GRAEV ULTRAMETRICS AND FREE PRODUCTS OF POLISH GROUPS

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ABSTRACT. We construct Graev ultrametrics on free products of groups with two-sided invariant ultrametrics and HNN extensions of such groups. We also introduce a notion of a free product of general Polish groups and prove, in particular, that two Polish groups G and H can be embedded into a Polish group T in such a way that the subgroup of T generated by G and H is isomorphic to the free product $G * H$.

1. INTRODUCTION

Mark Graev [Gra51] gave a construction of two-sided invariant metrics on free groups which now bear his name. Starting from a pointed metric space (X, d, e), the Graev metric δ is a two-sided invariant metric on the free group $F(X)$. The precise construction of δ will be explained below, but it is characterized by being *the largest two-sided invariant metric on $F(X)$ that extends d,* where we view X as being embedded into $F(X)$ in a natural way. In group theory, free groups are important, among other reasons, as surjectively universal objects: any group is a factor of a free group. Graev metrics and their generalizations proved to be very useful in constructing surjectively universal objects in various classes of metrizable groups.

For instance, let $\mathbb{N}^{\mathbb{N}}$ denote the *Baire space:* the space of infinite sequences of natural numbers with the metric

$$d(x, y) = \sup\{\, 2^{-n} \mid n \in \mathbb{N},\ x(n) \neq y(n) \,\}.$$

Let $\overline{F}(\mathbb{N}^{\mathbb{N}})$ be the group completion of $F(\mathbb{N}^{\mathbb{N}})$ endowed with the Graev metric (with respect to any distinguished point).

Theorem (Folklore, see Theorem 2.11 in [Kec94])**.** *The group $\overline{F}(\mathbb{N}^{\mathbb{N}})$ is surjectively universal in the class of Polish groups that admit compatible two-sided invariant metrics.*

An important question raised in [Kec94] and further advertised in [BK96] is whether there is a universal Polish group. Motivated by this question, L. Ding and S. Gao [DG07b] constructed generalized Graev metrics, and based on this construction Ding [Din12] answered the question of Kechris in the affirmative. In a recent paper Gao [Gao13] addressed the question of the existence of surjectively universal Polish *ultrametric* groups and gave yet another modification of Graev's original definition. The latter paper of Gao motivates our study of the Graev ultrametrics on free products of ultrametric groups.

1.1. Main results.

The main results of this work are twofold. In Section 2 we give the constructions of Graev ultrametrics for free products and HNN extensions of groups with two-sided invariant ultrametrics. In particular we prove

Theorem (see Theorem 2.13)**.** *Let (G, d_G) and (H, d_H) be groups with two-sided invariant ultrametrics, and let $A = G \cap H$ be a common closed subgroup. There exists a two-sided invariant ultrametric on the free product with amalgamation $G *_A H$ that extends the ultrametrics d_G and d_H.*

Theorem (see Theorem 2.15)**.** *Let (G, d) be a group with a two-sided invariant ultrametric d, A and B be closed subgroups of G and $\phi : A \to B$ be an isometric isomorphism. If $\operatorname{diam}(A) \leq K$, then there exists a two-sided invariant ultrametric δ on the HNN extension H of (G, ϕ) which extends d and such that $\delta(t, e) = K$, where t is the stable letter of H.*

While we follow closely the methods of [Slu12], the formalism for trivial words used in this paper is different. We introduce a new notion of a maximal evaluation forest and argue that it provides a more unified tool for studying Graev metrics on free products than the notion of an evaluation tree.

Key words and phrases. Graev metrics, ultrametrics, free products.

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In Section 3 we step outside of the two-sided invariant world and define a notion of a free product of general Polish groups. The results of Section 3 are new for both the metric and the ultrametric settings.

Our notion of a free product depends on a number of parameters. We show that when these parameters are "sufficiently large" the free product in our sense contains the usual free product of abstract groups as its dense subgroup. To establish this fact, we show, in particular, the following

Theorem (see Theorem 3.8)**.** *Let G and H be Polish groups. There are a Polish group T and embeddings $\psi_G : G \hookrightarrow T$, $\psi_H : H \hookrightarrow T$ such that $\langle \psi_G(G), \psi_H(H)\rangle$ is naturally isomorphic to the usual*[1] *free product $G * H$. Moreover, if G and H admit compatible left invariant ultrametrics, then T can be chosen to admit such a metric as well.*

1.2. **Notions and notations.** To establish the terminology, recall that an *ultrametric space* is a metric space (X, d) in which the metric satisfies a strong form of the triangle inequality:

$$d(x_1, x_2) \le \max\big\{d(x_1, x_3), d(x_3, x_2)\big\}$$

for all $x_1, x_2, x_3 \in X$. A *Polish space* is a separable completely metrizable topological space, and a *Polish group* is a topological group which is a Polish space. By a *metric group* we mean a pair (G, d), where G is a topological group, d is a metric on G, and the topology induced by d coincides with the topology of G; such metrics will be called *compatible*. A metric d on G is said to be *left invariant* if

$$d(fg_1, fg_2) = d(g_1, g_2)$$

for all $f, g_1, g_2 \in G$; the definition of a *right invariant* metric is symmetric. A metric d on G is *two-sided invariant* if it is both left and right invariant.

We also need the notion of a group completion. If (G, d) is a metric group with a left invariant metric d, we let D be the metric on G defined by

$$D(g_1, g_2) = d(g_1, g_2) + d\big(g_1^{-1},\, g_2^{-1}\big).$$

Note that D is compatible with the topology of G, but in general it is neither left nor right invariant. Let $(\overline{G}, D)$ denote the Hausdorff completion of the metric space (G, D). It turns out that the group operations on G admit a unique extension to $\overline{G}$, and the complete metric D turns $\overline{G}$ into a topological group. The group $\overline{G}$ is called the *group completion* of G. As a topological group, $\overline{G}$ does not depend on the choice of the compatible left invariant metric d on G.

Given two (ultra)metric spaces (X, d_X) and (Y, d_Y) and a common subspace $A = X \cap Y$, which is assumed to be closed in both X and Y, with

$$d_X(a_1, a_2) = d_Y(a_1, a_2) \quad \text{for all } a_1, a_2 \in A,$$

we define the *(ultra)metric amalgam of X and Y over A* to be the metric space (Z, d_Z), $Z = X \cup Y$, d_Z extends both d_X and d_Y, and for $x \in X$ and $y \in Y$

$$d_Z(x, y) = \begin{cases} \inf\limits_{a \in A} \big(d_X(x, a) + d_Y(a, y)\big) & \text{in the metric setting,} \\ \inf\limits_{a \in A} \max\big\{d_X(x, a), d_Y(a, y)\big\} & \text{in the ultrametric setting.} \end{cases}$$

Note that (Z, d_Z) is again an (ultra)metric space and that (X, d_X) and (Y, d_Y) are naturally subspaces of Z. By taking isometric copies of spaces we can define the amalgamation of (X, d_X) and (Y, d_Y) over (A, d_A) whenever we have two isometric inclusions $\iota_X : A \to X$ and $\iota_Y : A \to Y$.

Two-sided invariant (ultra)metrics are characterized among left invariant (ultra)metrics by the following inequality.

Proposition 1.1. *Let d be a left invariant metric on a group G. The metric d is two-sided invariant if and only if*

$$d(g_1 \cdots g_n, f_1 \cdots f_n) \le \sum_{i=1}^{n} d(g_i, f_i)$$

for all $g_i, f_i \in G$ and all $n \in \mathbb{N}$. If d is an ultrametric, then moreover

$$d(g_1 \cdots g_n, f_1 \cdots f_n) \le \max_{1 \le i \le n} \{d(g_i, f_i)\}.$$

[1] We mean a free product of abstract groups. No topology is considered on $G * H$ here.

With a left invariant pseudo-metric d on a group G we may associate a pseudo-norm $\mathrm{N} : G \to \mathbb{R}^+$ defined by $\mathrm{N}(f) = d(f, e)$ and satisfying for all $g, g_1, g_2 \in G$

(i) $\mathrm{N}(g) \geq 0$, $\mathrm{N}(e) = 0$; $\mathrm{N}(g) > 0$ for $g \neq e$ if and only if d is a metric;
(ii) $\mathrm{N}(g) = \mathrm{N}(g^{-1})$;
(iii) $\mathrm{N}(g_1 g_2) \leq \mathrm{N}(g_1) + \mathrm{N}(g_2)$;

If d is an ultrametric, then item (iii) becomes

(iii)′ $\mathrm{N}(g_1 g_2) \leq \max\{\mathrm{N}(g_1), \mathrm{N}(g_2)\}$.

If d is two-sided invariant, then

(iv) $\mathrm{N}(g g_1 g^{-1}) = \mathrm{N}(g_1)$.

The correspondence between left invariant metrics and norms is bijective: $d(g_1, g_2) = \mathrm{N}(g_1^{-1} g_2)$ is a left invariant metric on G for any norm N.

Expression $[m, n]$ will denote the interval of natural numbers $\{m, m+1, \dots, n-1, n\}$. For a set X we let $\mathrm{W}(X)$ to denote the set of nonempty words in the alphabet X. The length of a word $w \in \mathrm{W}(X)$ is denoted by $|w|$ and $w(i)$ denotes its ith letter. If $w \in \mathrm{W}(X)$ is a word of length n and $F \subseteq [1, n]$, $F = \{j_1, \dots, j_m\}$ with $j_1 < j_2 < \cdots < j_m$, then $w(F)$ denotes the word $w(j_1)w(j_2)\cdots w(j_m)$. The minimal element of F is denoted by $m(F)$, and $M(F)$ denotes its maximal element: $m(F) = j_1$, $M(F) = j_m$.

1.3. **Graev (ultra)metrics on free groups.** We now describe the construction of Graev metrics and Graev ultrametrics on free groups following [DG07b] and [Gao13]. Since these two constructions are very similar, we give them in parallel. A *pointed (ultra)metric space* is a triple (X, d, e), where $(X, d\,)$ is an (ultra)metric space and $e \in X$ is a distinguished point. Let X^{-1} denote a copy of X with elements of X^{-1} being formal inverses of the elements of X with the agreement $X \cap X^{-1} = \{e\}$, that is, $e^{-1} = e$. Extend d to an (ultra)metric on X^{-1} by declaring $d\big(x^{-1}, y^{-1}\big) = d(x, y)$ for all $x, y \in X$. Let $(\overline{X}, d, e)$ denote the (ultra)metric amalgam of $(X, d\,)$ and $(X^{-1}, d\,)$ over the subspace $\{e\} = X \cap X^{-1}$, see Figure 1. We can extend the inverse $x \mapsto x^{-1}$ to a function on $\overline{X}$ by setting $(x^{-1})^{-1} = x$. To summarize, starting from a pointed (ultra)metric space (X, d, e) we construct in a canonical way a pointed (ultra)metric space $(\overline{X}, d, e)$, X is a subspace of $\overline{X}$ and the function $\overline{X} \ni x \mapsto x^{-1}$ is an isometric involution. We shall say that $\overline{X}$ is obtained from X by *adding formal inverses*.

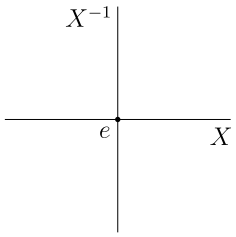

FIGURE 1. $\overline{X}$ is the amalgam of X and X^{-1} over $\{e\}$.

By $F(X)$ we denote the free group with generators $X \setminus \{e\}$ (therefore there is a slight abuse of notations, since a proper notation would be $F(X \setminus \{e\})$). The set $\overline{X}$ is viewed as a subset of $F(X)$, where $e \in \overline{X}$ is identified with the identity element of the free group. We have a natural *evaluation map* $\mathrm{W}(\overline{X}) \ni w \mapsto \hat{w} \in F(X)$, $\hat{w}$ being just the reduced form of w. This map is surjective. For two words $u_1, u_2 \in \mathrm{W}(\overline{X})$ which have equal lengths $|u_1| = n = |u_2|$ we define

$$\rho(u_1, u_2) = \begin{cases} \sum_{i=1}^{n} d\big(u_1(i), u_2(i)\big) & \text{in the metric case,} \\ \max_{i \leq n} \big\{ d\big(u_1(i), u_2(i)\big) \big\} & \text{in the ultametric case.} \end{cases}$$

Finally, the Graev (ultra)metric δ on $F(X)$ is defined by

$$\delta(f_1, f_2) = \inf\big\{ \rho(u_1, u_2) \;\big|\; u_i \in \mathrm{W}(\overline{X}),\ \hat{u}_i = f_i,\ |u_1| = |u_2| \big\}.$$

Theorem 1.2 (Graev [Gra51], Gao [Gao13])**.** *The function δ is a two-sided invariant (ultra)metric on the group $F(X)$. Moreover, δ extends d on $\overline{X}$.*

In order to describe an explicit formula for Graev metrics we need the notion of a match. Let F be a finite set of natural numbers $F = \{i_k\}_{k=1}^n$, $i_1 < i_2 < \cdots < i_n$. A *match* on F is a bijection $\theta : F \to F$ such that $\theta\big(\theta(i)\big) = i$ for all $i \in F$, and there are no $k < l$ such that $\theta(i_k) = i_p$, $\theta(i_l) = i_q$ and $k < l < p < q$. In other words we can think of a match as a set of arcs connecting elements of F such that two arcs are either disjoint, or one of the arcs is contained in the other one, see Figure 2. We shall sometimes say that θ is a match on a word w meaning that θ is a match on its letters, which we then identify with the set $\{1, \dots, |w|\}$. If θ is a match on F and $\{j_1, \dots, j_m\} \subseteq F$ such that $j_1 < j_2 < \cdots < j_m$, and $\theta(\{j_1, \dots, j_m\}) = \{j_1, \dots, j_m\}$, then the *restriction* of θ onto the set $\{j_1, \dots, j_m\}$ is a match on $\{j_1, \dots, j_m\}$, but we shall abuse the terminology and say that θ itself is a match on $\{j_1, \dots, j_m\}$ in this case.

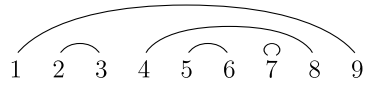

FIGURE 2. An example of a match on the set $\{1, \dots, 9\}$.

If $w \in \mathrm{W}(\overline{X})$ and θ is a match on w, the word w^θ is defined by

$$w^\theta(i) = \begin{cases} w(i) & \text{if } \theta(i) > i, \\ e & \text{if } \theta(i) = i, \\ w\big(\theta(i)\big)^{-1} & \text{if } \theta(i) < i. \end{cases}$$

Is is straightforward to check that $\hat{w}^\theta = e$ for any w and any θ. For example, if θ is the match in Figure 2, then for the following word w we have

$$\begin{array}{rcccccccccc} w & = & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ w^\theta & = & x_1 & x_2 & x_2^{-1} & x_4 & x_5 & x_5^{-1} & e & x_4^{-1} & x_1^{-1} \end{array}$$

Theorem 1.3 (Sipacheva–Uspenskij [SU87], Ding–Gao [DG07a], Gao [Gao13])**.** *Let δ be the Graev (ultra)metric on the free group $F(X)$. For any $f \in F(X)$*

$$\delta(f, e) = \min\big\{\, \rho\big(w, w^\theta\big) \;\big|\; w \textit{ is the reduced form of } f, \ \theta \textit{ is a match on } w \,\big\}.$$

In order to explain the generalized Graev metrics we need yet another tool.

Definition 1.4. A *scale* on a pointed set (X, e) is a function $\Gamma : X \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying for all $x \in X$ and all $r \in \mathbb{R}^+$

(i) $\Gamma(e, r) = r$, $\Gamma(x, r) \geq r$;
(ii) $\Gamma(x, r) = 0$ if and only if $r = 0$;
(iii) $\Gamma(x, \cdot)$ is a monotone increasing function with respect to the second variable;
(iv) $\lim\limits_{r \to 0} \Gamma(x, r) = 0$.

By a *scaled (ultra)metric space*, or for brevity just a *scaled space*, we mean a tuple $(\overline{X}, d, e, \Gamma)$, where $\overline{X}$ is obtained from some pointed (ultra)metric space X by adding formal inverses and Γ is a scale on $\overline{X}$. We shall denote scaled spaces with bold letters $\mathbf{X}$, $\mathbf{Y}$, etc. Let $\mathbf{X} = (\overline{X}, d, e, \Gamma)$ be a scaled space. Following [DG07b] and [Gao13], for a match θ on a word $w \in \mathrm{W}(\overline{X})$ we define the number $\mathrm{N}_\Gamma^\theta(w) = \mathrm{N}^\theta(w)$ by induction on the length of w as follows.

(i) If $w = x$ for some $x \in \overline{X}$, then $\mathrm{N}^\theta(w) = d(x, e)$; if $w = x_1 x_2$ and $\theta(x_1) = x_2$, then $\mathrm{N}^\theta(w) = d(x_1, x_2^{-1})$.
(ii) If $\theta(x_1) = x_k$ and $k < |w|$, then $w = u_1 u_2$ for some words $u_i \in \mathrm{W}(\overline{X})$ with $|u_1| = k$, $|u_2| = n - k$, θ is a match on both u_1 and u_2, and we set

$$\mathrm{N}^\theta(w) = \begin{cases} \mathrm{N}^\theta(u_1) + \mathrm{N}^\theta(u_2) & \text{in the metric case,} \\ \max\big\{\mathrm{N}^\theta(u_1), \mathrm{N}^\theta(u_2)\big\} & \text{in the ultrametric case.} \end{cases}$$

(iii) If $\theta(x_1) = x_n$, $n = |w|$, then let $w = x_1 u x_n$ for $u \in \mathrm{W}(\overline{X})$, $x_1, x_n \in \overline{X}$, θ is a match on u and we set

$$\mathrm{N}^\theta(w) = \begin{cases} d(x_1, x_n^{-1}) + \min\Big\{\Gamma\Big(x_1^{-1}, \mathrm{N}^\theta(u)\Big),\ \Gamma\Big(x_n, \mathrm{N}^\theta(u)\Big)\Big\} & \text{in the metric case,} \\ \max\Big\{d\big(x_1, x_n^{-1}\big), \min\Big\{\Gamma\Big(x_1^{-1}, \mathrm{N}^\theta(u)\Big),\ \Gamma\Big(x_n, \mathrm{N}^\theta(u)\Big)\Big\}\Big\} & \text{in the ultrametric case.} \end{cases}$$

The *Graev (ultra)norm* $\mathrm{N}_\Gamma = \mathrm{N}$ *of the scale* Γ is defined by

$$\mathrm{N}(f) = \inf\big\{\, \mathrm{N}^\theta(w) \;\big|\; w \in \mathrm{W}(\overline{X}),\ \hat{w} = f \text{ and } \theta \text{ is a match on } w \,\big\}.$$

Proposition 1.5 (Ding–Gao [DG07b], Gao [Gao13])**.** *Let (X, d, e) be a pointed (ultra)metric space. The function $f \mapsto \mathrm{N}(f)$ is an (ultra)norm on the group $F(X)$, and the latter is a topological group in the topology of* N*. The natural inclusion map $\overline{X} \hookrightarrow F(X)$ is an isometry.*

We denote by $F(\mathbf{X})$ the free group $F(X)$ together with the Graev norm N and view $\overline{X}$ as a subset of $F(\mathbf{X})$. A *canonical scale* on an (ultra)metric group (G, d) is a map $\mathcal{S}_G : G \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\mathcal{S}_G(g, r) = \max\Big\{r,\ \sup\big\{d(g^{-1}hg, e) \;\big|\; h \in G,\ d(h, e) \le r\big\}\Big\}.$$

Let $\mathbf{X} = (\overline{X}, d, e, \Gamma)$ be a scaled space, (G, d_G) be an (ultra)metric group, and Γ_G be a scale on G. A map $\phi : \overline{X} \to G$ is called a *Lipschitz morphism with respect to the scale* Γ_G if for all $x, y \in \overline{X}$ and all $r \in \mathbb{R}^+$

(i) $\phi(e) = e$;
(ii) $\phi(x^{-1}) = \phi(x)^{-1}$;
(iii) $d_G\big(\phi(x), \phi(y)\big) \le d(x, y)$;
(iv) $\Gamma_G(\phi(x), r) \le \Gamma(x, r)$.

We say that ϕ is a Lipschitz morphism if it is a Lipschitz morphism with respect to the canonical scale $\mathcal{S}_G$. In general we shall use the term Lipschitz to mean 1-Lipschitz.

Proposition 1.6 (Ding–Gao [DG07b], [Gao13])**.** *Let ϕ be a Lipschitz morphism from a scaled space $\mathbf{X}$ into an (ultra)metric group G. The map ϕ extends to a Lipschitz homomorphism $\phi : F(\mathbf{X}) \to G$. If G is completely metrizable, then ϕ can be further extended to a continuous homomorphism $\phi : \overline{F}(\mathbf{X}) \to G$.*

2. Graev ultrametrics on free products

Let (G, d_G) and (H, d_H) be ultrametric groups with two-sided invariant metrics, let $A = G \cap H$ be a common *closed* subgroup; we assume that the metrics agree: $d_G(a_1, a_2) = d_H(a_1, a_2)$ for all $a_1, a_2 \in A$. We shall define a two-sided invariant ultrametric δ on the amalgamated free product $G *_A H$.

The construction of Graev metrics on free products mimics that on the free groups. To start, we have a natural *evaluation map*:

$$\mathrm{W}(G \cup H) \ni w \mapsto \hat{w} \in G *_A H,$$

where $\hat{w}$ is just the product of letters of w. Note that this map is surjective. If $w \in \mathrm{W}(G \cup H)$ and $F \subseteq \big[1, |w|\big]$, then the evaluation of the subword $w(F)$ is denoted by $\hat{w}(F)$ (as opposed to $\widehat{w(F)}$). Let d be the ultrametric on the amalgam $G \cup H$ of (G, d_G) and (H, d_H) over A. If $u_1, u_2 \in \mathrm{W}(G \cup H)$ are two words of the same length $|u_1| = n = |u_2|$, we define $\rho(u_1, u_2)$ to be the maximum of distances between the corresponding letters:

$$\rho(u_1, u_2) = \max_{i \le n}\big\{d\big(u_1(i), u_2(i)\big)\big\}.$$

The *Graev ultrametric* on the free product $G *_A H$ is the function

$$\delta(f_1, f_2) = \inf\big\{\, \rho(u_1, u_2) \;\big|\; \hat{u}_i = f_i,\ |u_1| = |u_2| \,\big\}.$$

Our goal is to prove

Theorem 2.1. *The function δ is a two-sided invariant ultrametric on $G *_A H$. Moreover, δ extends d on $G \cup H$.*

As is typical for Graev metrics, it is straightforward to check that δ is a two-sided invariant pseudo-ultrametric. The main difficulty is to show that distinct elements are never glued: $\delta(f_1, f_2) > 0$ whenever $f_1 \neq f_2$.

Our arguments here are very similar to those in [Slu12], and we shall outline the proofs and give references for more details. Essentially the proofs are repetitions of the proofs fore the metric case when the summation operation is substituted with the operation of taking maximum, e.g., like in Proposition 1.1. Another, more

important difference is that in [Slu12] the notion of an evaluation tree was used. In our approach here we use instead the formalism of (maximal) evaluation forests, which has the advantage of working for both the amalgams and HNN extensions in a uniform way.

The following proposition is essentially obvious.

Proposition 2.2 (cf. Lemma 5.1 [Slu12])**.** *The function δ is a two-sided invariant pseudo-ultrametric on $G*_A H$.*

Let $\mathrm{N} : G *_A H \to \mathbb{R}^+$ be the pseudo-norm that corresponds to δ: $\mathrm{N}(f) = \delta(f, e)$. In order to show that $\delta(f_1, f_2) > 0$ for $f_1 \neq f_2$ it is enough to show that N is a genuine norm: $\mathrm{N}(f) > 0$ for $f \neq e$.

A pair of words (α, ζ), $\alpha, \zeta \in \mathrm{W}(G \cup H)$, is said to be an *$f$-pair* if $|\alpha| = |\zeta|$, $\hat{\zeta} = e$, and $\hat{\alpha} = f$. The definition of the function N can then be reformulated as

$$\mathrm{N}(f) = \inf\big\{\, \rho(\alpha, \zeta) \;\big|\; (\alpha, \zeta) \text{ is an } f\text{-pair} \,\big\}.$$

To get a better understanding of the function N, we shall gradually add restrictions on the f-pairs (α, ζ), while still keeping the equality above.

2.1. **Trivial words.** Before going any further we need to understand the structure of trivial words in the amalgam $G *_A H$. A word $\zeta \in \mathrm{W}(G \cup H)$ is said to be *trivial* if $\hat{\zeta} = e$.

We say that two letters $x, y \in G \cup H$ are *multipliable* if they both come from the same group: either $x, y \in G$ or $x, y \in H$. We also say that a word $w \in \mathrm{W}(G \cup H)$ is *multipliable* if all of its letters come from the same group.

Definition 2.3. Let $(\mathcal{T}, \preceq)$ be a poset, and let $s, t \in \mathcal{T}$. We say that s is an *immediate predecessor* of t if $s \prec t$ and for any $s \preceq s' \preceq t$ either $s' = s$ or $s' = t$. If s is an immediate predecessor of t, then t is also said to be an *immediate successor* of s. A *finite rooted tree*, or just a *tree*, is a finite poset $(\mathcal{T}, \preceq)$ with a distinguished element $\varnothing$, called the *root*, such that for any $t \in \mathcal{T}$

- $t \preceq \varnothing$, i.e., the root $\varnothing$ is the largest element;
- $\{\, s \in \mathcal{T} \mid t \preceq s \,\}$ is linearly ordered.

There is a natural graph structure on $\mathcal{T}$: we put an edge between s and t whenever s is an immediate predecessor of t or t is an immediate predecessor of s. With this assignment of edges $\mathcal{T}$ is a rooted tree in the sense of the graph theory. In a tree $\mathcal{T}$, any element $t \in \mathcal{T}$, except for the root, has a unique immediate successor, which we denote by t^+. A *leaf* in a tree is an element without predecessors.

A *finite forest* is a finite poset $(\mathcal{F}, \preceq)$ which is a disjoint union of rooted trees $\mathcal{F} = \sqcup_{i=1}^{r} \mathcal{T}_i$, where two elements are incomparable if they belong to different trees. The root of the tree $\mathcal{T}$ is denoted by $\varnothing(\mathcal{T})$ and $\mathcal{F}^{\varnothing}$ denotes the set of roots of trees in $\mathcal{F}$:

$$\mathcal{F}^{\varnothing} = \big\{\, \varnothing(\mathcal{T}) \;\big|\; \mathcal{T} \text{ is a tree in } \mathcal{F} \,\big\}.$$

An *evaluation forest* on an interval $[1, n]$ is a forest $\mathcal{F}$ together with an assignment $t \mapsto I_t \subseteq [1, n]$ such that for all $t, s \in \mathcal{F}$

(i) I_t is a non-empty subinterval of $[1, n]$;
(ii) $[1, n] = \sqcup_{\mathcal{T}} I_{\varnothing(\mathcal{T})}$, where the union is taken over all trees $\mathcal{T}$ in $\mathcal{F}$;
(iii) $I_s \cap I_t \neq \varnothing$ if and only if s and t are comparable in $\mathcal{F}$;
(iv) $s \preceq t$ if and only if $I_s \subseteq I_t$;
(v) if $s \prec t$, then $m(I_t) < m(I_s) \leq M(I_s) < M(I_t)$, and in particular $I_s \subset I_t$;

Let $\zeta \in \mathrm{W}(G \cup H)$ be a word of length n such that $\hat{\zeta} \in A$ and let $\mathcal{F}$ be an evaluation forest on $[1, n]$. We say that $\mathcal{F}$ is an *evaluation forest for* ζ if additionally for all $t \in \mathcal{F}$

(vi) $\hat{\zeta}(I_t) \in A$;
(vii) $\zeta(R_t)$ is multipliable, where $R_t = I_t \setminus \bigcup_{s \prec t} I_s$; the set R_t is called the *reminder* of the interval I_t.

We say that an interval $I \subseteq [1, n]$ is *decomposable* (relative to ζ) if one can write I as a disjoint union of non-trivial subintervals $I = J_1 \sqcup J_2$ with $\hat{\zeta}(J_1) \in A$ and $\hat{\zeta}(J_2) \in A$; otherwise we say that I is *indecomposable* (relative to ζ).

Let $\mathcal{F}$ be an evaluation forest for ζ. We say that $\mathcal{F}$ is *maximal* if for all $t \in \mathcal{F}$

(viii) I_t is indecomposable;
(ix) if $J \subset I_t$ is a non-empty subinterval with $\hat{\zeta}(J) \in A$, and for all $s \prec t$ either $I_s \subseteq J$ or $I_s \cap J = \varnothing$, then $J \subseteq \bigcup_{s \prec t} I_s$.

Example 2.4. The notion of a maximal evaluation forest is quite technical, and we illustrate it on a concrete example. Consider the word $\zeta \in \mathrm{W}(G \cup H)$ given by

$$\zeta = g_1\ b\ g_2\ h_1\ h_2\ g_3\ g_4\ h_3\ g_5\ g_6\ g_7\ h_4\ g_8\ g_9\ h_5\ h_6\ g_{10}\ h_7\ h_8\ g_{11}$$

where $g_i \in G \setminus A$, $h_j \in H \setminus A$, and $b \in A$. Suppose also that the following identities hold:

$$g_3g_4 = a_1 \qquad h_5h_6 = a_3 \qquad h_1h_2a_1h_3a_2h_4 = a_5 \qquad g_1bg_2a_5g_8 = a_7$$
$$g_5g_6g_7 = a_2 \qquad h_7h_8 = a_4 \qquad g_9a_3g_{10}a_4g_{11} = a_6$$

for some $a_i \in A$. In particular, $\hat{\zeta} = a_7 \cdot a_6 \in A$. Pictorially cancellations in ζ can be represented as follows.

$$\begin{array}{cccccccccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\zeta = & g_1 & b & g_2 & h_1 & h_2 & g_3 & g_4 & h_3 & g_5 & g_6 & g_7 & h_4 & g_8 & g_9 & h_5 & h_6 & g_{10} & h_7 & h_8 & g_{11}
\end{array}$$

The corresponding evaluation forest $\mathcal{F}_1$ for ζ is shown in Figure 3. But note that ζ has other evaluation forests as well (for instance, the forest $\mathcal{F}_2$ in Figure 3).

FIGURE 3. Two evaluation forests for ζ. None of them is maximal.

Intuitively speaking an evaluation forest $\mathcal{F}$ for ζ captures the combinatorial structure of cancellations: leaves of the forest are multipliable subwords that when multiplied produce an element from A, elements whose all predecessors are leaves correspond to subwords that after the evaluation of leaves become multipliable and when multiplied yield and element from A, etc. Informally $\mathcal{F}$ is a set of subintervals with multipliable remainders such that two intervals are either disjoint or one is contained in the other, and in the later case the containment is strict in the sense of item (v).

The forests $\mathcal{F}_1$ and $\mathcal{F}_2$ in Figure 3 are not maximal. In $\mathcal{F}_1$ item (ix) fails: for $I = [1, 13]$ we may add a subinterval $J = [2, 2]$. The forest $\mathcal{F}_2$, which consists of a single tree, is not maximal because of the failure of item (viii): we have $I_\varnothing = [1, 20]$ and $I_\varnothing = [1, 13] \sqcup [14, 20]$ with $\hat{\zeta}\big([1, 13]\big) \in A$ and $\hat{\zeta}\big([14, 20]\big) \in A$. But these are the only obstacles that prevent $\mathcal{F}_1$ and $\mathcal{F}_2$ from being maximal (provided that no further relations between elements g_i and h_j hold; for instance, if also $h_1h_2 \in A$, then the forest in Figure 4 is not maximal either). It is therefore easy to modify these forests to get a maximal forest $\mathcal{F}_3$, which is shown in Figure 4.

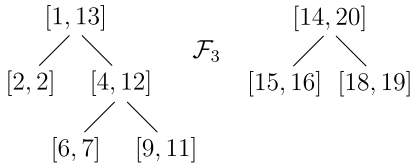

FIGURE 4. A maximal evaluation forest for ζ.

Item (ix) can be reformulated in a number of ways. If $J \subset I_t$ is a subinterval with $\hat{\zeta}(J) \in A$ such that $J \cap I_s = \varnothing$ or $I_s \subseteq J$ for any $s \prec t$, then the condition $J \subseteq \bigcup_{s \prec t} I_s$ is equivalent to saying that $J = \bigsqcup_{i=k}^{l} I_{s_i}$ for some $1 \leq k \leq l \leq m$, where $s_1, \dots, s_m$ are the immediate predecessors of t listed in the order $M(I_{s_i}) < m(I_{s_{i+1}})$. Yet another reformulation would be to say that $J \cap R_t = \varnothing$. So item (ix) prohibits the situation shown in Figure 5, where stars represent elements of R_t.

Remark 2.5. If $\mathcal{F}$ is a maximal forest on ζ and i is such that $\zeta(i) \in A$, then there must be a node $t_0 \in \mathcal{F}$ such that $I_{t_0} = [i, i]$. Indeed, if this were not the case, we would find the smallest $t \in \mathcal{F}$ such that $\zeta(i) \in I_t$ and obtain a contradiction with (ix) for I_t and $J = [i, i]$.

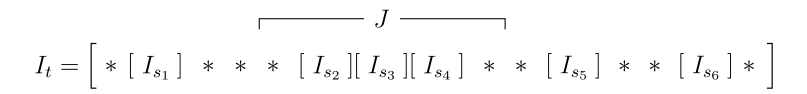

FIGURE 5. Such an interval J with $\hat{\zeta}(J) \in A$ is prohibited by item (ix).

Remark 2.6. While there is much less freedom in constructing maximal evaluation forests when compared to general evaluation forests, there is still some amount of flexibility. Here is a concrete example. Let $G = S_6$ -- the symmetric group on six elements, and let $A = \{e\}$ be the trivial subgroup. The group H does not matter, since our word will use only letters from G. Consider elements of G

$$g_1 = (12), \quad g_2 = (34), \quad g_3 = (12)(34), \quad f_1 = (12)(34)(56), \quad f_2 = (56).$$

The word

$$\begin{array}{rccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \zeta = & f_1 & g_1 & g_2 & g_2 & g_1 & g_3 & f_2 \end{array}$$

is trivial and it has two maximal evaluation forests (see Figure 6).

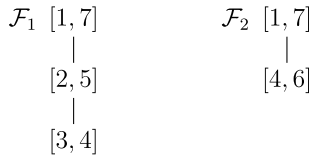

FIGURE 6. Two distinct maximal evaluation forests for ζ, each consisting of a single tree.

Proposition 2.7. *Any word $\zeta \in \mathrm{W}(G \cup H)$ with $\hat{\zeta} \in A$ has a maximal evaluation forest.*

Proof. We prove the statement by induction on the length of ζ. If $|\zeta| = 1$, then $\zeta = a$ for some $a \in A$, and therefore we may take $\mathcal{F}$ to consist of a single root $\mathcal{F} = \{\varnothing\}$ with $I_\varnothing = [1, 1]$.

Suppose the proposition has been proved for all words of length $< n$ and let ζ have length n.

Step 1: Decomposing $[1, n]$. If $[1, n]$ is decomposable and $[1, n] = J_1 \sqcup J_2$ with $\hat{\zeta}(J_1) \in A$ and $\hat{\zeta}(J_2) \in A$, then we may apply the assumption of induction to the words $\zeta_1 = \zeta(J_1)$, $\zeta_2 = \zeta(J_2)$ and obtain their maximal evaluation forests $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. The maximal evaluation forest for ζ is then just the union of $\mathcal{F}_1$ and $\mathcal{F}_2$ with the natural assignment of intervals.

We therefore may assume that $[1, n]$ is indecomposable.

Step 2: Typical case. Suppose that we can find a proper subinterval $J \subset [1, n]$ such that $|J| \geq 2$ and $\hat{\zeta}(J) = a \in A$. Let ζ_1 be the word obtained from ζ by evaluating $\zeta(J)$:

$$\zeta_1 = \zeta\big([1, m(J) - 1]\big)\, a\, \zeta\big([M(J) + 1, n]\big).$$

Since $|J| \geq 2$, the length of ζ_1 is less than n, hence by the inductive assumption we may find an evaluation forest $\mathcal{F}_1$ for ζ_1 with the assignment of intervals $t \mapsto J_t \subseteq \big[1, |\zeta_1|\big]$. Let now I_t be the subintervals of $[1, n]$ obtained from J_t by inserting $\zeta(J)$ back into ζ_1, or, more formally:

$$I_t = \begin{cases} \big[m(J_t), M(J_t)\big] & \text{if } m(J_t), M(J_t) < m(J), \\ \big[m(J_t), M(J_t) + |J| - 1\big] & \text{if } m(J_t) \leq m(J) \leq M(J_t), \\ \big[m(J_t) + |J| - 1, M(J_t) + |J| - 1\big] & \text{if } m(J) < m(J_t), M(J_t). \end{cases}$$

By the maximality of $\mathcal{F}_1$ for ζ_1, the intervals J_t are indecomposable relative to ζ_1, but this may no longer be true for the intervals I_t relative to ζ, because an interval I_t with $m(J) \in I_t$ has more possibilities for decomposition than the corresponding interval J_t.

The subword $\zeta(J)$ itself has length $< n$, and therefore the inductive assumption yields its maximal evaluation forest $\widetilde{\mathcal{F}}_2$. Since $\zeta_1\big(m(J)\big) = a \in A$, by Remark 2.5 there is some $t_0 \in \mathcal{F}_1$ such that $J_{t_0} = [m(J), m(J)]$. The naive approach would be "to put the forest $\widetilde{\mathcal{F}}_2$ instead of the node t_0" (see Figure 7). This does not work in general precisely because some of the intervals I_t may be decomposable.

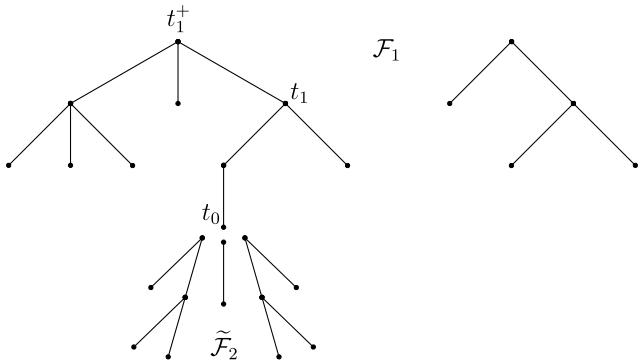

FIGURE 7. Naive approach of constructing $\mathcal{F}$.

In order to fix this let $t_1 \in \mathcal{F}_1$ be the maximal node with I_{t_1} being decomposable (note that $m(J) \in I_t$ for all decomposable intervals I_t, hence such intervals are comparable, and the largest node t_1 exists). If all I_t are indecomposable, we set $t_1 = t_0$. Note that t_1 is not the root of $\mathcal{F}_1$, since $I_\varnothing$ is assumed to be indecomposable. In particular $\zeta(I_{t_1})$ has length strictly less than n, and therefore by the assumption of induction it admits a maximal evaluation forest $\mathcal{F}_2$ with intervals $s \mapsto K_s$.

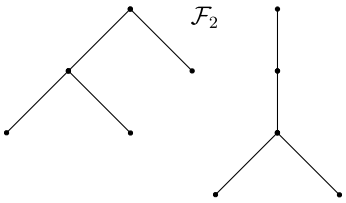

FIGURE 8. A possible example of the forest $\mathcal{F}_2$.

We define the forest $\mathcal{F}$ for ζ by

$$\mathcal{F} = \{s \in \mathcal{F}_1 \mid s \not\preceq t_1\} \sqcup \mathcal{F}_2,$$

with the ordering extending the orderings of $\mathcal{F}_1$ and $\mathcal{F}_2$ and $s \prec t_1^+$ for all $s \in \mathcal{F}_2$ (see Figure 8 and Figure 9). The assignment of intervals $\mathcal{F} \ni t \mapsto I_t$ is the natural one: we have already defined I_t for $t \in \mathcal{F}_1 \cap \mathcal{F}$, and for $s \in \mathcal{F}_2$ the interval I_s is just the interval K_s shifted by $m(I_{t_1}) - 1$:

$$I_s = \big[m(K_s) + m(I_{t_1}) - 1, M(K_s) + m(I_{t_1}) - 1\big].$$

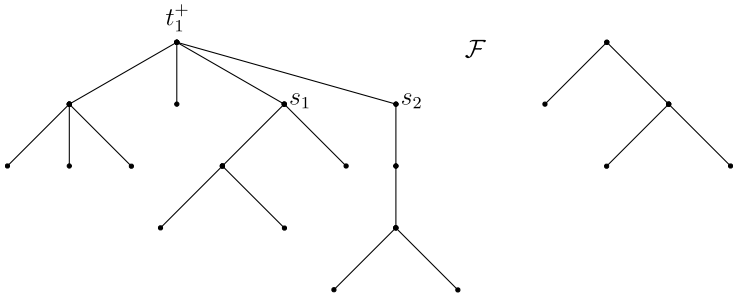

FIGURE 9. The forest $\mathcal{F} = \{\, s \in \mathcal{F}_1 \mid s \not\preceq t_1 \,\} \sqcup \mathcal{F}_2$.

We claim that $\mathcal{F}$ with $s \mapsto I_s$ is a maximal evaluation forest for ζ. It is straightforward to check that $\mathcal{F}$ is an evaluation forest and item (viii) follows immediately from the construction. It remains to check item (ix). For $s \neq t_1^+$ item (ix) follows from the maximality of $\mathcal{F}_1$ and $\mathcal{F}_2$, we need to check it only for t_1^+.

Suppose we have a subinterval $L \subset I_{t_1^+}$ such that $\hat{\zeta}(L) \in A$, for all $s \prec t_1^+$ either $I_s \cap L = \varnothing$ or $I_s \subseteq L$ and $L \cap R_{t_1^+} \neq \varnothing$. Let $s_1, \dots, s_m$ be the immediate predecessors of t_1^+, and let $s_k, \dots, s_l$ be those of the predecessors of t_1^+ that correspond to the roots of $\mathcal{F}_2$. Note that the intervals I_{s_i} are adjacent for $k \le i < l$: $M(I_{s_i}) + 1 = m(I_{s_{i+1}})$. The idea is to construct an interval $\widetilde{L}$ that will contradict item (ix) for t_1^+ in the forest $\mathcal{F}_1$. We have several cases.

Case 1: $L \cap I_{s_i} = \varnothing$ for all $k \le i \le l$. In this case the interval L naturally corresponds to a subword of ζ_1; let

$$\widetilde{L} = \begin{cases} L & \text{if } M(L) < m(I_{s_k}), \\ \big[m(L) - |I_{t_1}| + 1, M(L) - |I_{t_1}| + 1\big] & \text{if } M(I_{s_l}) < m(L). \end{cases}$$

We now get a contradiction with item (ix) of the maximality of $\mathcal{F}_1$ for ζ_1 with $J_{t_1^+}$ and $\widetilde{L}$.

We therefore may assume that either $I_{s_k} \subseteq L$ or $I_{s_l} \subseteq L$. In either case, we may enlarge L to an interval L' defined by (see Figure 10)

$$L' = L \cup \Big(\bigsqcup_{i=k}^{l} I_{s_i} \Big),$$

$$I_{t_1^+} = \Big[* \, [\, I_{s_1} \,] \;\; * \;\; * \;\; * \;\; \overbrace{[\, I_{s_2} \,][\, I_{s_3} \,][\, I_{s_4} \,]}^{L} \;\; * \;\; * \;\; [\, I_{s_5} \,] \;\; * \;\; * \;\; [\, I_{s_6} \,] \, * \Big]$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{L'}$$

FIGURE 10. The construction of the interval L'. Here $k = 2$ and $l = 4$.

Case 2: $L' = I_{t_1^+}$. In this case L is either an initial subinterval of $I_{t_1^+}$, or a terminal subinterval. In both cases $I_{t_1^+}$ is decomposable contrary to the choice of t_1.

Case 3: $L' \neq I_{t_1^+}$. Since $I_{s_i} \subset L'$ for all s_i that correspond to the roots of $\mathcal{F}_2$, we may let $\widetilde{L}'$ be the subinterval of $J_{t_1^+}$ that corresponds to L':

$$\widetilde{L}' = \big[m(L'), M(L') - |I_{t_1}| + 1\big].$$

We again get a contradiction with item (ix) and maximality of $\mathcal{F}_1$ for ζ_1, since $\widetilde{L}'$ is a proper subinterval of $J_{t_1^+}$ by the assumptions of this case.

Step 3: Degenerate case. In the last step we suppose that one cannot find any $J \subset [1, n]$ such that $|J| \ge 2$ and $\hat{\zeta}(J) \in A$. It is easy to see that in this case ζ must be multipliable. Let $i_1, \dots, i_m$ be the list of letters from A: $\zeta(i_k) \in A$ for all $k \le m$. We must have $i_k + 1 < i_{k+1}$, because if we had two consecutive letters from A at indices, say, i and $i + 1$, there would be a contradiction with the assumptions of this step for $J = [i, i + 1]$. (To be precise, we get a contradiction if also $n \ge 3$; if $n = 2$, and $\zeta = a_1\, a_2$, then the maximal evaluation forest consists of two trivial trees $[1, 1]$ and $[2, 2]$.) Note also that $1 < i_1$ and $i_m < n$, because otherwise $[1, n]$ is decomposable. Put now $\mathcal{F} = \{\varnothing, t_1, \dots, t_m\}$ and $I_\varnothing = [1, n]$, $I_{t_k} = [i_k, i_k]$. It is straightforward to check that $\mathcal{F}$ is a maximal forest for ζ. □

2.2. **Reductions.** An f-pair (α, ζ) is said to be *multipliable* if $\alpha(i)$ is multipliable with $\zeta(i)$ for any i. (We therefore use the word multipliable in two senses: a word is multipliable if all of its letters come from the same group, while a pair of words is multipliable if for each index corresponding letters of two words are from the same group.) Our first reduction states that in the definition of the norm function N one may take only multipliable pairs.

Lemma 2.8 (cf. Lemma 5.2 [Slu12])**.** *For any $f \in G *_A H$*

$$\mathrm{N}(f) = \inf \big\{ \rho(\alpha, \zeta) \;\big|\; (\alpha, \zeta) \textit{ is a multipliable } f\textit{-pair} \big\}.$$

Proof. The idea of the proof is simple. Let (α, ζ) be an f-pair. Fix an $\epsilon > 0$. If we have letters $\alpha(i)$ and $\zeta(i)$ which are not multipliable, then by the definition of the metric d on $G \cup H$ we can find an element $a \in A$ such that

$$d\big(\alpha(i), \zeta(i)\big) \ge \max\big\{ d\big(\alpha(i), a\big),\, d\big(a, \zeta(i)\big) \big\} - \epsilon.$$

Let $x = \alpha(i) \cdot a^{-1}$. We now substitute the word '$x\ a$' into α for the letter $\alpha(i)$ and the word '$e\ \zeta(i)$' into ζ for the letter $\zeta(i)$. In other words, if (α, ζ) is written as

$$\begin{array}{rccccc} & & & \overset{i-1}{} & \overset{i}{} & \overset{i+1}{} & \\ \alpha = & \cdots & \alpha(i-1) & \alpha(i) & \alpha(i+1) & \cdots \\ \zeta = & \cdots & \zeta(i-1) & \zeta(i) & \zeta(i+1) & \cdots \end{array}$$

then the pair (α_1, ζ_1) after this substitution can be written as

$$\begin{array}{rccccc} & & \overset{i-1}{} & \overset{i}{} & \overset{i+1}{} & \overset{i+2}{} & \\ \alpha_1 = & \cdots & \alpha(i-1) & \alpha(i) \cdot a^{-1} & a & \alpha(i+1) & \cdots \\ \zeta_1 = & \cdots & \zeta(i-1) & e & \zeta(i) & \zeta(i+1) & \cdots \end{array}$$

One now does this procedure for all i such that $\alpha(i)$ and $\zeta(i)$ are not multipliable. The resulting pair (β, ξ) is multipliable, and by the two-sided invariance of the metric d we have $\rho(\beta, \xi) \le \rho(\alpha, \zeta) + \epsilon$. Since the pair (α, ζ) and ϵ were arbitrary, we get

$$\mathrm{N}(f) = \inf\big\{\, \rho(\beta, \xi) \;\big|\; (\beta, \xi) \text{ is a multipliable } f\text{-pair} \,\big\}. \qquad \square$$

Lemma 2.9 (cf. Lemma 5.4 and Lemma 5.5 [Slu12])**.** *Let (α, ζ) be a multipliable f-pair, and let $\mathcal{F}$ be a maximal evaluation forest for ζ. There exists a multipliable f-pair (β, ξ) such that*

(i) $|\xi| = |\zeta|$*;*
(ii) $\mathcal{F}$ *is a maximal evaluation forest for* ξ *(with the same assignment* $t \mapsto I_t$*);*
(iii) $\rho(\alpha, \zeta) = \rho(\beta, \xi)$*;*
(iv) $\hat{\xi}(I_t) = e$ *for all* $t \in \mathcal{F}$*;*

Proof. The proof is based on the following observation. Let $a \in A$, $i < |\alpha|$, and define a pair (α_1, ζ_1) by changing $\alpha(i)$ to $\alpha(i) \cdot a^{-1}$, $\alpha(i+1)$ to $a \cdot \alpha(i+1)$ and also $\zeta(i)$ to $\zeta(i) \cdot a^{-1}$ and $\zeta(i+1)$ to $a \cdot \zeta(i+1)$:

$$\begin{array}{rccc} & & \overset{i}{} & \overset{i+1}{} & \\ \alpha = & \cdots & \alpha(i) & \alpha(i+1) & \cdots \\ \zeta = & \cdots & \zeta(i) & \zeta(i+1) & \cdots \\ & & \downarrow & & \\ \alpha_1 = & \cdots & \alpha(i) \cdot a^{-1} & a \cdot \alpha(i+1) & \cdots \\ \zeta_1 = & \cdots & \zeta(i) \cdot a^{-1} & a \cdot \zeta(i+1) & \cdots \end{array}$$

We call this operation a *transfer operation*. Observe that (α_1, ζ_1) is also a multipliable f-pair, $\rho(\alpha, \zeta) = \rho(\alpha_1, \zeta_1)$, and $\mathcal{F}$ is still a maximal evaluation forest for ζ_1, since $\zeta_1(j) \in A$ if and only if $\zeta(j) \in A$. A typical application of the transfer is for $a = \hat{\zeta}(I_t)$ and $i = M(I_t)$, which yields a pair (α_1, ζ_1) with $\hat{\zeta}_1(I_t) = e$.

To obtain the desired pair (β, ξ), we apply the transfer operation for all intervals I_t, $t \in \mathcal{F} \setminus \mathcal{F}^{\varnothing}$. But this has to be done in a consistent order. We traverse the forest $\mathcal{F}$ "from leaves to roots" and "from left to right". Somewhat more formally, we can define a function $h : \mathcal{F} \to \mathbb{N}$ by $h(t)$ being "the longest downward path to a leaf". For example, $h(t) = 0$ if and only if t is a leaf, $h(t) = 1$ if and only if all the predecessors of t are leaves, etc.

Let $t_1, \dots, t_m$ be all the leaves of $\mathcal{F}$ ordered in such a way that $M(I_{t_i}) < m(I_{t_{i+1}})$. First we apply the transfer for $a = \hat{\zeta}(I_{t_1})$ at the index $M(I_{t_1})$ and obtain a pair (α_1, ζ_1) such that $\hat{\zeta}_1(I_{t_1}) = e$; next we apply the transfer with $a = \hat{\zeta}_1(I_{t_2})$ to this new pair at the index $M(I_{t_2})$ and get (α_2, ζ_2) with $\hat{\zeta}_2(I_{t_1}) = e$ and $\hat{\zeta}_2(I_{t_2}) = e$, etc. It is important that at the second step we take $a = \hat{\zeta}_1(I_{t_2})$ as opposed to $a = \hat{\zeta}(I_{t_2})$, since these may not be equal when $M(I_{t_1}) + 1 = m(I_{t_2})$. Once we get (α_m, ζ_m), we continue with nodes t such that $h(t) = 1$, again ordering them "from left to right."

From item (v) of the definition of the evaluation forest it follows that transfers at nodes with higher values of $h(t)$ do not ruin the equalities $\hat{\zeta}_k(I_s) = e$ for nodes s with smaller $h(s)$.

Note that transfer operations within different trees commute with each other. We continue the above process for all $t \in \mathcal{F} \setminus \mathcal{F}^{\varnothing}$, and let (β_1, ξ_1) be the resulting pair. It satisfies $\rho(\beta_1, \xi_1) = \rho(\alpha, \zeta)$, $\mathcal{F}$ is a maximal

evaluation forest for ξ_1, and $\hat{\xi}_1(I_t) = e$ for all $t \notin \mathcal{F}^{\varnothing}$. To achieve the latter equality for roots, we again apply the transfer. Let $\varnothing_1, \dots, \varnothing_p$ be the list of roots of $\mathcal{F}$. As usually we assume that $M(I_{\varnothing_i}) < m(I_{\varnothing_{i+1}})$. We let (β_2, ξ_2) be the transfer of (β_1, ξ_1) with $a = \hat{\xi}_1(I_{\varnothing_1})$ at $i = M(I_{\varnothing_1})$; let (β_3, ξ_3) be the transfer of (β_2, ξ_2) with $a = \hat{\xi}_2(I_{\varnothing_2})$ at $i = M(I_{\varnothing_2})$; etc. We continue this process until the penultimate root $\varnothing_{p-1}$: the pair (β_p, ξ_p) is obtained from (β_{p-1}, ξ_{p-1}) by transfer with $a = \hat{\xi}_{p-1}(I_{\varnothing_{p-1}})$ at $i = M(I_{\varnothing_{p-1}})$.

We set (β, ξ) to be the pair (β_p, ξ_p) and claim that it satisfies the conclusion of the lemma. All the items follow immediately from the construction with one exception: we have to explain why is it the case that $\hat{\xi}(I_{\varnothing_p}) = e$. This follows from the observation that

$$\hat{\xi} = \hat{\xi}(I_{\varnothing_1}) \cdot \hat{\xi}(I_{\varnothing_2}) \cdots \hat{\xi}(I_{\varnothing_{p-1}}) \cdot \hat{\xi}(I_{\varnothing_p}) = \hat{\xi}(I_{\varnothing_p}),$$

and from $\hat{\xi} = e$, since $\hat{\xi} = \hat{\zeta} = e$. □

Remark 2.10. In the context of the above lemma it follows that $\xi(i) = e$ whenever $\xi(i) \in A$, since $\mathcal{F}$ is maximal and any i with $\xi(i) \in A$ corresponds to an interval $I_t = [i, i]$ for some $t \in \mathcal{F}$ by Remark 2.5.

An f-pair (α, ζ) with a maximal evaluation forest $\mathcal{F}$ for ζ is said to be *simple* if for all $t \in \mathcal{F}$ one has $\hat{\zeta}(I_t) = e$. Lemma 2.9 then implies that for any $f \in G *_A H$

$$\mathrm{N}(f) = \inf \big\{ \rho(\alpha, \zeta) \;\big|\; (\alpha, \zeta) \text{ is a simple } f\text{-pair} \big\}.$$

2.3. **Symmetrization.** Simple pairs are important, because they allow for the following symmetrization operation. Let (α, ζ) be a multipliable f-pair with an evaluation forest $\mathcal{F}$ and let $t \in \mathcal{F}$ be a node with the reminder R_t. Let $i_1 < i_2 < \cdots < i_m$ be some of the elements of this reminder $i_k \in R_t$ and suppose that:

- $\zeta(i_k) \notin A$ for all k;
- $\zeta(j) = e$ for all $j \in R_t \setminus \{i_k\}_{k=1}^m$;
- $\hat{\zeta}(I_s) = e$ for all immediate predecessors $s \prec t$;
- $\hat{\zeta}(I_t) = e$.

A typical example of such a situation comes from a simple pair (α, ζ) with a maximal evaluation forest $\mathcal{F}$: for some $t \in \mathcal{F}$ with $|I_t| \geq 2$ we may set $\{i_k\}_{k=1}^m = R_t$. Under these assumptions the symmetrization of (α, ζ) with respect to $\{i_k\}_{k=1}^m$ and k_0, $1 \leq k_0 \leq m$, is the pair (α, ξ), where ξ is defined by

$$\xi(i) = \begin{cases} \zeta(i) & \text{if } i \neq i_k \text{ for all } k, \\ \alpha(i) & \text{if } i = i_k \text{ for } k \neq k_0, \\ \alpha(i_{k_0-1})^{-1} \cdots \alpha(i_1)^{-1} \cdot \alpha(i_m)^{-1} \cdots \alpha(i_{k_0+1})^{-1} & \text{if } i = i_{k_0}. \end{cases}$$

Schematically symmetrization is showed on the following diagram:

$$\begin{array}{lccccccccccccc} & & & {\scriptstyle i_1} & & {\scriptstyle i_2} & & {\scriptstyle i_{k_0-1}} & & {\scriptstyle i_{k_0}} & & {\scriptstyle i_{k_0+1}} & & {\scriptstyle i_n} & \\ \alpha = & \cdots & g_1 & \cdots & g_2 & \cdots\cdots & g_{k_0-1} & \cdots & g_{k_0} & \cdots & g_{k_0+1} & \cdots\cdots & g_n & \cdots \\ \zeta = & \cdots & * & \cdots & * & \cdots\cdots & * & \cdots & * & \cdots & * & \cdots\cdots & * & \cdots \\ & & & & & & & \downarrow & & & & & & \\ \xi = & \cdots & g_1 & \cdots & g_2 & \cdots\cdots & g_{k_0-1} & \cdots & x & \cdots & g_{k_0+1} & \cdots\cdots & g_n & \cdots \end{array}$$

where x is such that

$$\hat{\xi}(I_t) = \hat{\xi}(R_t) = g_1 \cdots g_{k_0-1} x g_{k_0+1} \cdots g_m = e,$$

i.e., $x = g_{k_0-1}^{-1} \cdots g_1^{-1} \cdot g_m^{-1} \cdots g_{k_0+1}^{-1}$.

If (α, ζ) is a multipliable pair, $t \in \mathcal{F}$, and the list $i_1 < \cdots < i_m$ of elements in R_t satisfies the requirements for symmetrization, we call such a list *symmetrization admissible*.

Lemma 2.11 (cf. Lemma 5.6 [Slu12])**.** *If (α, ζ) is a multipliable f-pair with an evaluation forest $\mathcal{F}$, and (α, ξ) is obtained from (α, ζ) by symmetrization according to a symmetrization admissible list $\{i_k\}_{k=1}^m$, then (α, ξ) is also a multipliable f-pair, $\mathcal{F}$ is an evaluation forest for ξ and $\rho(\alpha, \xi) \leq \rho(\beta, \zeta)$.*

Proof. The proof follows from the following calculations:

$$\begin{aligned} d\big(\alpha(i_{k_0}), x\big) &= d\big(\alpha(i_{k_0}), \alpha(i_{k_0-1})^{-1} \cdots \alpha(i_1)^{-1} \cdot \alpha(i_m)^{-1} \cdots \alpha(i_{k_0+1})^{-1}\big) \\ &= d\big(\alpha(i_1) \cdots \alpha(i_n), e\big) \quad \text{[by the two-sided invariance of } d \text{]} \\ &= d\big(\alpha(i_1) \cdots \alpha(i_n), \zeta(i_1) \cdots \zeta(i_n)\big) \qquad \text{[since } \hat{\zeta}(R_t) = e \text{]} \\ &\le \max_{k \le n} d\big(\alpha(i_k), \zeta(i_k)\big) \qquad \text{[by Proposition 1.1]} \end{aligned}$$

□

Recall that a word $\alpha \in \mathrm{W}(G \cup H)$ is said to be a *reduced form* of $f \in G *_A H$ if $\hat{\alpha} = f$ and α is reduced in the sense that for no i letters $\alpha(i)$ and $\alpha(i+1)$ are multipliable. We say that a simple f-pair with a maximal evaluation forest (α, ζ) is *reduced* if α is a reduced form of f. Note that when $A \neq \{e\}$, the reduced form of an element is not unique, but the length of the reduced form is nevertheless well-defined (see, for instance, [Slu12, Lemma 2.4]).

Lemma 2.12 (cf. Lemma 5.8 [Slu12])**.** *For any $f \in G *_A H$*

$$\mathrm{N}(f) = \inf\big\{\, \rho(\alpha, \zeta) \;\big|\; (\alpha, \zeta) \textit{ is a reduced } f\textit{-pair} \,\big\}.$$

Proof. We start with an observation. Let (α, β) be a multipliable f-pair and suppose that there is an index i such that letters $\alpha(i)$, $\alpha(i+1)$, $\zeta(i)$ and $\zeta(i+1)$ are pairwise multipliable. We may shorten the pair (α, ζ) by considering the products $\alpha(i) \cdot \alpha(i+1)$ and $\zeta(i) \cdot \zeta(i+1)$ as single letters. More formally, we let the word β to be defined by

$$\beta(j) = \begin{cases} \alpha(j) & \text{if } j < i, \\ \alpha(i) \cdot \alpha(i+1) & \text{if } j = i, \\ \alpha(j+1) & \text{if } j > i. \end{cases}$$

The word ξ is defined similarly using ζ instead of α. The pair (β, ξ) is also a multipliable f-pair, $|\beta| < |\alpha|$, and $\rho(\beta, \xi) \le \rho(\alpha, \zeta)$, since by Proposition 1.1

$$d\big(\alpha(i) \cdot \alpha(i+1), \zeta(i) \cdot \zeta(i+1)\big) \le \max\big\{d\big(\alpha(i), \zeta(i)\big), d\big(\alpha(i+1), \zeta(i+1)\big)\big\}.$$

Note that a word $\alpha \in \mathrm{W}(G \cup H)$ with $\hat{\alpha} = f$ is a reduced form of f if and only if α is the shortest word that evaluates to f: if $\alpha_1 \in \mathrm{W}(G \cup H)$ is such that $\hat{\alpha}_1 = f$, then $|\alpha_1| \ge |\alpha|$. Based on this Lemma 2.9 implies that if (α, ζ) is a multipliable f-pair in which α is a reduced form of f, then there exists a *simple* f-pair (β, ξ) such that $\rho(\beta, \xi) \le \rho(\alpha, \zeta)$ and $|\beta| = |\alpha|$, i.e., (β, ξ) is a reduced f-pair. Hence to prove the lemma it is enough to show that for any non-reduced simple f-pair (α, β) there is a multipliable f-pair (β, ξ) such that $\rho(\beta, \xi) \le \rho(\alpha, \zeta)$ and $|\beta| < |\alpha|$.

Pick a non-reduced simple f-pair (α, ζ). If i such that $\alpha(i)$ and $\alpha(i+1)$ are multipliable and $\alpha(i), \alpha(i+1) \notin A$, then $\alpha(i), \alpha(i+1), \zeta(i)$, and $\zeta(i+1)$ are pairwise multipliable and we may shorten the pair by our observation above. We therefore need to consider the case $\alpha(i) \in A$ for some i. Let $t \in \mathcal{F}$ be such that $i \in R_t$.

In a typical situation $|R_t| \ge 2$ and we may choose $j \in R_t$ such that $j \neq i$. Let (α, ξ) be the symmetrization of (α, ζ) according to R_t at j. By Lemma 2.11 $\rho(\alpha, \xi) \le \rho(\alpha, \zeta)$ and also $\xi(i) = \alpha(i) \in A$. Since (α, ξ) is also a multipliable f-pair, all the elements $\alpha(i)$, $\alpha(i+1)$, $\xi(i)$, and $\xi(i+1)$ are pairwise multipliable, and we may finish the proof as before by shortening the pair (α, ξ).

Finally, if $|R_t| = 1$, then $R_t = I_t = [i, i]$, hence $\zeta(i) = e$, and again $\alpha(i)$, $\alpha(i+1)$, $\zeta(i)$, and $\zeta(i+1)$ must be pairwise multipliable. □

Theorem 2.13 (cf. Proposition 5.9 and Theorem 5.10 [Slu12])**.** *The function δ is a two-sided invariant ultrametric on $G *_A H$. Moreover, δ extends d on $G \cup H$.*

Proof. By Lemma 2.12 we have

$$\mathrm{N}(f) = \inf\big\{\, \rho(\alpha, \zeta) \;\big|\; (\alpha, \zeta) \text{ is a reduced } f\text{-pair} \,\big\}.$$

First we show that δ extends d. If $f \in G \cup H$, then the unique reduced f-pair is the pair (f, e), whence

$$\delta(f, e) = \mathrm{N}(f) = d(f, e).$$

If $g \in G$ and $h \in H$, then reduced gh^{-1}-pairs are of the form $(g_1\, h_1^{-1}, e\, e)$, where $g_1 = g \cdot a$ and $h_1^{-1} = a^{-1} \cdot h^{-1}$ for some $a \in A$. Therefore

$$d(g_1, e) = d\big(g, a^{-1}\big), \quad d\big(h_1^{-1}, e\big) = d\big(h^{-1}, a\big) = d\big(a^{-1}, h\big).$$

Since d by definition is the ultrametric amalgam of metrics d_G and d_H on A, it follows that $\delta(g,h) = d(g,h)$. Thus δ extends d on $G \cup H$.

We show that $\mathrm{N}(f) > 0$ for any $f \neq e$. Since we already know that δ extends d, it is enough to consider the case $f \notin A$. Pick a reduced form α_0 of f and let ϵ be such that $d\big(\alpha_0(i), A\big) \geq \epsilon > 0$ for all i (here we use that A is closed in both G and H). Note that if α is any other reduced form of f, then $|\alpha| = |\alpha_0|$ and $A\alpha(i)A = A\alpha_0(i)A$ for all i, [Slu12, Lemma 2.4].

So let (α, ζ) be any reduced f-pair, and let $\mathcal{F}$ be an evaluation forest for ζ. Pick a leaf $t \in \mathcal{F}$. The subword $\zeta(I_t)$ is multipliable. Since $\alpha(i)$ is multipliable with $\zeta(i)$ for all i, and since $\alpha(i)$ is *not* multipliable with $\alpha(i+1)$ (because α is reduced), we get that either $\zeta\big(m(I_t)\big) \in A$, or $\zeta\big(m(I_t)+1\big) \in A$. In any case, there is an index j such that $\zeta(j) \in A$. This shows that

$$\rho(\alpha,\zeta) = \max_{i \leq |\alpha|} \big\{ d\big(\alpha(i), \zeta(i)\big) \big\} \geq d\big(\alpha(j), \zeta(j)\big) \geq d\big(\alpha(j), A\big) = d\big(\alpha_0(j), A\big) \geq \epsilon > 0.$$

And therefore also $\mathrm{N}(f) \geq \epsilon$. This proves that N is a genuine ultranorm on $G *_A H$. □

Remark 2.14. The above result is valid for any number of factors: if $(G_\lambda, d_\lambda)_{\lambda \in \Lambda}$ is a family of ultrametric groups with two-sided invariant ultrametrics d_λ, A is a common closed subgroup of the groups G_λ, metrics d_λ agree on A, then one can define in a similar way a two-sided invariant Graev ultrametric δ on the free product $*_A G_\lambda$ over all $\lambda \in \Lambda$, which extends metrics d_λ.

2.4. **Graev ultrametrics on HNN extensions.** Let G be a group, $A, B < G$ be its subgroups, and $\phi : A \to B$ be an isomorphism. One way to construct the HNN extension of (G, ϕ) is as follows. We start with free products $G * \langle u \rangle$ and $G * \langle v \rangle$, where $\langle u \rangle$ and $\langle v \rangle$ are free groups on one generator. The map ϕ gives rise to an isomorphism $G * uAu^{-1} \to G * vBv^{-1}$. Let $\widetilde{H}$ be the amalgam of the groups $G * \langle u \rangle$ and $G * \langle v \rangle$ over the subgroups $G * uAu^{-1}$ and $G * vBv^{-1}$ (which are canonically isomorphic to $\langle G, uAu^{-1} \rangle$ and $\langle G, vBv^{-1} \rangle$ respectively). The HNN extension of (G, ϕ) is the subgroups of $\widetilde{H}$ generated by G and the element $v^{-1}u$, called the *stable letter* of the HNN extension.

The construction of Graev metrics given in Sections 8 and 9 of [Slu12] can be carried in the ultrametric setting, essentially by substituting the max operation for the operation of summation. In particular, one can prove the following.

Theorem 2.15 (cf. Theorem 9.1 [Slu12])**.** *Let (G, d) be an ultrametric group with a two-sided invariant metric d, A and B be closed subgroups of G and $\phi : A \to B$ be a d-isometric isomorphism. If $\operatorname{diam}(A) \leq K$, then there exists a two-sided invariant ultrametric δ on the HNN extension H of (G, ϕ) which extends d and such that $\delta(t, e) = K$, where t is the stable letter of H.*

3. Free products of Polish groups

In this section we introduce and investigate a notion of a free product of Polish groups. Our construction goes as follows. First we define unions of scaled spaces and argue that the union of scaled spaces $\mathbf{X}$ and $\mathbf{Y}$ gives rise to a natural notion of the free product of the free groups $F(\mathbf{X})$ and $F(\mathbf{Y})$. Next using the surjective universality of groups $\overline{F}(\mathbf{X})$ we define free products of Polish groups as factors of $\overline{F}(\mathbf{X} \cup \mathbf{Y})$. Our construction of a free product of Polish groups G and H is not canonical. It takes for input two scaled spaces $\mathbf{X}$, $\mathbf{Y}$, left invariant metrics d_G and d_H on G and H respectively, and surjective Lipschitz morphisms $\phi_G : X \to G$ and $\phi_H : Y \to H$. Universal properties of our construction that are reminiscent of the universal properties for free products of abstract groups are given in Proposition 3.6.

While the results of the previous section are companions of the corresponding earlier result in the metric setting, the free product construction of this section is new for both metric and ultrametric cases.

Definition 3.1. Given two scaled (ultra)metric spaces $\mathbf{X} = \big(\overline{X}, d_X, e, \Gamma_X\big)$ and $\mathbf{Y} = \big(\overline{Y}, d_Y, e, \Gamma_Y\big)$ we define their union $\mathbf{X} \cup \mathbf{Y} = (\overline{Z}, d, e, \Gamma)$ to be the (ultra)metric amalgam of $\overline{X}$ and $\overline{Y}$ over $\{e\}$ (see Figure 11). More precisely, if Z is the (ultra)metric amalgam of the (ultra)metric spaces X and Y over the subspace $\{e\}$, then as an (ultra)metric space $\mathbf{X} \cup \mathbf{Y}$ is obtained from Z by adding formal inverses. Note that $\overline{Z}$ is also the amalgam of $\overline{X}$ and $\overline{Y}$ over $\{e\}$; in other words $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$. The scale Γ on $\overline{Z}$ is the union of scales Γ_X and Γ_Y:

$$\Gamma(z,r) = \begin{cases} \Gamma_X(z,r) & \text{if } z \in \overline{X}, \\ \Gamma_Y(z,r) & \text{if } z \in \overline{Y}. \end{cases}$$

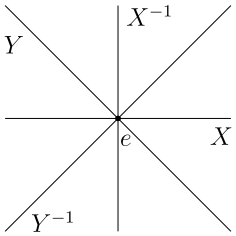

FIGURE 11. Union of scaled spaces.

Let $\mathbf{X}$ and $\mathbf{Y}$ be scaled (ultra)metric spaces and let $\pi_X : \overline{X \cup Y} \to \overline{X}$ be the retract map:

$$\pi_X(z) = \begin{cases} z & \text{if } z \in \overline{X}, \\ e & \text{if } z \in \overline{Y}. \end{cases}$$

This map is Lipschitz and it extends to a surjective group homomorphism $\pi_X : F(\mathbf{X} \cup \mathbf{Y}) \to F(\mathbf{X})$.

Proposition 3.2. *The homomorphism $\pi_X : F(\mathbf{X} \cup \mathbf{Y}) \to F(\mathbf{X})$ is Lipschitz.*

Proof. Let $f \in F(\mathbf{X} \cup \mathbf{Y})$, N_X be the Graev (ultra)norm on $F(\mathbf{X})$, and N be the Graev (ultra)norm on $F(\mathbf{X} \cup \mathbf{Y})$. Pick $w \in \mathrm{W}(\overline{X \cup Y})$, $\hat{w} = f$, and a match θ on w. We need to show that $\mathrm{N}_X\big(\pi_X(f)\big) \leq \mathrm{N}(f)$, and for this it is enough to find a word $u \in \mathrm{W}(\overline{X})$ and a match μ on u such that $\hat{u} = \pi_X(f)$ and $\mathrm{N}^\mu(u) \leq \mathrm{N}^\theta(w)$ (note that $\mathrm{N}^\mu_X(u) = \mathrm{N}^\mu(u)$ because $\mathbf{X}$ is a subspace of $\mathbf{X} \cup \mathbf{Y}$, and therefore we omit the subscript). If $w = z_1 \cdots z_n$ with $z_i \in \overline{X \cup Y}$, set $u = \tilde{z}_1 \cdots \tilde{z}_n$ with $\tilde{z}_i = \pi_X(z_i)$. We can view θ as being also a match on u. Since $\pi_X : \overline{X \cup Y} \to \overline{X}$ is Lipschitz, we have $d\big(z_i, z_j^{-1}\big) \geq d_X\big(\tilde{z}_i, \tilde{z}_j^{-1}\big)$ for all i, j. By item (i) of the definition of the scale, $\Gamma(z, r) \geq \Gamma\big(\pi_X(z), r\big)$ for all $r \in \mathbb{R}^+$ and all $z \in \overline{X \cup Y}$. It now follows from item (iii) of the scale and from the definition of the norm that $\mathrm{N}^\theta(u) \leq \mathrm{N}^\theta(w)$. □

The homomorphism π_X is therefore continuous and extends to a continuous homomorphism

$$\pi_X : \overline{F}(\mathbf{X} \cup \mathbf{Y}) \to \overline{F}(\mathbf{X}).$$

Note that $\pi_X(f) = f$ for any $f \in \overline{F}(\mathbf{X})$ and that $\pi_X(f) = e$ holds true for all $f \in \overline{F}(\mathbf{Y})$.

Corollary 3.3. *Let $\mathbf{X}$ and $\mathbf{Y}$ be scaled (ultra)metric spaces. The inclusion $F(\mathbf{X}) \hookrightarrow F(\mathbf{X} \cup \mathbf{Y})$ is isometric.*

Proof. Let N denote the Graev (ultra)norm on $F(\mathbf{X} \cup \mathbf{Y})$, and N_X be the Graev (ultra)norm on $F(\mathbf{X})$. We need to show that for all $f \in F(\mathbf{X})$ one has $\mathrm{N}(f) = \mathrm{N}_X(f)$. By definition

$$\mathrm{N}_X(f) = \inf\Big\{\mathrm{N}^\theta(w) : w \in \mathrm{W}(\overline{X}),\ \hat{w} = f \text{ and } \theta \text{ is a match on } w\Big\},$$
$$\mathrm{N}(f) = \inf\Big\{\mathrm{N}^\theta(w) : w \in \mathrm{W}(\overline{X \cup Y}),\ \hat{w} = f \text{ and } \theta \text{ is a match on } w\Big\},$$

and therefore $\mathrm{N}(f) \leq \mathrm{N}_X(f)$. The reverse inequality follows immediately from $\pi_X(f) = f$ for $f \in F(\mathbf{X})$ and Proposition 3.2. □

Corollary 3.4. *Inclusion $F(\mathbf{X}) \hookrightarrow F(\mathbf{X} \cup \mathbf{Y})$ extends to $\overline{F}(\mathbf{X}) \hookrightarrow \overline{F}(\mathbf{X} \cup \mathbf{Y})$.*

It is natural to regard $\overline{F}(\mathbf{X} \cup \mathbf{Y})$ as being the free product of groups $\overline{F}(\mathbf{X})$ and $\overline{F}(\mathbf{Y})$.

Let (G, d_G) and (H, d_H) be Polish groups with compatible left invariant (ultra)metrics, let $\mathbf{X}$ and $\mathbf{Y}$ be separable scaled (ultra)metric spaces, and let $\phi_G : \mathbf{X} \to G$ and $\phi_H : \mathbf{Y} \to H$ be *surjective* Lipschitz morphisms. By Proposition 1.6 they extend to surjective homomorphisms

$$\phi_G : \overline{F}(\mathbf{X}) \to G, \quad \phi_H : \overline{F}(\mathbf{Y}) \to H$$

with kernels K_G and K_H respectively. We note that as proved in [DG07b, Theorem 3.10], for any Polish group G there are plenty of surjective Lipschitz morphisms $\phi : \overline{X} \to G$, and moreover, one may always take $X = \mathbb{N}^\mathbb{N}$. Note also that G is isomorphic to $\overline{F}(\mathbf{X})/\mathrm{K}_G$ with the quotient topology (see, for instance, [BK96, Theorem

1.2.6]). We shall identify G with $\overline{F}(\mathbf{X})/\mathrm{K}_G$ and H with $\overline{F}(\mathbf{Y})/\mathrm{K}_H$. By Corollary 3.4 we may view K_G and K_H as subgroups of $\overline{F}(\mathbf{X}\cup\mathbf{Y})$. Let K°_{G*H} be the normal subgroup of $\overline{F}(\mathbf{X}\cup\mathbf{Y})$ generated by K_G and K_H:

$$\mathrm{K}^\circ_{G*H}=\Big\{f_1h_1f_1^{-1}\cdots f_nh_nf_n^{-1}\;\Big|\;n\in\mathbb{N},\ f_i\in\overline{F}(\mathbf{X}\cup\mathbf{Y}),\ h_i\in\big\langle\mathrm{K}_G,\mathrm{K}_H\big\rangle\Big\},$$

and let K_{G*H} be the closure of K°_{G*H}, i.e., K_{G*H} is the *closed* normal subgroup in $\overline{F}(\mathbf{X}\cup\mathbf{Y})$ generated by K_G and K_H.

Lemma 3.5. *In the setting above $\pi_X(\mathrm{K}_{G*H})=\mathrm{K}_G$.*

Proof. If $g\in\mathrm{K}^\circ_{G*H}$ is of the form

$$g=f_1h_1f_1^{-1}\cdots f_nh_nf_n^{-1},\quad f_i\in\overline{F}(\mathbf{X}\cup\mathbf{Y}),\ h_i\in\big\langle\mathrm{K}_G,\mathrm{K}_H\big\rangle,$$

then

$$\pi_X(g)=\pi_X(f_1)\pi_X(h_1)\pi_X(f_1)^{-1}\cdots\pi_X(f_n)\pi_X(h_n)\pi_X(f_n)^{-1}\in\mathrm{K}_G,$$

because $\pi_X(h_i)\in\mathrm{K}_G$, $\pi_X(f_i)\in\overline{F}(\mathbf{X})$ and K_G is a normal subgroup in $\overline{F}(\mathbf{X})$. Thus $\pi_X(\mathrm{K}^\circ_{G*H})=\mathrm{K}_G$ and therefore also $\pi_X(\mathrm{K}_{G*H})=\mathrm{K}_G$, since K_G is closed in $\overline{F}(\mathbf{X})$, and hence $\pi_X^{-1}(\mathrm{K}_G)$ is also closed in $\overline{F}(\mathbf{X}\cup\mathbf{Y})$. □

A *Polish free product of G and H over ϕ_G and ϕ_H* is the group $\overline{F}(\mathbf{X}\cup\mathbf{Y})/\mathrm{K}_{G*H}$; we denote it by $G*_{\phi_G\phi_H}H$. The free product comes with homomorphisms $\iota_G:G\to G*_{\phi_G\phi_H}H$, $\iota_H:H\to G*_{\phi_G\phi_H}H$ and $\pi_G:G*_{\phi_G\phi_H}H\to G$, $\pi_H:G*_{\phi_G\phi_H}H\to H$ given by

$$\iota_G(f\mathrm{K}_G)=f\mathrm{K}_{G*H},\quad \iota_H(f\mathrm{K}_H)=f\mathrm{K}_{G*H},$$
$$\pi_G(f\mathrm{K}_{G*H})=\pi_X(f)\mathrm{K}_G,\quad \pi_H(f\mathrm{K}_{G*H})=\pi_Y(f)\mathrm{K}_H.$$

Note that π_G and π_H are well-defined by Lemma 3.5. Note also that $\pi_G\big(\iota_G(g)\big)=g$ and $\pi_H\big(\iota_H(h)\big)=h$ for all $g\in G$ and $h\in H$.

Proposition 3.6. *Let $\mathbf{X}$, $\mathbf{Y}$, ϕ_G, ϕ_H and $G*_{\phi_G\phi_H}H$ be as above. Let d_G and d_H be compatible left invariant (ultra)metrics on G and H with respect to which ϕ_G and ϕ_H are Lipschitz morphisms.*

- *(i) ι_G and ι_H are injective;*
- *(ii) ι_G and ι_H are continuous;*
- *(iii) π_G and π_H are continuous;*
- *(iv) $\iota_G(G)\cap\iota_H(H)=\{e\}$;*
- *(v) $\langle\iota_G(G),\iota_H(H)\rangle$ is a dense subgroup of $G*_{\phi_G\phi_H}H$.*

Recall that $\mathcal{S}_G$ denotes the canonical scale on G.

- *(vi) If (T,d_T) is a Polish (ultra)metric group, $\psi_G:G\to T$, $\psi_H:H\to T$ are Lipschitz homomorphisms and $\mathcal{S}_T(\psi_G(g),r)\le\mathcal{S}_G(g,r)$, $\mathcal{S}_T(\psi_H(h),r)\le\mathcal{S}_H(h,r)$ for all $g\in G$, $h\in H$ and $r\in\mathbb{R}^+$, then there exists a unique continuous homomorphism $\psi:G*_{\phi_G\phi_H}H\to T$ such that $\psi\circ\iota_G=\psi_G$ and $\psi\circ\iota_H=\psi_H$.*

Proof. (i) To show that ι_G is injective it is enough to check that $\mathrm{K}_{G*H}\cap\overline{F}(\mathbf{X})=\mathrm{K}_G$. If $f\in\mathrm{K}_{G*H}\cap\overline{F}(\mathbf{X})$, then $f=\pi_X(f)\in\pi_X(\mathrm{K}_{G*H})=\mathrm{K}_G$ by Lemma 3.5; hence $f\in\mathrm{K}_G$.

(ii) Let d be a compatible right-invariant (ultra)metric on $\overline{F}(\mathbf{X}\cup\mathbf{Y})$. It induces compatible right-invariant (ultra)metrics on the factor groups $\overline{F}(\mathbf{X}\cup\mathbf{Y})/\mathrm{K}_{G*H}$ and $\overline{F}(\mathbf{X})/\mathrm{K}_G$ (see [Gao09, Lemma 2.2.8])

$$d_1(f_1\mathrm{K}_{G*H},f_2\mathrm{K}_{G*H})=\inf\big\{d(f_1k_1,f_2k_2):k_1,k_2\in\mathrm{K}_{G*H}\big\},$$
$$d_2(f_1\mathrm{K}_G,f_2\mathrm{K}_G)=\inf\big\{d(f_1k_1,f_2k_2):k_1,k_2\in\mathrm{K}_G\big\}.$$

With respect to the (ultra)metrics d_1 and d_2 the homomorphism ι_G is Lipschitz, hence continuous.

(iii) It is enough to prove that π_G is continuous at the identity, i.e., that $f_n\mathrm{K}_{G*H}\to\mathrm{K}_{G*H}$ implies $\pi_X(f_n)\mathrm{K}_G\to\mathrm{K}_G$. The sequence $f_n\mathrm{K}_{G*H}$ converges to K_{G*H} if and only if there is a sequence $h_n\in\mathrm{K}_{G*H}$ such that $f_nh_n\to e$. This implies $\pi_X(f_n)\pi_X(h_n)\to e$ with $\pi_X(h_n)\in\mathrm{K}_G$ by Lemma 3.5, and therefore $\pi_G(f_n\mathrm{K}_{G*H})=\pi_X(f_n)\mathrm{K}_G\to\mathrm{K}_G$.

(iv) If $f\mathrm{K}_{G*H}\in\iota_G(G)\cap\iota_H(H)$, then $f\mathrm{K}_{G*H}=f_1\mathrm{K}_{G*H}=f_2\mathrm{K}_{G*H}$ for some $f_1\in\overline{F}(\mathbf{X})$ and $f_2\in\overline{F}(\mathbf{Y})$. Therefore $\pi_X(f\mathrm{K}_{G*H})=\pi_X(f_1\mathrm{K}_{G*H})=\pi_X(f_2\mathrm{K}_{G*H})$, but $\pi_X(f_1\mathrm{K}_{G*H})=f_1\mathrm{K}_G$ and $\pi_X(f_2\mathrm{K}_{G*H})=\mathrm{K}_G$, whereby $f_1\in\mathrm{K}_G$ and thus $f\mathrm{K}_{G*H}=\mathrm{K}_{G*H}$.

(v) This item is obvious, since the group generated by the images of ι_G and ι_H is nothing else but $\langle \overline{F}(\mathbf{X}), \overline{F}(\mathbf{Y})\rangle \mathrm{K}_{G*H}$.

(vi) Maps $\psi_G \circ \phi_G : \overline{X} \to T$ and $\psi_H \circ \phi_H : \overline{Y} \to T$ are Lipschitz morphisms and so is the map $\zeta : \overline{X \cup Y} \to T$ given by

$$\zeta(z) = \begin{cases} \psi_G \circ \phi_G(z) & \text{if } z \in \overline{X}, \\ \psi_H \circ \phi_H(z) & \text{if } z \in \overline{Y}. \end{cases}$$

By Proposition 1.6 the map ζ extends to a continuous homomorphism $\zeta : \overline{F}(\mathbf{X} \cup \mathbf{Y}) \to T$. Since ζ extends both $\psi_G \circ \phi_G$ and $\psi_H \circ \phi_H$, the kernel of ζ contains K_G and K_H, and therefore also K_{G*H}. Thus ζ factors to a continuous homomorphism $\psi : G {}_{\phi_G}*_{\phi_H} H \to T$. Uniqueness follows from item (v). □

Note that items (i), (ii) and (iii) imply that ι_G and ι_H are embeddings. The homomorphisms ι_G, ι_H can be extended to a homomorphism from the free product of abstract groups $\iota : G * H \to G {}_{\phi_G}*_{\phi_H} H$. Is the homomorphism ι injective? We shall show in Corollary 3.9 that the answer is yes when ϕ_G and ϕ_H are "large enough".

Lemma 3.7. *Let $f \in G * H$ be a non-trivial element in the abstract free product. There are a Polish group T and two embeddings $\psi_G : G \to T$ and $\psi_H : H \to T$ such that for the common extension of these homomorphisms $\psi : G * H \to T$ one has $\psi(f) \neq e$. Moreover, if G and H admit compatible left invariant ultrametrics, then we may find T that also admits a compatible left invariant ultrametric.*

Proof. Here we deal with metric and ultrametric cases separately. First assume that G and H are general Polish groups.

Let $f \in G * H$ be given. By conjugating f if necessary we may assume without loss of generality that f "starts with g", i.e, it is of the form $f = g_{n-1}h_{n-1}\cdots g_0 h_0$ for some $n \geq 1$ with non-trivial $g_i \in G$ and $h_j \in H$. By a theorem of Uspenskij [Usp86] (see also [Kec95, Theorem 9.18]) the group $\mathrm{Homeo}\big([0,1]^\omega\big)$ of homeomorphisms of the Hilbert Cube with the compact-open topology is a universal Polish group in the following sense: any Polish group can be embedded into $\mathrm{Homeo}\big([0,1]^\omega\big)$. In particular, G and H can be embedded into $\mathrm{Homeo}\big([0,1]^\omega\big)$; to simplify notations we assume that G and H are actual subgroups of $\mathrm{Homeo}\big([0,1]^\omega\big)$. Note that $\alpha H \alpha^{-1}$ is a copy of H inside $\mathrm{Homeo}\big([0,1]^\omega\big)$ for any $\alpha \in \mathrm{Homeo}\big([0,1]^\omega\big)$. To prove the lemma it is therefore sufficient to construct a homeomorphism $\alpha \in \mathrm{Homeo}\big([0,1]^\omega\big)$ such that for some $x_0 \in [0,1]^\omega$

$$g_{n-1}\alpha h_{n-1}\alpha^{-1}\cdots g_0 \alpha h_0 \alpha^{-1}(x_0) \neq x_0.$$

Pick any $x_0 \in [0,1]^\omega$ and any $x_1 \in [0,1]^\omega$ such that $x_1 \neq x_0$ and $h_0(x_1) \notin \{x_1, x_0\}$; set $x_2 = h_0(x_1)$. Pick any $x_3 \in [0,1]^\omega$ such that $x_3 \notin \{x_0, x_1, x_2\}$ and $g_0(x_3) \notin \{x_0, x_1, x_2, x_3\}$; set $x_4 = g_0(x_3)$. We continue in this fashion and construct a sequence $(x_k)_{k=1}^{4n}$ such that

(i) $x_i \neq x_j$ for $i \neq j$;
(ii) $h_k(x_{4k+1}) = x_{4k+2}$ for $k = 0, \ldots, n-1$;
(iii) $g_k(x_{4k+3}) = x_{4k+4}$ for $k = 0, \ldots, n-1$.

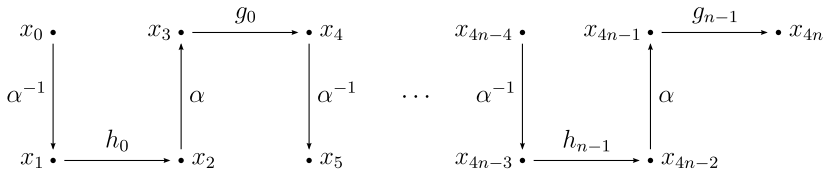

FIGURE 12. Construction of the homeomorphism α.

For all $m \in \mathbb{N}$ the space $[0,1]^\omega$ is m-homogeneous: for tuples $(y_1, \ldots, y_m)$ and $(z_1, \ldots, z_m)$ of distinct elements there is a homeomorphism $\alpha \in \mathrm{Homeo}\big([0,1]^\omega\big)$ such that $\alpha(y_i) = z_i$ (see, for example, [vM89, Exercise 2, p. 261]). Whence there is some $\alpha \in \mathrm{Homeo}\big([0,1]^\omega\big)$ such that $\alpha(x_{4k+1}) = x_{4k}$ and $\alpha(x_{4k+2}) = x_{4k+3}$ for all $k = 0, \ldots, n-1$. For such an α we have

$$g_{n-1}\alpha h_{n-1}\alpha^{-1}\cdots g_0 \alpha h_0 \alpha^{-1}(x_0) = x_{4n},$$

and $x_{4n} \neq x_0$ by construction.

In the ultrametric setting a similar argument works with the following modifications. For the injectively universal group we take S_∞ (see [BK96, Theorem 1.5.1]) and note that without loss of generality we may assume that $G, H < S_\infty$, and every non-trivial element in G and H has infinite support. This is so, because we can embed diagonally S_∞ into $\prod_{n\in\mathbb{N}} S_\infty$ and view the latter again as a subgroup of S_∞ by partitioning the natural numbers into infinitely many infinite pieces.

Now for $f = g_{n-1}h_{n-1}\cdots g_0h_0$ in $G * H$ using that supports of g_i and h_j are all infinite, we can easily find $\alpha \in S_\infty$ such that

$$g_{n-1}\alpha h_{n-1}\alpha^{-1}\cdots g_0\alpha h_0\alpha^{-1} \neq e.$$

Such an element α is again constructed as shown in Figure 12. □

Theorem 3.8. *There are a Polish group T and embeddings $\psi_G : G \hookrightarrow T$, $\psi_H : H \hookrightarrow T$ such that the group $\langle \psi_G(G), \psi_H(H)\rangle$ is naturally isomorphic to the group $G * H$. Moreover, if G and H admit compatible left invariant ultrametrics, then T can be chosen to also admit a compatible left invariant ultrametric.*

Proof. Lemma 3.7 implies that for any non-trivial $f \in G*H$ we may fix a Polish group T_f and a homomorphism $\psi_f : G * H \to T_f$ such that $\psi_f|_G : G \to T_f$ and $\psi_f|_H : H \to T_f$ are embeddings and $\psi_f(f) \neq e$.

Let $f \in G * H$ be given and assume that f has form $g_{n-1}h_{n-1}\cdots g_0h_0$ for some non-trivial $g_i \in G$, $h_j \in H$ and $n \geq 1$. By continuity of $\psi_f|_G$ and $\psi_f|_H$, and since $\psi_f(f) \neq e$, there are neighborhoods $U_i^{(f)} \subseteq G$ of g_i and $V_j^{(f)} \subseteq H$ of h_j such that $e \notin \psi_G(U_{n-1}^{(f)})\psi_H(V_{n-1}^{(f)})\cdots\psi_G(U_0^{(f)})\psi_H(V_0^{(f)})$. Therefore we can select a countable family $(f_m)_{m=1}^\infty$ of elements $f_m \in G * H$ such that for any $f \in G * H$ there is some m with $\psi_{f_m}(f) \neq e$.

Let $T = \prod_m T_{f_m}$ be the direct product of the groups T_{f_m} and let $\psi : G * H$ be given by $\psi(f)(m) = \psi_{f_m}(f)$. The homomorphisms $\psi|_G : G \to T$ and $\psi|_H : H \to T$ are embeddings. By the choice of the family (f_m) we also have $\psi(f) \neq e$ for any non-trivial $f \in G * H$ and therefore ψ is injective. □

Corollary 3.9. *There are left invariant compatible (ultra)metrics d_G and d_H on G and H respectively and scales Γ_G and Γ_H on (G, d_G) and (H, d_H) with the following property: if $\mathbf{X}$ and $\mathbf{Y}$ are (ultra)metric scaled spaces and $\phi_G : \mathbf{X} \to G$, $\phi_H : \mathbf{Y} \to H$ are surjective Lipschitz morphisms with respect to the scales Γ_G and Γ_H, then the canonical homomorphism $\iota : G * H \to G *_{\phi_G,\phi_H} H$ is injective.*

Proof. By Theorem 3.8 we may assume that G and H are closed subgroups of a Polish group T and that $\langle G, H\rangle$ is isomorphic to $G * H$. Let d be a compatible left invariant (ultra)metric on T and let d_G and d_H be the restrictions of d onto G and H respectively. Finally, let Γ_G and Γ_H be the restrictions of $\mathcal{S}_T$ onto G and H. By item (vi) of Proposition 3.6 the maps ϕ_G and ϕ_H extend to a homomorphism $\phi : G *_{\phi_G,\phi_H} H \to T$. Since $\langle G, H\rangle \cong G * H$, the homomorphism $\iota : G * H \to G *_{\phi_G,\phi_H} H$ must be injective. □

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