c))$ (see Figure 3). Similarly to Theorem 5(v), item (v) says that we need to consider only countably many different diffeomorphisms $\phi_m|_{W_m(c')}$. The only purpose of this property is to make it easy for us to argue that the flow $\mathfrak{F}'$, which will be constructed later in this section, is Borel.

Proof of Lemma 10. The construction goes by induction on n, and we begin with its base. Set $t_1(c) = 1$ for all $c \in \mathcal{C}_1$. By item (i) of Theorem 5, each region $\widetilde{W}_1(c)$, $c \in \mathcal{C}_1$, is a smooth disk. So for $\phi_1 : V_1 \to B(1)$ we pick any map satisfying (ii) and (v), which can be done since there are only countably many shapes $\widetilde{W}_1(c)$ by 5(v).

For the inductive step consider a region $W_n(c)$. We pick points $c_1, \ldots, c_l \in \bigcup_{i<n} \mathcal{C}_i$, and integers $i(j)$ defined by $c_j \in \mathcal{C}_{i(j)}$, that correspond to a maximal family of subregions of $W_n(c)$ as per Lemma 6. For such points c_j we have $\rho(c, c_j) + \widetilde{W}_{i(j)}(c_j) \subset \operatorname{int} \widetilde{W}_n(c)$, as guaranteed by 5(iii). An example of such a region $W_n(c)$ is shown in Figure 4 on page 11. By inductive assumption regions $\widetilde{W}_{i(j)}(c_j)$ are diffeomorphic to balls $B(t_{i(j)}(c_j))$ via the diffeomorphisms $\vec{r} \mapsto \phi_{i(j)}(c_j + \vec{r})$. We shift these balls along the x-axis to make them disjoint, and view them inside a sufficiently large ball in $\mathbb{R}^d$ (see Figure 3). More specifically, let

$$W'_{i(j),n} = \{(c', x) \in W_{i(j)} : c' \in \operatorname{proj}_1(P_{i(j),n}) \text{ and } c' \notin \operatorname{proj}_1(P_{i(j),k}) \text{ for all } i(j) < k < n\},$$

and put $V'_{i(j),n} = \operatorname{proj}_2(W'_{i(j),n})$; in other words, $W'_{i(j),n}$ consists of those regions $W_{i(j)}(c')$ for which n is the smallest index to satisfy $W_{i(j)}(c') \subset W_n(c)$ for some $c \in \mathcal{C}_n$. Set for $1 \leq j \leq l$

$$s_{i(j),n}(c_j, c) = (j-1) + t_{i(j)}(c_j) + 2 \sum_{1 \leq k < j} t_{i(k)}(c_k),$$

and consider the map $\phi'_{i(j)} : V'_{i(j),n} \to \mathbb{R}^d$ to be given for $x \in W_{i(j)}(c_j)$ by $\phi'_{i(j)}(x) = \phi_{i(j)}(x) + \vec{s}_{i(j),n}(c_j, c)$. Note that restrictions $\phi'_{i(j)}|_{W_{i(j)}(c_j)}$ are diffeomorphisms onto disks $B(t_{i(j)}(c_j)) + \vec{s}_{i(j),n}(c_j, c)$. The radius $t_n(c)$ is taken to be sufficiently large to contain these disks: $t_n(c) = \max\big\{n,\ s_{i(l),n}(c_l, c) + t_{i(l)}(c_l) + 1\big\}$. This ensures that images $\phi'_{i(j)}(W_{i(j)}(c_j))$ are inside $B(t_n(c))$, and are furthermore at least 1 unit of distance away from its boundary, which yields item (iv). Item (i) also continues to be satisfied by this choice of $t_n(c)$.

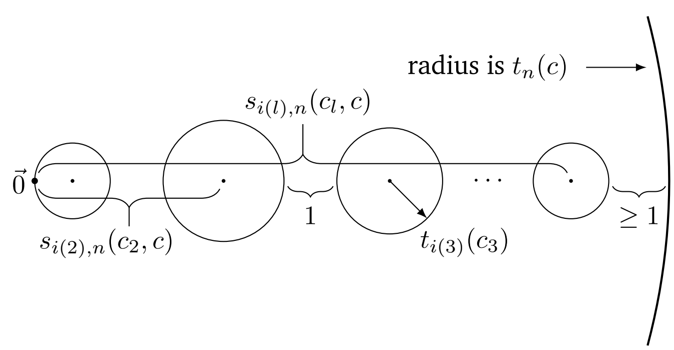

FIGURE 3. Alignment of disks $\phi_{i(j)}(W_{i(j)}(c_j)) + \vec{s}_{i(j),n}(c_j, c)$.

Extension Lemma 2 can now be applied to diffeomorphisms $\rho(c, c_j) + \widetilde{W}_{i(j)}(c_j) \ni \vec{r} \mapsto \phi'_{i(j)}(c + \vec{r})$, since they are defined on disjoint disks $\rho(c, c_j) + \widetilde{W}_{i(j)}(c_j)$ and have disjoint images $B(t_{i(j)}(c_j)) + \vec{s}_{i(j),n}(c_j, c)$, $1 \leq j \leq l$. All the domains of these maps lie in the interior of the disk $\widetilde{W}_n(c)$, while the images are subsets of $\operatorname{int} B(t_n(c))$. We therefore can find a common extension to a diffeomorphism $\phi_n|_{W_n(c)} : W_n(c) \to B(t_n(c))$ that satisfies

$$\phi_n(x) = \phi'_{i(j)}(x) = \phi_{i(j)}(x) + \vec{s}_{i(j),n}(c_j, c)$$

for all $x \in W_{i(j)}(c_j)$ and all $1 \leq j \leq l$, thus implying (iii).

We are not quite done yet though. The construction above defined diffeomorphisms ϕ_n and radii $t_n(c)$ for all $c \in \mathcal{C}_n$, but shifts $s_{m,n}(c', c)$ are currently defined only for those $c' \in \mathcal{C}_m$ that belong to the maximal family of subregions of $W_n(c)$. Nonetheless, values $s_{m,n}(c', c)$ satisfying item (iii), are uniquely specified for all $(c', c) \in P_{m,n}$ based on the following observation. Pick any $c' \in \mathcal{C}_m$, $m < n$, such that $W_m(c') \subseteq W_n(c)$, let j be the unique index $1 \leq j \leq l$ such that $W_m(c') \subseteq W_{i(j)}(c_j)$. Suppose c' does not belong to the maximal family of subregions of $W_n(c)$, hence $m < i(j)$. Using the inductive assumption (iii), we find that for any $x \in W_m(c')$

$$\begin{aligned} \phi_n(x) &= \phi_{i(j)}(x) + \vec{s}_{i(j),n}(c_j, c) \\ &= \phi_m(x) + \vec{s}_{m,i(j)}(c', c_j) + \vec{s}_{i(j),n}(c_j, c). \end{aligned}$$

Thus, for $s_{m,n}(c', c) = s_{m,i(j)}(c', c_j) + s_{i(j),n}(c_j, c)$ item (iii) holds for all $m < n$ and all $c' \in \mathcal{C}_m$ such that $W_m(c') \subseteq W_n(c)$.

We check that (iv) continues to hold. Let us assume that this property has been verified for regions at levels below n, and by construction we have also established

$$t_n(c) - s_{i(j),n}(c_j, c) \geq t_{i(j)}(c_j) + 1. \tag{2}$$

Using the additivity of values $s_{m,n}(c', c)$ shown above we get

$$\begin{aligned} t_n(c) - t_m(c') - s_{m,n}(c', c) &= t_n(c) - s_{i(j),n}(c_j, c) - s_{m,i(j)}(c', c_j) - t_m(c') \\ \text{Eq. (2)} &\geq t_{i(j)}(c_j) + 1 - s_{m,i(j)}(c', c_j) - t_m(c') \\ \text{inductive assumption} &\geq t_m(c') + 2 - t_m(c') = 2. \end{aligned}$$

Therefore (iv) holds for all $m < n$ and all $(c', c) \in P_{m,n}$.

Finally, to guarantee item (v) note that diffeomorphisms ϕ_n have been chosen using Lemma 2 based on the shapes of regions $\widetilde{W}_n(c)$, as well as shapes of subregions $\widetilde{W}_{i(j)}(c_j)$, and their locations inside $\widetilde{W}_n(c)$ specified by values $\rho(c, c_j)$. By Theorem 5, the union $\bigcup_k \mathcal{C}_k$ is on a rational grid, so all the values $\rho(c, c_j)$ are rational. Also, by 5(v), there are only countably many possible shapes for regions $W_{i(j)}(c_j)$. We may therefore choose the same diffeomorphism $\phi_n|_{W_n(c)}$ whenever the inputs to Lemma 2 are the same, which guarantees fulfillment of item (v). $\square$

The item 10(iv) above guarantees that the image of $W_m(c')$ under ϕ_n is at least 1 unit of distance away from the boundary of $B(t_n(c))$ whenever $W_m(c') \subseteq W_n(c)$. The following two lemmas show that we can find such n and $c \in \mathcal{C}_n$ for which the set $\phi_n(W_m(c'))$ is as far from the boundary of $B(t_n(c))$ as we desire.

Lemma 11. *Let $m_1 < m_2 < m_3 < \cdots$ be an increasing sequence of integers and $c_j \in \mathcal{C}_{m_j}$ be elements such that $W_{m_j}(c_j) \subseteq W_{m_{j+1}}(c_{j+1})$ (such a sequence is produced by Lemma 9). For all $j \geq 2$ one has*

$$t_{m_j}(c_j) - \sum_{1 \leq k < j} s_{m_k, m_{k+1}}(c_k, c_{k+1}) - t_{m_1}(c_1) \geq j - 1.$$

Proof. The argument is a simple induction coupled with item 10(iv), which, in particular, gives the base

$$t_{m_2}(c_2) - s_{m_1,m_2}(c_1, c_2) - t_{m_1}(c_1) \geq 1.$$

Suppose the statement has been established for $j - 1$:

$$t_{m_{j-1}}(c_{j-1}) - \sum_{1 \leq k < j-1} s_{m_k, m_{k+1}}(c_k, c_{k+1}) - t_{m_1}(c_1) \geq j - 2. \tag{3}$$

By item 10(iv) we have $t_{m_j}(c_j) - s_{m_{j-1},m_j}(c_{j-1}, c_j) \geq 1 + t_{m_{j-1}}(c_{j-1})$, and therefore

$$\begin{aligned} t_{m_j}(c_j) - &\sum_{1\leq k<j} s_{m_k,m_{k+1}}(c_k, c_{k+1}) - t_{m_1}(c_1) \\ &= t_{m_j}(c_j) - s_{m_{j-1},m_j}(c_{j-1}, c_j) - \sum_{1\leq k<j-1} s_{m_k,m_{k+1}}(c_k, c_{k+1}) - t_{m_1}(c_1) \\ &\geq 1 + t_{m_{j-1}}(c_{j-1}) - \sum_{1\leq k<j-1} s_{m_k,m_{k+1}}(c_k, c_{k+1}) - t_{m_1}(c_1) \\ \text{Eq. (3)} &\geq 1 + j - 2 = j - 1, \end{aligned}$$

which yields the step of induction. □

Lemma 12. *For any $x \in \Omega$ and any $R \in \mathbb{R}^{\geq 0}$ there exist n and $c \in \mathcal{C}_n$ such that $x \in W_n(c)$ and*

$$||\phi_n(x)|| + R \leq t_n(c).$$

Proof. In view of 5(iv) there is some m_1 and $c_1 \in \mathcal{C}_{m_1}$ such that $x \in W_{m_1}(c_1)$. By Lemma 9 there exist levels $m_1 < m_2 < \cdots$ and points $c_j \in \mathcal{C}_{m_j}$ such that $W_{m_j}(c_j) \subseteq W_{m_{j+1}}(c_{j+1})$. For each j we have in view of 10(iii)

$$\begin{aligned} t_{m_j}(c_j) - ||\phi_{m_j}(x)|| &= t_{m_j}(c_j) - ||\phi_{m_{j-1}}(x) + \vec{s}_{m_{j-1},m_j}(c_{j-1}, c_j)|| \\ &= \cdots \quad j-2 \text{ further applications of 10(iii)} \\ &= t_{m_j}(c_j) - \Big|\Big|\phi_{m_1}(x) + \sum_{1\leq k<j} \vec{s}_{m_k,m_{k+1}}(c_k, c_{k+1})\Big|\Big| \\ &\geq t_{m_j}(c_j) - t_{m_1}(c_1) - \sum_{1\leq k<j} s_{m_k,m_{k+1}}(c_k, c_{k+1}) \\ \text{Lemma 11} &\geq j - 1. \end{aligned}$$

Therefore $n = m_j$ and $c = c_j$ satisfy the conclusions of the lemma as long as $j - 1 \geq R$. □

We now define a new flow $\mathfrak{F}'$ on the same phase space Ω. Notation $x \oplus \vec{r}$ will be used to distinguish the action of $\mathfrak{F}'$ from the action of the original flow $\mathfrak{F}$. For $x \in \Omega$ and $\vec{r} \in \mathbb{R}^d$ let $c \in \mathcal{C}_n$ be such that $x \in W_n(c)$ and $\phi_n(x) + \vec{r} \in B(t_n(c))$ (such n and c exist by Lemma 12). The $\mathfrak{F}'$ action of $\vec{r}$ upon x is defined by

$$x \oplus \vec{r} = (\phi_n|_{W_n(c)})^{-1}(\phi_n(x) + \vec{r}),$$

or, equivalently, $x \oplus \vec{r}$ is the element of $W_n(c)$ for which $\phi_n(x \oplus \vec{r}) = \phi_n(x) + \vec{r}$. The geometric interpretation of the action is as follows. We use the diffeomorphism ϕ_n to identify $W_n(c)$ with $B(t_n(c))$. One acts upon $\phi_n(x) \in B(t_n(c)) \subseteq \mathbb{R}^d$ by translation. Assuming the image lies within the same ball $B(t_n(c))$, we can pull it back to an element of $W_n(c)$, which is what $x \oplus \vec{r}$ is defined to be. As we argue below, this definition does not depend on the choice of n and c due to the coherence of diffeomorphisms ϕ_n provided by item 10(iii). Having this simple picture of the action in their mind will make it easy for the reader to follow the somewhat tedious but elementary computations that constitute a large portion of the remainder of this section.

Lemma 13. *The definition of $x \oplus \vec{r}$ does not depend on the choice of n and c.*

Proof. Let $c' \in \mathcal{C}_m$ be another element that can be used in the definition of $x \oplus \vec{r}$, i.e., $x \in W_m(c')$ and $\phi_m(x) + \vec{r} \in B(t_m(c'))$. Item 5(ii) implies $m \neq n$, and 5(iii) guarantees that either $W_m(c') \subseteq W_n(c)$ or $W_n(c) \subseteq W_m(c')$. Since roles of m and n are symmetric, we may assume without loss of generality that the former is the case. Consider the chain of equalities

$$\begin{aligned} \phi_n(x) + \vec{r} &= \phi_m(x) + \vec{s}_{m,n}(c', c) + \vec{r} \\ &= \phi_m\Big((\phi_m|_{W_m(c')})^{-1}\big(\phi_m(x) + \vec{r}\big)\Big) + \vec{s}_{m,n}(c', c) \\ \text{item 10(iii)} &= \phi_n\Big((\phi_m|_{W_m(c')})^{-1}\big(\phi_m(x) + \vec{r}\big)\Big). \end{aligned}$$

Applying $(\phi_n|_{W_n(c)})^{-1}$ to the first and the last expressions above yields

$$(\phi_n|_{W_n(c)})^{-1}(\phi_n(x) + \vec{r}) = (\phi_m|_{W_m(c')})^{-1}(\phi_m(x) + \vec{r}),$$

which finishes the proof of the lemma. □

Having established that $x \oplus \vec{r}$ is well-defined, we can now verify it to be a free Borel flow.

Lemma 14. *The map $\Omega \times \mathbb{R}^d \ni (x, \vec{r}) \mapsto x \oplus \vec{r} \in \Omega$ defines a free Borel action of $\mathbb{R}^d$ on Ω. This flow is smoothly equivalent to $\mathfrak{F}$. Moreover, the identity map $\mathrm{id} : \Omega \to \Omega$ is a smooth equivalence between $\mathfrak{F}$ and $\mathfrak{F}'$.*

Proof. Pick $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^d$, and use Lemma 12 to choose n, $c \in \mathcal{C}_n$, such that

$$||\phi_n(x)|| + ||\vec{r}_1|| + ||\vec{r}_2|| \leq t_n(c).$$

This inequality guarantees that $(\phi_n|_{W_n(c)})^{-1}$ is defined in the following terms:

$$\begin{aligned}(x \oplus \vec{r}_1) \oplus \vec{r}_2 &= (\phi_n|_{W_n(c)})^{-1}\big(\phi_n(x \oplus \vec{r}_1) + \vec{r}_2\big)\\ &= (\phi_n|_{W_n(c)})^{-1}\big(\phi_n(x) + \vec{r}_1 + \vec{r}_2\big) = x \oplus (\vec{r}_1 + \vec{r}_2).\end{aligned}$$

Coupled with the straightforward $x \oplus \vec{0} = x$, these computations show that $\oplus$ defines a flow on Ω. This flow is free, because the maps $\phi_n|_{W_n(c)}$ are injective.

It is easy to see that orbits of $\mathfrak{F}'$ coincide with those of $\mathfrak{F}$. The inclusion $E_{\mathfrak{F}'} \subseteq E_{\mathfrak{F}}$ is guaranteed by the condition $W_n \subseteq E_{\mathfrak{F}}$. For the inverse direction, let $x, y \in \Omega$ be such that $x E_{\mathfrak{F}} y$. By 5(iv), there are n and $c \in \mathcal{C}_n$ such that $x, y \in W_n(c)$. One has

$$\begin{aligned}x \oplus \big(\phi_n(y) - \phi_n(x)\big) &= (\phi_n|_{W_n(c)})^{-1}\big(\phi_n(x) + \phi_n(y) - \phi_n(x)\big)\\ &= (\phi_n|_{W_n(c)})^{-1}(\phi_n(y)) = y,\end{aligned}$$

and thus $E_{\mathfrak{F}} = E_{\mathfrak{F}'}$.

The flow $\mathfrak{F}'$ is Borel. To justify this set $\alpha : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ to be defined by $x \oplus \vec{r} = x + \alpha(x, \vec{r})$; it suffices to show that α is Borel. In general, we would have to verify that the value $(\phi_n|_{W_n(c)})^{-1}\big(\phi_n(x) + \vec{r}\big)$ depends in a Borel way on $x \in \Omega$ and $\vec{r} \in \mathbb{R}^d$, which requires going into the details of the way maps $\phi_n|_{W_n(c)}$ are constructed based on n and $c \in \mathcal{C}_n$. However, we ensured in item 10(v) that there is a countable Borel partition of each cross section $\mathcal{C}_n = \bigsqcup_k \mathcal{C}_{n,k}$ such that $\widetilde{W}_n(c_1) = \widetilde{W}_n(c_2)$, $t_n(c_1) = t_n(c_2)$, and $\phi_n|_{W_n(c_1)}(c_1 + \vec{r}) = \phi_n|_{W_n(c_2)}(c_2 + \vec{r})$ for all $c_1, c_2 \in \mathcal{C}_{n,k}$ and all $\vec{r} \in \widetilde{W}(c_1)$. Let $\widetilde{W}_{n,k}$ denote the common shape of $\widetilde{W}_n(c)$, and similarly let $t_{n,k}$ be the radius $t_n(c)$ common for all $c \in \mathcal{C}_{n,k}$. Likewise, maps $\phi_n|_{W_n(c)}(c + \cdot)$ produce the same diffeomorphism $\varphi_{n,k} : \widetilde{W}_{n,k} \to B(t_{n,k})$ regardless of the choice of $c \in \mathcal{C}_{n,k}$.

Recall that $\pi_n : V_n \to \mathcal{C}_n$ is the map that associates the distinguished point c to every x in $W_n(c)$. Let

$$X_{n,k} = \{(x, \vec{r}) \in \Omega \times \mathbb{R}^d : \pi_n(x) \in \mathcal{C}_{n,k},\ x \in V_n,\ \text{and } \varphi_{n,k}(\rho(\pi_n(x), x)) + \vec{r} \leq t_{n,k}\}$$

denote the set of those pairs $(x, \vec{r})$ for which $x \oplus \vec{r}$ can be defined using $W_n(c)$ for some $c \in \mathcal{C}_{n,k}$. To conclude that α is Borel, we observe that for $(x, \vec{r}) \in X_{n,k}$ its value equals

$$\alpha(x, \vec{r}) = \rho(x, \pi_n(x)) + \varphi_{n,k}^{-1}\big(\varphi_{n,k}(\rho(\pi_n(x), x)) + \vec{r}\big), \tag{4}$$

which is a composition of Borel functions. It is at this point that our efforts in ensuring stability of the construction in 5(v) and 10(v) yield their fruits. We have only one diffeomorphism $\varphi_{n,k}$ in the definition of α which applies to all arguments $(x, \vec{r}) \in X_{n,k}$. Thus $\alpha|_{X_{n,k}}$ is Borel regardless of how this diffeomorphism was chosen. Since $\bigcup_{n,k} X_{n,k} = \Omega \times \mathbb{R}^d$, we may conclude that α is Borel on all of $\Omega \times \mathbb{R}^d$.

We have already established that $\mathrm{id} : \Omega \to \Omega$ is an orbit equivalence, and it remains to verify that it is smooth, which amounts to showing that $\alpha(x, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism for each $x \in \Omega$. It is a bijection, since $\mathfrak{F}$, $\mathfrak{F}'$ are free and $E_{\mathfrak{F}} = E_{\mathfrak{F}'}$, and it is smooth and orientation preserving, since according to Eq. (4) for any fixed $x \in \Omega$ the function $\alpha(x, \cdot)$ is a composition of $\varphi_{n,k}^{-1}$ with translation maps. □

The goal of this section is to show that every free flow is smoothly equivalent to a special one. So far starting with a free flow $\mathfrak{F}$ we have constructed a smoothly equivalent flow $\mathfrak{F}'$, and it remains to verify that the latter is special. We need a subset $\Omega_1 \subseteq \Omega$ invariant under the shifts $\Omega_1 \oplus s \times \vec{0}^{d-1} = \Omega_1$ for all $s \in \mathbb{R}$. Set $L_n \subseteq W_n$ to correspond to the preimages of the x-axis inside $B(t_n(c))$ under the diffeomorphisms ϕ_n:

$$L_n = \big\{(c, x) \in W_n : \phi_n(x) \in \big[-t_n(c), t_n(c)\big] \times \vec{0}^{d-1}\big\}.$$

Sets $L_n(c)$ represent line segments inside regions $W_n(c)$ (see Figure 4). Set $L = \bigcup_n L_n$, and let Ω_1 be the projection of L onto the second coordinate:

$$\Omega_1 = \Big\{x : (c, x) \in L \text{ for some } c \in \bigcup_n \mathcal{C}_n\Big\}.$$

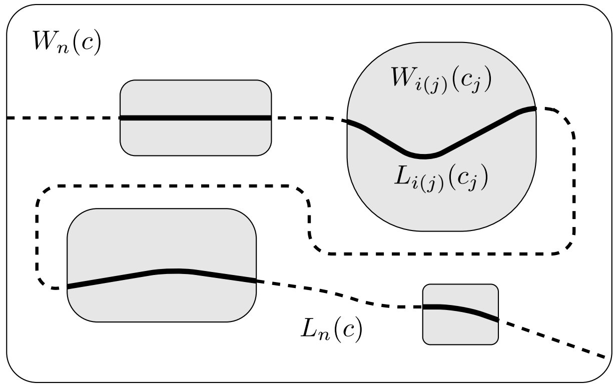

FIGURE 4. Region $W_n(c)$, containing four subregions $W_{i(j)}(c_j)$ marked in gray. Each of the subregions has a line segment $L_{i(j)}(c_j)$, which are all contained inside $L_n(c)$.

Lemma 15. *The set $\Omega_1 \subseteq \Omega$ is Borel and $\Omega_1 \oplus s \times \vec{0}^{d-1} = \Omega_1$ for all $s \in \mathbb{R}$.*

Proof. The set $\Omega_1 = \bigcup_n \mathrm{proj}_2(L_n)$ is Borel, since projections $\mathrm{proj}_2 : L_n \to \Omega_1$ are injective, hence have Borel images (see [Kec95, Corollary 15.2]). To check shift invariance, pick $x \in \Omega_1$ and $s \in \mathbb{R}$; let $\vec{s}$ denote the vector $s \times \vec{0}^{d-1}$. There has to exist some m and $c' \in \mathcal{C}_m$ such that $(c', x) \in L_m$. Pick $n > m$ and $c \in \mathcal{C}_n$ to satisfy $x, x \oplus \vec{s} \in W_n(c)$, which exist by 5(iv). Note that $(c, x) \in L_n$ as according to 10(iii) $\phi_n(x) = \phi_m(x) + \vec{s}_{m,n}(c', c)$, and therefore $\phi_m(x) \in [-t_m(c'), t_m(c')] \times \vec{0}^{d-1}$ implies

$$\begin{aligned} \phi_n(x) &\in \bigl[-t_m(c') - s_{m,n}(c', c),\ t_m(c') + s_{m,n}(c', c)\bigr] \times \vec{0}^{d-1} \\ \text{item 10(iv)} &\subseteq [-t_n(c), t_n(c)] \times \vec{0}^{d-1}. \end{aligned}$$

Since $x \oplus \vec{s} \in W_n(c)$, we have $\phi_n(x) + \vec{s} \in [-t_n(c), t_n(c)] \times \vec{0}^{d-1}$, thus $x \oplus \vec{s} \in \Omega_1$ as claimed. □

The following lemma will be helpful in establishing that $\mathfrak{F}'$ is special.

Lemma 16. *For any $x \in \Omega$ there exists $\vec{q} \in \mathbb{R}^{d-1}$ such that $x \oplus 0 \times \vec{q} \in \Omega_1$.*

Proof. Pick some $x \in \Omega$ and, as usual, let n, $c \in \mathcal{C}_n$ be chosen to satisfy $x \in W_n(c)$. Set $\vec{q} \in \mathbb{R}^{d-1}$ to be the negative of the projection of $\phi_n(x)$ onto the last $(d-1)$-many coordinates: $\vec{q} = -\mathrm{proj}_{[2,d]}(\phi_n(x))$. By the definition of the action,

$$\begin{aligned} x \oplus 0 \times \vec{q} &= (\phi_n|_{W_n(c)})^{-1}(\phi_n(x) + 0 \times \vec{q}) \\ &= (\phi_n|_{W_n(c)})^{-1}\bigl(\phi_n(x) - 0 \times \mathrm{proj}_{[2,d]}(\phi_n(x))\bigr) \\ &= (\phi_n|_{W_n(c)})^{-1}\bigl(\mathrm{proj}_1(\phi_n(x)) \times \vec{0}^{d-1}\bigr) \in L_n(c) \subseteq \Omega_1, \end{aligned}$$

whence $\vec{q}$ satisfies the conclusion of the lemma. □

We have established that Ω_1 is invariant under the $\mathbb{R}$-flow corresponding to the actions by vectors $s \times \vec{0}$, and we may therefore naturally define a special flow $\overline{\mathfrak{F}}'$ on $\Omega_1 \times \mathbb{R}^{d-1}$ by

$$(x, \vec{q}) \boxplus (r_1, r_2, \ldots, r_d) = \bigl(x \oplus r_1 \times \vec{0}^{d-1},\ \vec{q} + (r_2, \ldots, r_d)\bigr).$$

Note that $\boxplus$ is used for the action to distinguish it from both the actions given by $\mathfrak{F}$ and $\mathfrak{F}'$. We are going to verify that $\overline{\mathfrak{F}}'$ is isomorphic to $\mathfrak{F}'$, and to this end we define two Borel maps $\mu : \Omega \to \Omega_1$ and $\nu : \Omega \to \mathbb{R}^{d-1}$ such that $\Omega \ni x \mapsto (\mu(x), \nu(x)) \in \Omega_1 \times \mathbb{R}^{d-1}$ will be the desired isomorphism. For $x \in \Omega$, n and $c \in \mathcal{C}_n$, $x \in W_n(c)$, set

$$\begin{aligned} \mu(x) &= (\phi_n|_{W_n(c)})^{-1}\bigl(\mathrm{proj}_1(\phi_n(x)) \times \vec{0}^{d-1}\bigr) \in \Omega_1 \\ \nu(x) &= \mathrm{proj}_{[2,d]}(\phi_n(x)) \in \mathbb{R}^{d-1}. \end{aligned}$$

Lemma 17. *Maps μ and ν are well-defined in the sense that their values do not depend on the choice of n and $c \in \mathcal{C}_n$.*

Proof. Let $m < n$ and $c' \in \mathcal{C}_m$ be other elements such that $x \in W_m(c)$. Once again, $\phi_n(x) = \phi_m(x) + \vec{s}_{m,n}(c', c)$, by item 10(iii). In particular, the projections of $\phi_n(x)$ and $\phi_m(x)$ onto the last $(d-1)$-many coordinates are equal, because $\vec{s}_{m,n}(c', c) = s_{m,n}(c', c) \times \vec{0}^{d-1}$ This shows that $\nu(x)$ is well-defined.

For $s = \mathrm{proj}_1(\phi_n(x))$, $s' = \mathrm{proj}_1(\phi_m(x))$, and $\vec{s} = s \times \vec{0}^{d-1}$, $\vec{s}\,' = s' \times \vec{0}^{d-1}$, we have

$$\begin{aligned} \vec{s} &= \vec{s}\,' + \vec{s}_{m,n}(c', c) \\ &= \phi_m\big((\phi_m|_{W_m(c')})^{-1}(\vec{s}\,')\big) + \vec{s}_{m,n}(c', c) \\ \text{item 10(iii)} &= \phi_n\big((\phi_m|_{W_m(c')})^{-1}(\vec{s}\,')\big). \end{aligned}$$

Applying $(\phi_n|_{W_n(c)})^{-1}$ to both sides yields

$$(\phi_n|_{W_n(c)})^{-1}(\vec{s}) = (\phi_m|_{W_m(c')})^{-1}(\vec{s}\,'),$$

which shows that the value $\mu(x) \in \Omega_1$ does not depend on the choice of n and $c \in \mathcal{C}_n$. □

Lemma 18. *The map $\Omega \ni x \mapsto (\mu(x), \nu(x)) \in \Omega_1 \times \mathbb{R}^{d-1}$ is a bijection.*

Proof. For injectivity, let $x, y \in \Omega$ be distinct; recall that the orbit equivalence relations of the flows $\mathfrak{F}$ and $\mathfrak{F}'$ coincide by Lemma 14, and we denote it by E. Note that $xE\mu(x)$ and $yE\mu(y)$, so if $\neg xEy$, then $\mu(x) \neq \mu(y)$. Thus we need to consider the case xEy and by 5(iv) there is some n and $c \in \mathcal{C}_n$ such that $x, y \in W_n(c)$. Since $\phi_n|_{W_n(c)} : W_n(c) \to B(t_n(c))$ is injective, $\phi_n(x) \neq \phi_n(y)$, and thus either

$$\begin{aligned} \mathrm{proj}_1(\phi_n(x)) &\neq \mathrm{proj}_1(\phi_n(y)) \qquad \text{or} \\ \mathrm{proj}_{[2,d]}(\phi_n(x)) &\neq \mathrm{proj}_{[2,d]}(\phi_n(y)), \end{aligned}$$

and hence either $\mu(x) \neq \mu(y)$ or $\nu(x) \neq \nu(y)$.

For surjectivity, pick $(x, \vec{q}) \in \Omega_1 \times \mathbb{R}^{d-1}$, and n, $c \in \mathcal{C}_n$, with $x \in W_n(c)$ and $t_n(c) - ||\phi_n(x)|| \geq ||\vec{q}||$, which exist by Lemma 12. Note that by the coherence property of L_n established in the proof of Lemma 15, we have $(c, x) \in L_n$, which is equivalent to $\phi_n(x) \in \mathbb{R} \times \vec{0}^{d-1}$. These conditions ensure that $(\phi_n|_{W_n(c)})^{-1}$ is defined on $\phi_n(x) + 0 \times \vec{q}$ and

$$(\mu, \nu)\big((\phi_n|_{W_n(c)})^{-1}\big(\phi_n(x) + 0 \times \vec{q}\big)\big) = (x, \vec{q}),$$

witnessing surjectivity. □

At last, we can check that the flows $\mathfrak{F}'$ and $\overline{\mathfrak{F}}'$ are isomorphic.

Lemma 19. *The map $\Omega \ni x \mapsto (\mu(x), \nu(x)) \in \Omega_1 \times \mathbb{R}^{d-1}$ is an isomorphism of flows $\mathfrak{F}'$ and $\overline{\mathfrak{F}}'$.*

Proof. Once Lemma 18 is available, our remaining goal is to show that $(\mu, \nu)(x \oplus \vec{r}) = (\mu(x), \nu(x)) \boxplus \vec{r}$ for all $x \in \Omega$ and all $\vec{r} \in \mathbb{R}^d$. We first verify this for those $\vec{r}$ that satisfy $\mathrm{proj}_1(\vec{r}) = 0$, i.e., $\vec{r} = 0 \times \vec{q}$, for some $\vec{q} \in \mathbb{R}^{d-1}$. One has

$$\begin{aligned} \nu(x \oplus 0 \times \vec{q}) &= \mathrm{proj}_{[2,d]}\big(\phi_n(x) + 0 \times \vec{q}\big) \\ &= \mathrm{proj}_{[2,d]}\big(\phi_n(x)\big) + \vec{q} = \nu(x) + \vec{q}, \\ \mu(x \oplus 0 \times \vec{q}) &= (\phi_n|_{W_n(c)})^{-1}\big(\mathrm{proj}_1\big(\phi_n(x) + 0 \times \vec{q}\big) \times \vec{0}^{d-1}\big) \\ &= (\phi_n|_{W_n(c)})^{-1}\big(\mathrm{proj}_1(\phi_n(x)) \times \vec{0}^{d-1}\big) = \mu(x). \end{aligned}$$

We have thus shown that

$$(\mu, \nu)(x \oplus 0 \times \vec{q}) = \big((\mu, \nu)(x)\big) \boxplus 0 \times \vec{q} \quad \text{for all } \vec{q} \in \mathbb{R}^{d-1} \text{ and all } x \in \Omega. \tag{5}$$

Note also that $(\mu, \nu)(y) = (y, \vec{0})$ for all $y \in \Omega_1 \subseteq \Omega$, and therefore by Lemma 15 for all $s \in \mathbb{R}$ and all $y \in \Omega_1$

$$(\mu, \nu)(y \oplus s \times \vec{0}^{d-1}) = (y \oplus s \times \vec{0}^{d-1}, \vec{0}) = (\mu, \nu)(y) \boxplus (s \times \vec{0}^{d-1}). \tag{6}$$

For any $x \in \Omega$, Lemma 16 gives an element $\vec{q}_0 \in \mathbb{R}^{d-1}$ such that $x \oplus 0 \times \vec{q}_0 \in \Omega_1$. Let $\vec{r} \in \mathbb{R}^d$ be arbitrary, and write it as $\vec{r} = s \times \vec{0}^{d-1} + 0 \times \vec{q}$ for some $s \in \mathbb{R}$ and $\vec{q} \in \mathbb{R}^{d-1}$. We have

$$\begin{aligned}
(\mu,\nu)(x \oplus \vec{r}) &= (\mu,\nu)\big(x \oplus 0 \times \vec{q}_0 \oplus (-0 \times \vec{q}_0) \oplus s \times \vec{0}^{d-1} \oplus 0 \times \vec{q}\big) \\
&= (\mu,\nu)\big(x \oplus 0 \times \vec{q}_0 \oplus s \times \vec{0}^{d-1} \oplus 0 \times (\vec{q} - \vec{q}_0)\big) \\
\text{Eq. (5)} &= (\mu,\nu)\big(x \oplus 0 \times \vec{q}_0 \oplus s \times \vec{0}^{d-1}\big) \boxplus 0 \times (\vec{q} - \vec{q}_0) \\
\text{Eq. (6)} &= (\mu,\nu)(x \oplus 0 \times \vec{q}_0) \boxplus s \times \vec{0}^{d-1} \boxplus 0 \times (\vec{q} - \vec{q}_0) \\
\text{Eq. (5)} &= (\mu,\nu)(x) \boxplus 0 \times \vec{q}_0 \boxplus s \times \vec{0}^{d-1} \boxplus 0 \times (\vec{q} - \vec{q}_0) \\
&= (\mu,\nu)(x) \boxplus (s \times \vec{q}) = (\mu,\nu)(x) \boxplus \vec{r}.
\end{aligned}$$

Thus (μ,ν) is an isomorphism between flows $\mathfrak{F}'$ and $\overline{\mathfrak{F}}'$. □

The following theorem summarizes the analysis that has been conducted in Lemmas 10 through 19.

Theorem 20. *Every free Borel $\mathbb{R}^d$-flow, $d \geq 2$, is smoothly equivalent to a special flow.*

4. Smooth Equivalence of Flows

We are finally ready for the proof of the main result of this article—smooth equivalence of all non-tame Borel $\mathbb{R}^d$-flows. For this we just need to combine Theorem 20 with the result of Miller–Rosendal on one-dimensional flows.

Theorem 21. *All non-tame free Borel $\mathbb{R}^d$-flows, $d \geq 2$, are smoothly equivalent.*

Proof. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be non-tame free Borel $\mathbb{R}^d$-flows. By Theorem 20, each of them is smoothly equivalent to a special flow $\mathbb{R}^d \curvearrowright \Omega_i \times \mathbb{R}^{d-1}$, $i = 1, 2$. Note that neither of the $\mathbb{R}$-flows $\mathbb{R} \curvearrowright \Omega_i$ can be tame, for otherwise the corresponding $\mathbb{R}^d$-flow would also be tame. By the Miller–Rosendal result (see Theorem 1), there is a smooth equivalence $\bar{\xi} : \Omega_1 \to \Omega_2$ between the $\mathbb{R}$-flows. Let $\alpha_{\bar{\xi}} : \Omega_1 \times \mathbb{R} \to \mathbb{R}$ be the corresponding family of diffeomorphisms defined by $\alpha_{\bar{\xi}}(x, s) = \rho(\bar{\xi}(x), \bar{\xi}(x + s))$. The map $\xi : \Omega_1 \times \mathbb{R}^{d-1} \to \Omega_2 \times \mathbb{R}^{d-1}$ defined by $\xi(x, \vec{q}) = (\bar{\xi}(x), \vec{q})$ is a smooth equivalence between the special flows, because

$$\alpha_\xi\big((x, \vec{q}), (r_1, \ldots, r_d)\big) = (\alpha_{\bar{\xi}}(x, r_1), r_2, \ldots, r_d) \in \mathbb{R}^d$$

is a C^∞-smooth orientation preserving diffeomorphism for all $x \in \Omega_1$ and all $\vec{q} \in \mathbb{R}^{d-1}$. Flows $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are smoothly equivalent by transitivity of the smooth equivalence relation. □

Our approach to the construction of smooth equivalence between multidimensional flows is different from those taken in the ergodic theoretical antecedents. Of particular interest is the technique used in [Fel91]. Their strategy is to start with an orbit equivalence between cross sections of flows[2], and extend such a map to a smooth equivalence by a back-and-forth argument. It would be interesting to establish the possibility of such an extension in the descriptive set-theoretical context. In [Slu19], such an extension was constructed up to a compressible set. It was also proven therein that if a complete extension is always possible, then it will imply smooth equivalence of all flows (Theorem 21 above).

Conjecture 22. *Let $\mathbb{R}^d \curvearrowright \Omega_1$ and $\mathbb{R}^d \curvearrowright \Omega_2$ be free Borel flows, $d \geq 2$, let $\mathcal{C}_i \subseteq \Omega_i$ be cocompact cross sections, and let $\zeta : \mathcal{C}_1 \to \mathcal{C}_2$ be an orbit equivalence (i.e., a Borel bijection such that $cE_1c' \iff \zeta(c)E_2\zeta(c')$). There exists a smooth equivalence $\xi : \Omega_1 \to \Omega_2$ that extends ζ.*

Rudolph's Theorem [Rud79, Proposition 1.1] produces a smooth equivalence of $\mathbb{R}^d$-flows which is also a Lebesgue orbit equivalence, i.e., a map that preserves the Lebesgue measure between orbits (see [Slu17] for the discussion of the notion of Lebesgue orbit equivalence). This is a significant strengthening of the smooth equivalence, and it is therefore interesting to see if such a strengthening is possible in the descriptive set-theoretical context as well. Any Lebesgue orbit equivalence produces an isomorphism between the spaces of invariant measures of the flows [Slu17, Theorem 4.5], so we need to restrict ourselves to flows that have the same cardinalities of sets of ergodic invariant probability measures. According to [Slu17, Theorem 9.1],

[2] The conditions of when such an orbit equivalence exists are well understood and follow from Dye's Theorem [Dye59, Dye63] and Dougherty–Jackson–Kechris classification of the hyperfinite equivalence relations [DJK94].

this would ensure the existence of a Lebesgue orbit equivalence between the flows and, of course, Theorem 21 guarantees that there is a smooth equivalence as well. The question is whether the two can be combined.[3]

Question 23. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be free non-tame Borel $\mathbb{R}^d$-flows, $d \geq 2$, having the same numbers of invariant ergodic probability measures. Is there a smooth equivalence between these flows which is also a Lebesgue orbit equivalence?*

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[3]We would like to thank the referee for suggesting this question.