

AUTOMATIC CONTINUITY FOR HOMOMORPHISMS INTO FREE PRODUCTS

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ABSTRACT. A homomorphism from a completely metrizable topological group into a free product of groups whose image is not contained in a factor of the free product is shown to be continuous with respect to the discrete topology on the range. In particular, any completely metrizable group topology on a free product is discrete.

1. INTRODUCTION AND BACKGROUND

The main concept of automatic continuity is to establish conditions on topological groups G and H under which a homomorphism $\phi : G \rightarrow H$ is necessarily continuous. In many cases one of the groups is assumed to be very special, while the conditions on the other group are relatively mild. An important example of this phenomenon is a theorem of R. M. Dudley [Dud61] from 1961:

Theorem 1.1 (Dudley). *Any homomorphism from a completely metrizable group or a Hausdorff locally compact group into a free group with the discrete topology is continuous.*

In fact Dudley's result is more general: the target group does not necessarily have to be a free group, it is enough to assume that it admits a special length function (which is called a norm in [Dud61]). In the case of a free group this length function is given by the length of the reduced form of an element. An important corollary of this theorem is that any Hausdorff locally compact or completely metrizable group topology on a free group is necessarily discrete. In 1976 S. A. Morris and P. Nickolas [MN76] investigated continuity of homomorphisms $\phi : G \rightarrow \Gamma$ between a Hausdorff locally compact G and free products $\Gamma = A * B$.

Theorem 1.2 (Morris–Nickolas). *Given a homomorphism $\phi : G \rightarrow A * B$ from a Hausdorff locally compact group into a free product with the discrete topology either ϕ is continuous or its image $\phi(G)$ is contained (up to a conjugation) in one of the factors. In particular, any Hausdorff locally compact group topology on a non-trivial free product is discrete.*

The next step was made six years later by R. Alperin in [Alp82], where he studied actions of Hausdorff locally compact groups on trees in the sense of Bass–Serre theory. Automatic continuity implications of Alperin's work are summarized in the following theorem.

Theorem 1.3 (Alperin). *Let G be a locally compact group and G_0 be the connected component of the identity.*

- (i) *If G can be written in a non-trivial way as an amalgam $A *_C B$, then A , B and C are necessarily open.*
- (ii) *If G/G_0 is compact and $\phi : G \rightarrow A * B$ is a homomorphism into a free product, then up to a conjugation $\phi(G)$ is contained in one of the factors.*

Methods used by Morris–Nickolas and Alperin rely on the structure theory for locally compact groups, and therefore are not applicable for general completely metrizable groups. C. Rosendal [Ros11] developed further the Bass–Serre theory approach and proved the following.

Theorem 1.4 (Rosendal). *Let G be a completely metrizable group, and let \mathbb{D} be the set of element that generate a finite or non-discrete subgroup:*

$$\mathbb{D} = \{g \in G : \langle g \rangle \text{ is finite or non-discrete}\}.$$

- (i) *If \mathbb{D} is somewhere dense and if G is written in a non-trivial way as an amalgam $A *_C B$, then the three subgroups A , B and C are open in G .*
- (ii) *If \mathbb{D} is dense in G and if $\phi : G \rightarrow A * B$ is a homomorphism into a free product, then up to a conjugation $\phi(G)$ is contained in one of the factors.*

The group Γ is	The group G is locally compact	The group G is completely metrizable
a free group	Dudley 1961	Dudley 1961
a free product $A * B$	Morris–Nickolas 1976	—
an amalgam $A *_C B$	Alperin 1982	Rosendal 2011

TABLE 1. Automatic continuity results for homomorphisms $\phi : G \rightarrow \Gamma$.

The aforementioned articles can be put into a table:

Our goal in this paper is to fill the gap in the table and to prove the main result of Morris–Nickolas [MN76] for completely metrizable groups.

We would like to mention that none of the results in the table fully imply any other result in it. For example, let us take a locally compact group G and a homomorphism $\phi : G \rightarrow F$ into a free group F . If we view the free group F as a free product of copies of \mathbb{Z} and apply the Morris–Nickolas Theorem (which is, in fact, true for any number of factors in the free product), then we may conclude that either ϕ is continuous or $\phi(G)$ is contained in a cyclic subgroup of F . But to get Dudley’s Theorem we still need to know that all homomorphisms $\phi : G \rightarrow \mathbb{Z}$ are continuous.

We now state our main results. Let $\Gamma = *_{a \in A} \Gamma_a$ and let $\phi : G \rightarrow \Gamma$ be a homomorphism from a completely metrizable group into a free product of non-trivial groups. We say that *the image of ϕ is contained in a factor of Γ* if there are $a \in A$ and $\gamma \in \Gamma$ such that $\phi(G) \subseteq \gamma \Gamma_a \gamma^{-1}$. We shall prove (see Theorem 5.6 below)

Theorem 1.5. *Any homomorphism $\phi : G \rightarrow *_{a \in A} \Gamma_a$ from a completely metrizable group into a free product with the discrete topology is continuous unless its image is contained in one of the factors.*

By taking ϕ to be the identity map we obtain the following corollary (which, in fact, will be proved before Theorem 1.5 and will serve as a step in its proof).

Theorem 1.6. *Any completely metrizable group topology on a non-trivial free product $*_{a \in A} \Gamma_a$ is discrete.*

Our proof of Theorems 1.5 and 1.6 will take the original construction of Dudley as an important ingredient, so we find it useful to recall briefly the argument from [Dud61] for completely metrizable groups.

Sketch of proof of Dudley’s theorem. Let $\phi : G \rightarrow F$ be a homomorphism from a completely metrizable group G into a free group F and let $\|\cdot\| : F \rightarrow \mathbb{N}$ be the length function given by the length of the reduced form of an element. The crucial property of this length function is the inequality $\|\gamma^n\| \geq \max\{n, \|\gamma\|\}$ for all non-trivial $\gamma \in F$ all $n \geq 1$.

If the homomorphism ϕ is discontinuous, then its kernel cannot be open, and we can find a sequence $(g_m)_{m=1}^\infty$ of elements in G such that $\phi(g_m) \neq e$ and for

$$\tilde{h}_{n,m} = g_m(g_{m+1}(\cdots(g_{n-1}(g_n)^{r_{n-1}})^{r_{n-2}} \cdots)^{r_{m+1}})^{r_m},$$

where $r_n = n + \sum_{l=1}^n \|\phi(g_l)\|$, the sequence $\tilde{h}_{n,m}$ converges for all m . For the limit $h_m = \lim_{n \rightarrow \infty} \tilde{h}_{n,m}$ we have $h_m = g_m h_{m+1}^{r_m}$. We claim that $\|\phi(h_m)^{r_{m-1}}\| \geq r_{m-1}$. Indeed, if $\phi(h_{m+1}) \neq e$, then

$$\|\phi(h_m)^{r_{m-1}}\| \geq \|\phi(h_m)\| \geq \|\phi(h_{m+1})^{r_m}\| - \|\phi(g_m)\| \geq r_{m-1} + 1.$$

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If on the other hand $\phi(h_{m+1}) = e$, then $\phi(h_m)^{r_{m-1}} = \phi(g_m)^{r_{m-1}}$ and $\|\phi(g_m)^{r_{m-1}}\| \geq r_{m-1}$, since $\phi(g_m) \neq e$ by the choice of g_m . And so

$$\begin{aligned} \|\phi(h_1)\| &= \|\phi(g_1)\phi(h_2)^{r_1}\| \geq \|\phi(h_2)^{r_1}\| - \|\phi(g_1)\| \geq \|\phi(h_2)\| - \|\phi(g_1)\| \\ &= \|\phi(g_2)\phi(h_3)^{r_2}\| - \|\phi(g_1)\| \geq \|\phi(h_3)^{r_2}\| - \|\phi(g_1)\| - \|\phi(g_2)\| \geq \|\phi(h_3)\| - \|\phi(g_1)\| - \|\phi(g_2)\| \\ &= \dots \\ &= \|\phi(g_m)\phi(h_{m+1})^{r_m}\| - \sum_{l=1}^{m-1} \|\phi(g_l)\| \geq \|\phi(h_{m+1})^{r_m}\| - \sum_{l=1}^m \|\phi(g_l)\| \geq r_m - \sum_{l=1}^m \|\phi(g_l)\| = m \end{aligned}$$

is true for all m , which is impossible. \square

In the above argument we used the freeness of the target group only in the amount that it implies the existence of a length function with the special property. In general, a *length function* on a group Γ is a function $\|\cdot\| : \Gamma \rightarrow \mathbb{N}$ that for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$ satisfies:

- (i) $\|e\| = 0$ (where e is the identity element);
- (ii) $\|\gamma^{-1}\| = \|\gamma\|$;
- (iii) $\|\gamma_1\gamma_2\| \leq \|\gamma_1\| + \|\gamma_2\|$.

Suppose now that we have a homomorphism $\phi : G \rightarrow \Gamma$ and a sequence $(g_m)_{m=1}^\infty$ of elements in G such that the limit

$$h_m = \lim_{n \rightarrow \infty} g_m(g_{m+1}(\dots(g_{n-1}(g_n)^{r_{n-1}})^{r_{n-2}} \dots)^{r_{m+1}})^{r_m}$$

exists for every m , where r_n is an increasing sequence of natural numbers (possibly different from the sequence used in the argument above). To obtain the inequalities $\|h_1\| \geq M$ for all M with respect to a given length function $\|\cdot\|$ on Γ we need the following:

- (i) $\|\phi(h_m)^{r_{m-1}}\| \geq r_{m-1}$ for all $m \geq 2$;
- (ii) $\|\phi(h_m)^{r_{m-1}}\| \geq \|\phi(h_m)\|$ for all $m \geq 2$;
- (iii) numbers $r_m - \sum_{l=1}^m \|\phi(g_l)\|$ are unbounded from above.

In our setting the group $\Gamma = \ast_{a \in A} \Gamma_a$ will be a free product of non-trivial groups and the length function $\|\cdot\| : \ast_{a \in A} \Gamma_a \rightarrow \mathbb{N}$ will be given by the length of the reduced form of an element. For such a length function the inequality $\|\gamma^n\| \geq \max\{n, \|\gamma\|\}$ is false for some non-trivial $\gamma \in \Gamma$, so we shall need to take special care to fulfil the above list of conditions.

In Section 2 we study the length function on $\ast_{a \in A} \Gamma_a$ and the set of elements on which the inequality $\|\gamma^n\| \geq \max\{n, \|\gamma\|\}$ fails. In Section 3 we show that any completely metrizable group topology on a free product is discrete. Section 4 does some further analysis of the behaviour of the length function and Section 5 contains the proof of the main result.

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2. WORD LENGTH FUNCTIONS ON FREE PRODUCTS

Throughout this section let $\Gamma = \ast_{a \in A} \Gamma_a$ be a free product of non-trivial groups Γ_a with A being an index set of size $|A| \geq 2$. The identity element of any group is denoted by e ; we hope that there will be no confusion to which of the groups we refer. Let $X_\Gamma = \bigcup_{a \in A} \Gamma_a$ viewed as a subset of Γ . The set X_Γ depends not only on the group Γ , but also on the representation of Γ as a free product $\ast_{a \in A} \Gamma_a$, but we abuse notations and write just X_Γ ; the choice of Γ_a will be clear from the context. We say that two element $x, y \in X_\Gamma$ are *multipliable* if there is an $a \in A$ such that $x, y \in \Gamma_a$; in this case there is a unique $z \in X_\Gamma$ such that $xy = z$. Any element $\gamma \in \Gamma$ can be written uniquely as a reduced product $\gamma = x_1 x_2 \dots x_n$, $x_i \in X_\Gamma$, where reduced means x_i and x_{i+1} are not multipliable. The word length function on Γ is then defined by $\|\gamma\| = n$. We also make an agreement that $e \in \Gamma$ is represented by the empty word and $\|e\| = 0$. The product $\gamma = \gamma_1 \gamma_2 \dots \gamma_n$ for $\gamma_i \in \Gamma$ is said to be reduced if $\|\gamma\| = \sum_{i=1}^n \|\gamma_i\|$. An element $\gamma \in \Gamma$ is called *cyclically reduced* if in its reduced form $\gamma = x_1 \dots x_n$ either $n \leq 1$ or $x_1 \neq x_n^{-1}$.

Lemma 2.1. *Any $\gamma \in \Gamma$ can be uniquely written as a reduced product of the form $\gamma = \alpha \gamma_c \alpha^{-1}$, $\alpha, \gamma_c \in \Gamma$, where γ_c is cyclically reduced.*

Proof. Let $\gamma \in \Gamma$ be a non-trivial element with the reduced form $y_1 \dots y_N$, $y_i \in X_\Gamma$. Let $k \leq N/2$ be maximal such that $y_k = y_{N-k+1}^{-1}$ and set $\alpha = y_1 \dots y_k$, $\gamma_c = y_{k+1} \dots y_{N-k}$. Then $\gamma = \alpha \gamma_c \alpha^{-1}$ is reduced and $y_{k+1} y_{N-k} \neq e$ by the choice of k unless $N - k = k + 1$, in which case $k = (N - 1)/2$ and $\|\gamma_c\| = 1$. \square

We introduce the following sets to control the behaviour of the length function on Γ :

$$\begin{aligned} S_\Gamma &= \{\gamma \in \Gamma : \|\gamma^n\| \geq n \text{ for all } n \geq 1\}, \\ \tilde{F}_\Gamma &= \{\gamma \in \Gamma : \|\gamma^n\| \geq \|\gamma\| \text{ for all } n \geq 1\}, \\ F_\Gamma &= \{\gamma \in \Gamma : \gamma^n = e \text{ for some } n \geq 1\}. \end{aligned}$$

Complements of F_Γ and S_Γ in Γ will be denoted by F_Γ^c and S_Γ^c respectively.

Lemma 2.2. *For any group $\Gamma = \ast_{a \in A} \Gamma_a$*

- (i) $S_\Gamma^c = \{\alpha x \alpha^{-1} : \alpha \in \Gamma, x \in X_\Gamma\}$, and in particular $\gamma \in S_\Gamma^c$ if and only if $\gamma^n \in S_\Gamma^c$ for all n .
- (ii) For any $n \geq 2$ and any $\gamma \in \Gamma$ the inequality $\|\gamma^n\| < \|\gamma\|$ holds if and only if $\gamma = \alpha x \alpha^{-1}$, $x \in X_\Gamma$, with $x^n = e$ and $x \neq e$. In other words $\|\gamma^n\| \geq \|\gamma\|$ unless $\gamma^n = e$.
- (iii) $\tilde{F}_\Gamma^c = \{\alpha x \alpha^{-1} : \alpha \in \Gamma, x \in X_\Gamma, x^n = e \text{ for some } n \geq 1, x \neq e\} = F_\Gamma \setminus \{e\}$.

Proof. (i) If $\gamma = \alpha x \alpha^{-1}$ for some $x \in X_\Gamma$, then $\|\gamma^n\| = \|\alpha x^n \alpha^{-1}\| \leq \|\alpha x \alpha^{-1}\| = \|\gamma\|$ and therefore $\gamma \notin S_\Gamma$. If $\gamma = \alpha \gamma_c \alpha^{-1}$ is reduced and $\|\gamma_c\| \geq 2$, then

$$\|\gamma^n\| = \|\alpha \gamma_c^n \alpha^{-1}\| \geq 2\|\alpha\| + n\|\gamma_c\| - (n-1) > n,$$

and $\gamma \in S_\Gamma$.

(ii) If $\gamma = \alpha x \alpha^{-1}$ with $x \in X_\Gamma$, $x \neq e$ and $x^n = e$, then $\gamma^n = e$ and in particular $\|\gamma^n\| < \|\gamma\|$. If $x^n \neq e$, then $\|\gamma^n\| = \|\gamma\|$. Finally, if $\gamma = \alpha \gamma_c \alpha^{-1}$ and $\|\gamma_c\| \geq 2$, then for any $n \geq 2$ we have

$$\|\gamma^n\| \geq 2\|\alpha\| + n\|\gamma_c\| - (n-1) = 2\|\alpha\| + n(\|\gamma_c\| - 1) + 1 \geq 2\|\alpha\| + 2\|\gamma_c\| - 1 \geq \|\gamma\|.$$

(iii) The equality $F_\Gamma = \{\alpha x \alpha^{-1} : \alpha \in \Gamma, x \in X_\Gamma, x^n = e \text{ for some } n \geq 1\}$ is classical, so $\tilde{F}_\Gamma^c = F_\Gamma \setminus \{e\}$ follows from item (ii). \square

We now come to a rather technical lemma that describes elements that can be written as a product of two elements from S_Γ^c . Corollaries of this lemma will serve as supplements to our modifications of Dudley's argument.

Lemma 2.3. *Let $\gamma \in S_\Gamma^c \cdot S_\Gamma^c$ be a non-trivial element. There are $\mu, \nu, \eta \in \Gamma$ such that the reduced form of γ is one of the following:*

- (i) $\eta z \eta^{-1}$, for some $z \in X_\Gamma \setminus \{e\}$;
- (ii) $\eta \mu z_0 \mu^{-1} \nu z_1 \nu^{-1} \eta^{-1}$, for some $z_0, z_1 \in X_\Gamma \setminus \{e\}$;
- (iii) $\eta \delta_1 \nu z \nu^{-1} \delta_2 \eta^{-1}$, for some $\delta_1, \delta_2, z \in X_\Gamma \setminus \{e\}$, δ_1 and δ_2 are multipliable and $\delta_1 \cdot \delta_2 \neq e$;
- (iv) $\eta \delta_1 \mu z_0 \mu^{-1} \delta_2 \nu z_1 \nu^{-1} \delta_3 \eta^{-1}$, for $\delta_i, z_j \in X_\Gamma \setminus \{e\}$, $\delta_1, \delta_2, \delta_3$ are multipliable and $\delta_2 = \delta_1^{-1} \cdot \delta_3^{-1}$.

Remember that we treat identity as an empty word, so any of the μ, ν, η are allowed to be absent.

Proof. Fix $\gamma \in S_\Gamma^c \cdot S_\Gamma^c$, $\gamma \neq e$, $\gamma = \alpha z_0 \alpha^{-1} \cdot \beta z_1 \beta^{-1}$, where $z_0, z_1 \in X_\Gamma$ and both $\alpha z_0 \alpha^{-1}$ and $\beta z_1 \beta^{-1}$ are reduced. If either $z_0 = e$ or $z_1 = e$, then the reduced form of γ is described by item (i) and we shall assume that z_0 and z_1 are non-trivial. Let $\alpha = x_1 \dots x_m$, $\beta = y_1 \dots y_n$, $x_i, y_j \in X_\Gamma$ be the reduced forms. By taking γ^{-1} instead of γ we may and shall assume that $m \leq n$. We are going to show that no matter how many reductions take place in the product $\alpha z_0 \alpha^{-1} \cdot \beta z_1 \beta^{-1}$, the element's γ reduced representation has form given by one of the items above. To make it easier to recognise that a given product can indeed be written in a certain form we shall underline elements η, μ, ν and their inverses; elements z_j and δ_i are then the letters in between.

If $\gamma = \alpha z_0 \alpha^{-1} \beta z_1 \beta^{-1}$ is reduced, then, of course, γ has the form given by item (ii). So we assume that this product is not reduced. If $m = 0$ and $n = 0$, then γ is a single letter, which fits in (i). If $m = 0$, but $n \geq 1$, then, depending on whether $z_0 \cdot y_1 = e$, the reduced form of γ is one of the two:

$$\begin{aligned} \gamma &= \underbrace{y_2 \dots y_n}_{\mu} z_1 \underbrace{y_n^{-1} \dots y_2^{-1}}_{\mu^{-1}} z_0, \\ \gamma &= u \underbrace{y_2 \dots y_n}_{\nu} z_1 \underbrace{y_n^{-1} \dots y_2^{-1}}_{\nu^{-1}} y_1^{-1}, \quad u = z_0 \cdot y_1, \end{aligned}$$

which satisfy items (ii) and (iii) respectively.

So we assume $m \geq 1$, $n \geq 1$, and x_1, y_1 are multipliable. If $x_1^{-1} \cdot y_1 \neq e$, then the product

$$\gamma = x_1 \underbrace{x_2 \cdots x_m}_{\mu} z_0 \underbrace{x_m^{-1} \cdots x_2^{-1}}_{\mu^{-1}} u \underbrace{y_2 \cdots y_n}_{\nu} z_1 \underbrace{y_n^{-1} \cdots y_2^{-1}}_{\nu^{-1}} y_1^{-1}, \quad u = x_1^{-1} \cdot y_1,$$

is reduced and γ satisfies (iv). If $x_1 = y_1$, let $k \leq m$ be maximal such that $x_i = y_i$ for $i \leq k$. If $k < m$, then, depending on whether x_{k+1} and y_{k+1} are multipliable or not, one of the two products

$$\gamma = \underbrace{y_1 \cdots y_k}_{\eta} \underbrace{x_{k+1} \cdots x_m}_{\mu} z_0 \underbrace{x_m^{-1} \cdots x_{k+1}^{-1}}_{\mu^{-1}} \underbrace{y_{k+1} \cdots y_n}_{\nu} z_1 \underbrace{y_n^{-1} \cdots y_{k+1}^{-1}}_{\nu^{-1}} \underbrace{y_k^{-1} \cdots y_1^{-1}}_{\eta^{-1}},$$

$$\gamma = \underbrace{y_1 \cdots y_k}_{\eta} \underbrace{x_{k+1} x_{k+2} \cdots x_m}_{\mu} z_0 \underbrace{x_m^{-1} \cdots x_{k+2}^{-1}}_{\mu^{-1}} u \underbrace{y_{k+2} \cdots y_n}_{\nu} z_1 \underbrace{y_n^{-1} \cdots y_{k+2}^{-1}}_{\nu^{-1}} \underbrace{y_{k+1}^{-1} y_k^{-1} \cdots y_1^{-1}}_{\eta^{-1}}, \quad u = x_{k+1}^{-1} \cdot y_{k+1},$$

is reduced and they are of the form (ii) and (iv) respectively.

We therefore consider the case $k = m$. If $m = n$, then depending on whether z_0 and z_1 are multipliable the reduced form of γ is one of the following two

$$\begin{aligned} \gamma &= \underbrace{y_1 \cdots y_m}_{\eta} z_0 z_1 \underbrace{y_m^{-1} \cdots y_1^{-1}}_{\eta^{-1}}, \\ \gamma &= \underbrace{y_1 \cdots y_m}_{\eta} u \underbrace{y_m^{-1} \cdots y_1^{-1}}_{\eta^{-1}}, \quad u = z_0 \cdot z_1 \neq e, \end{aligned}$$

and they satisfy (ii) and (i) respectively (here we use $\gamma \neq e$ to exclude $z_0 \cdot z_1 = e$).

We may now assume that $m < n$ and γ can be written (in a not necessarily reduced way) as

$$\gamma = \underbrace{y_1 \cdots y_m}_{\eta} z_0 \cdot \underbrace{y_{m+1} \cdots y_n}_{\nu} z_1 \underbrace{y_n^{-1} \cdots y_{m+1}^{-1}}_{\nu^{-1}} \underbrace{y_m^{-1} \cdots y_1^{-1}}_{\eta^{-1}},$$

where \cdot represents the only position where reductions can take place. If z_0 is not multipliable with y_{m+1} , then the above representation is reduced and satisfies (ii). If it is not reduced, but $z_0 \cdot y_{m+1} \neq e$, then the reduced form of γ is

$$\gamma = \underbrace{y_1 \cdots y_m}_{\eta} u \underbrace{y_{m+2} \cdots y_n}_{\nu} z_1 \underbrace{y_n^{-1} \cdots y_{m+2}^{-1}}_{\nu^{-1}} \underbrace{y_{m+1}^{-1} y_m^{-1} \cdots y_1^{-1}}_{\eta^{-1}}, \quad u = z_0 \cdot y_{m+1},$$

in accordance with (iii). If $z_0 = y_{m+1}^{-1}$, then setting $s_0 = n + 1$, $s_1 = s_0 - (n - m) = m + 1$ we have

$$\gamma = y_1 \cdots y_{s_1-1} \cdot y_{s_1+1} \cdots y_{s_0-1} z_1 y_{s_0-1}^{-1} \cdots y_{s_1+1}^{-1} z_0 y_{s_1-1}^{-1} \cdots y_1^{-1}.$$

If the block $y_{s_1+1} \cdots y_{s_0-1} z_1$ cancels the block $y_{s_2} \cdots y_{s_1-1}$, where $s_2 = s_1 - (n - m) = s_0 - 2(n - m)$, then

$$\gamma = y_1 \cdots y_{s_2-1} \cdot y_{s_2+1} \cdots y_{s_1-1} z_0 y_{s_1-1}^{-1} \cdots y_{s_2+1}^{-1} z_1 y_{s_2-1}^{-1} \cdots y_1^{-1}.$$

If again the block $y_{s_2+1} \cdots y_{s_1-1} z_0$ cancels the block $y_{s_3} \cdots y_{s_2-1}$ we can proceed in the same way and build a sequence $s_k = s_{k-1} - (n - m) = s_0 - k(n - m)$ such that

$$\gamma = y_1 \cdots y_{s_k-1} \cdot y_{s_k+1} \cdots y_{s_{k-1}-1} z_{\epsilon_k} y_{s_{k-1}-1}^{-1} \cdots y_{s_k+1}^{-1} z_{1-\epsilon_k} y_{s_k-1}^{-1} \cdots y_1^{-1}, \quad \epsilon_k = k \pmod{2}.$$

Note that we use $m < n$ to ensure that $s_k < s_{k-1}$, in which case there is the largest k for which this process works (it is possible that $k = 1$). For the largest possible k the block $y_{s_k+1} \cdots y_{s_1-1} z_{\epsilon_k}$ does not cancel completely and so there are $0 \leq p \leq s_k - 1$ and $s_k + 1 \leq q \leq s_{k-1}$ such that

$$\gamma = y_1 \cdots y_p \cdot y_q \cdots y_{s_k-1-1} z_{\epsilon_k} y_{s_k-1-1}^{-1} \cdots y_q^{-1} y_{p+1} \cdots y_{s_k-1} z_{1-\epsilon_k} y_{s_k-1}^{-1} \cdots y_1^{-1},$$

and this product is ‘‘almost reduced’’: it is either reduced or there is exactly one reduction. If $q < s_{k-1}$ and y_p, y_q are not multipliable (including the case $p = 0$), then the reduced form of γ is

$$\gamma = \underbrace{y_1 \cdots y_p}_{\eta} \underbrace{y_q \cdots y_{s_k-1-1}}_{\mu} z_{\epsilon_k} \underbrace{y_{s_k-1-1}^{-1} \cdots y_q^{-1}}_{\mu^{-1}} \underbrace{y_{p+1} \cdots y_{s_k-1}}_{\nu} z_{1-\epsilon_k} \underbrace{y_{s_k-1}^{-1} \cdots y_{p+1}^{-1}}_{\nu^{-1}} \underbrace{y_p^{-1} \cdots y_1^{-1}}_{\eta^{-1}},$$

and it has the form of (ii). If $q < s_{k-1}$, $p > 0$ and y_p, y_q are multipliable and $y_p \cdot y_q \neq e$, then the reduced form of γ is

$$\gamma = \underbrace{y_1 \cdots y_{p-1}}_{\eta} u \underbrace{y_{q+1} \cdots y_{s_k-1-1}}_{\mu} z_{\epsilon_k} \underbrace{y_{s_k-1-1}^{-1} \cdots y_{q+1}^{-1}}_{\mu^{-1}} y_q^{-1} \underbrace{y_{p+1} \cdots y_{s_k-1}}_{\nu} z_{1-\epsilon_k} \underbrace{y_{s_k-1}^{-1} \cdots y_{p+1}^{-1}}_{\nu^{-1}} \underbrace{y_{p-1}^{-1} \cdots y_1^{-1}}_{\eta^{-1}},$$

where $u = y_p \cdot y_q$ and γ satisfies item (iv), because $u^{-1}(y_p^{-1})^{-1} = y_q^{-1}$. If $q = s_{k-1}$, then the ‘‘almost reduced’’ form of γ is

$$\gamma = \underbrace{y_1 \cdots y_p}_{\eta} \cdot z_{\epsilon_k} \underbrace{y_{p+1} \cdots y_{s_{k-1}}}_{\nu} z_{1-\epsilon_k} \underbrace{y_{s_{k-1}}^{-1} \cdots y_{p+1}^{-1}}_{\nu^{-1}} \underbrace{y_p^{-1} \cdots y_1^{-1}}_{\eta^{-1}}.$$

We see that γ satisfies (ii) if the above product is reduced (including the case $p = 0$), and finally if $p \geq 1$, y_p and z_{ϵ_k} are multipliable and $y_p \cdot z_{\epsilon_k} \neq e$, then the reduced form of γ is

$$\gamma = \underbrace{y_1 \cdots y_{p-1}}_{\eta} u \underbrace{y_{p+1} \cdots y_{s_{k-1}}}_{\nu} z_{1-\epsilon_k} \underbrace{y_{s_{k-1}}^{-1} \cdots y_{p+1}^{-1}}_{\nu^{-1}} \underbrace{y_p^{-1} y_{p-1}^{-1} \cdots y_1^{-1}}_{\eta^{-1}}, \quad u = y_p \cdot z_{\epsilon_k},$$

which also has the form described by the item (iii), because $y_p z_{\epsilon_k} y_p^{-1} = e$ if and only if $z_{\epsilon_k} = e$.

This exhausts all the possibilities and therefore the lemma is proved. \square

Remark 2.4. It is easy to see that any element in Γ with the reduced form satisfying one of these items is, in fact, an element in $S_\Gamma^c \cdot S_\Gamma^c$ and so the description above is complete, but we shall not use this.

Corollary 2.5. *If $\gamma \in S_\Gamma^c \cdot S_\Gamma^c$ has the reduced form $\gamma = x_1 \cdots x_n$ with $n \geq 2$ and x_1, x_n are not multipliable, then the reduced form of γ can be written as $\mu z_0 \mu^{-1} \nu z_1 \nu^{-1}$ for some $\mu, \nu \in \Gamma$ and some $z_0, z_1 \in X_\Gamma \setminus \{e\}$.*

Lemma 2.6. *For $\Gamma = *_{a \in A} \Gamma_a$ the inclusion $S_\Gamma^c \cdot S_\Gamma^c \subset \Gamma$ is proper, unless $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$.*

Proof. The proof splits into three cases.

Case 1. Suppose there is an $a \in A$ and a non-trivial $x \in \Gamma_a$ such that order of x is not two, i.e., $x \neq x^{-1}$. Pick $b \in A \setminus \{a\}$ and a non-trivial $y \in \Gamma_b$ and let $\gamma = (xy)^3 = xyxyxy$. We claim that $\gamma \notin S_\Gamma^c \cdot S_\Gamma^c$. Indeed, if γ were in this set, then by Corollary 2.5 it would have a reduced form $\mu z_0 \mu^{-1} \nu z_1 \nu^{-1}$ and using $x \neq x^{-1}$ it is straightforward to see that this is not the case.

The remaining cases deal with all Γ_a consisting of elements of order 2.

Case 2. Suppose there is Γ_a such that $|\Gamma_a| \geq 4$, and pick $b \in A \setminus \{a\}$. Let $x_1, x_2, x_3 \in \Gamma_a$ be distinct non-trivial elements and pick a non-trivial $y \in \Gamma_b$. Using Corollary 2.5 and since $x_i \neq x_j^{-1}$ for $i \neq j$, one sees that $\gamma = x_1 y x_2 y x_3 y \notin S_\Gamma^c \cdot S_\Gamma^c$.

Case 3. We are left with the case when each of Γ_a is isomorphic to \mathbb{Z}_2 . Since $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$ is excluded in the statement of the theorem, we may assume that there are at least three factors. Let x, y, z be elements of order 2 from three different factors and set $\gamma = xyz$. Note that the reduced of γ cannot be written in the form $\mu z_0 \mu^{-1} \nu z_1 \nu^{-1}$ and therefore Corollary 2.5 finishes the proof. \square

Remark 2.7. It is easy to check that for the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$ the equality $S_{\mathbb{Z}_2 * \mathbb{Z}_2}^c \cdot S_{\mathbb{Z}_2 * \mathbb{Z}_2}^c = \mathbb{Z}_2 * \mathbb{Z}_2$ indeed holds: $S_{\mathbb{Z}_2 * \mathbb{Z}_2}^c$ consists of all elements of odd length plus the identity, so its square is the whole group.

3. COMPLETELY METRIZABLE FREE PRODUCTS ARE DISCRETE

In this section the group $\Gamma = \Gamma_1 * \Gamma_2$ is assumed to have exactly two factors unless stated otherwise.

Definition 3.1. For a pair $(i, j) \in \{1, 2\} \times \{1, 2\}$ we say that $\gamma \in \Gamma \setminus \{e\}$ is of type (i, j) if for the reduced form of $\gamma = x_1 \cdots x_n$ one has $x_1 \in \Gamma_i$ and $x_n \in \Gamma_j$. The type (i, j) is called *symmetric* if $i = j$ and *asymmetric* otherwise. Since $\Gamma = \Gamma_1 * \Gamma_2$ has only two factors, whether an element $\gamma \in \Gamma$ has a symmetric or an asymmetric type depends only upon the parity of its length. Type (i, j) is said to be the *opposite* of type (j, i) .

Lemma 3.2. *Let $\Gamma \neq \mathbb{Z}_2 * \mathbb{Z}_2$ be a topological group and let γ_n be a sequence such that $\gamma_n \neq e$, $\gamma_n \rightarrow e$ and all the elements γ_n have the same type. There is a subsequence γ_{n_k} satisfying the following property: for any type (i, j) there is a sequence $\lambda_{n_k} \in \Gamma$, such that $\lambda_{n_k} \rightarrow e$, each λ_{n_k} has type (i, j) , and $\|\gamma_{n_k}\| \leq \|\lambda_{n_k}\| \leq 4\|\gamma_{n_k}\|$ for all k .*

Proof. Depending on the type of γ_n we have two cases.

Case 1: γ_n have symmetric type (i, i) . If $x \in \Gamma_j$ is non-trivial for $j \neq i$, then elements $\lambda_n = x \gamma_n x^{-1}$ have type (j, j) , converge to identity and $\|\lambda_n\| = \|\gamma_n\| + 2 \leq 3\|\gamma_n\|$. The sequence $\lambda_n \gamma_n$ has type (j, i) , and $\gamma_n \lambda_n$ is of type (i, j) , and elements in both sequences have length at most $4\|\gamma_n\|$. Note that in this case we do not have to pass to a subsequence of γ_n .

Case 2: γ_n have asymmetric type (i, j) , $i \neq j$. One of Γ_1, Γ_2 is not isomorphic to \mathbb{Z}_2 , so let $\Gamma_1 \neq \mathbb{Z}_2$. Note that γ_n is of type (i, j) implies γ_n^{-1} is of the opposite type (j, i) . So there is no loss in generality in assuming γ_n is of type $(1, 2)$.

Subcase 1: there is $x \in \Gamma_1$ such that for infinitely many n the first letter of γ_n is x . Let (n_k) be such that all elements γ_{n_k} start with x and pick a $y \in \Gamma_1$ such that $x \cdot y \neq e$, then $\lambda_{n_k} = y\gamma_{n_k}y^{-1}$ converges to identity, all the elements λ_{n_k} have type $(1, 1)$ and $\|\gamma_{n_k}\| \leq \|\lambda_{n_k}\| \leq 2\|\gamma_{n_k}\|$. Similarly to the first case one can conjugate λ_{n_k} by a non-trivial element from Γ_2 to get a sequence of type $(2, 2)$.

Subcase 2: for any $x \in \Gamma_1$ there are only finitely many γ_n which start with x . Let $x \in \Gamma_1$ be any non-trivial element, then $x\gamma_n x^{-1}$ converges to identity, $\|\gamma_n\| \leq \|x\gamma_n x^{-1}\| \leq 2\|\gamma_n\|$ and for all but possibly finitely many n elements $x\gamma_n x^{-1}$ have type $(1, 1)$. Passing to a subsequence to avoid this finite set of n 's we obtain λ_{n_k} of type $(1, 1)$, which again can be conjugated by any non-trivial element from Γ_2 into a sequence of type $(2, 2)$. \square

Lemma 3.3. *If $\Gamma = \Gamma_1 * \Gamma_2$ is a non-discrete metrizable topological group, then the word length function is unbounded on any non-empty open set.*

Proof. If $\Gamma \neq \mathbb{Z}_2 * \mathbb{Z}_2$, then by Lemma 3.2 for any open set U and any $\gamma \in U$ we can find $\lambda \neq e$ such that the product $\lambda\gamma$ is reduced and $\lambda\gamma \in U$, so $\|\lambda\gamma\| = \|\lambda\| + \|\gamma\|$, and thus $\|\cdot\|$ is unbounded on U .

For $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$ the statement is obvious, since for such a Γ the set of elements of length at most n is finite for any n and Γ being non-discrete implies finite sets cannot be open. \square

Recall that a subset of a metric space is called *separated* if there is a non-zero lower bound on the distance between its distinct points.

Lemma 3.4. *If $\Gamma = \Gamma_1 * \Gamma_2$ is completely metrizable and non-discrete, then the interior of $S_\Gamma^c \cdot S_\Gamma^c$ is empty.*

Proof. First of all Γ being completely metrizable and non-discrete must be uncountable, so there is no need to worry about Γ being isomorphic to the infinite dihedral group. Fix a left-invariant metric d on Γ and set $L_n = \{\gamma \in \Gamma : \|\gamma\| \leq n\}$, $\Gamma = \bigcup_n L_n$. Since a separated set is necessarily closed and nowhere dense (by non-discreteness of the group), Baire's theorem ensures the existence of N such that L_N is not separated: there are sequences $\gamma'_n, \gamma''_n \in L_N$ such that $\gamma'_n \neq \gamma''_n$ for all n and $d(\gamma'_n, \gamma''_n) \rightarrow 0$. By left-invariance of d we see that $\gamma_n = (\gamma''_n)^{-1}\gamma'_n$ converges to e , and $\gamma_n \in L_{2N}$. By passing to a subsequence and using Lemma 3.2 we may assume that M is such that for any type (i, j) there is a sequence $\tilde{\gamma}_n \in L_M$, $\tilde{\gamma}_n \rightarrow e$ that consists of elements of type (i, j) .

Pick a non-empty open U ; we shall find an element in U but not in $S_\Gamma^c \cdot S_\Gamma^c$. By Lemma 3.3 find $\lambda \in U$ with the reduced form $\lambda = x_1 \cdots x_l$, $l \geq 8M + 2$. We now construct inductively elements γ_k for $k = 1, \dots, l$ such that for

$$\lambda_k = x_1\gamma_1x_2\gamma_2 \cdots x_k\gamma_kx_{k+1}x_{k+2} \cdots x_l$$

and for all k we have

- (i) the product in the definition of λ_k is reduced;
- (ii) $\lambda_k \in U$;
- (iii) $\|\gamma_k\| \leq M$;
- (iv) if W_k denotes the set of letters in the reduced form of λ_k , then there is a letter y_k in the reduced form of γ_k such that $y_k \notin W_{k-1}^{-1}$;
- (v) the last element λ_l has an asymmetric type $(1, 2)$ or $(2, 1)$.

The first step of the construction is not any different from the later steps, so we show how to find γ_k . Suppose that $\gamma_1, \dots, \gamma_{k-1}$ have been constructed. By the choice of L_M we can find a sequence $\tilde{\gamma}_n \rightarrow e$, such that $\|\tilde{\gamma}_n\| \leq M$ and $x_k\tilde{\gamma}_n x_{k+1}$ is reduced for all n (and we take x_{l+1} to be x_1 in the last step to satisfy item (v)). Throwing away finitely elements from the sequence we may assume that $x_1\gamma_1 \cdots x_{k-1}\gamma_{k-1}x_k\tilde{\gamma}_n x_{k+1} \cdots x_l \in U$ for all n . Since W_{k-1}^{-1} is finite and since there are only finitely many words in the alphabet W_{k-1}^{-1} of length at most M , there must be $\tilde{\gamma}_{n_0}$ that has a letter not in W_{k-1}^{-1} in its reduced form; we set $\gamma_k = \tilde{\gamma}_{n_0}$.

We finally claim that $\lambda_l \notin S_\Gamma^c \cdot S_\Gamma^c$. Indeed, suppose λ_l were in this set. Since λ_l is of asymmetric type, Corollary 2.5 implies the reduced form of λ_l can be written as $\mu z_0 \mu^{-1} \nu z_1 \nu^{-1}$. Assume for definiteness that $\|\mu\| \geq \|\nu\|$, then $\|\mu\| \geq \frac{8M+2-2}{4} \geq 2M$ and therefore $\|\gamma_k\| \leq M$ implies there is $1 \leq k \leq l$ such that $\gamma_k \subseteq \mu^{-1}$, but then every letter in γ_k is also in W_{k-1}^{-1} , contradicting the construction.

Since $\lambda_l \in U$, we have proved that $U \not\subseteq S_\Gamma^c \cdot S_\Gamma^c$, and since U was arbitrary, the conclusion follows. \square

Theorem 3.5. *Any completely metrizable group topology on Γ is discrete.*

Proof. Suppose the statement is false and Γ is equipped with a non-discrete completely metrizable group topology. Let $F_n = \{f \in \Gamma : f^n = e\}$, and therefore $F_\Gamma = \bigcup_n F_n$, where each of F_n is closed. Recall that $F_\Gamma \subseteq S_\Gamma^c$.

Fix a complete metric d on Γ . As in the original construction of Dudley we build a sequence $g_n \in \Gamma$ such that for $r_n = n + \sum_{k=1}^n \|g_k\|$ and

$$\tilde{h}_{n,m} = g_m(g_{m+1}(\cdots(g_{n-1}(g_n)^{r_{n-1}})^{r_{n-2}} \cdots)^{r_{m+1}})^{r_m}, \quad 1 \leq m \leq n,$$

we have

- (i) $g_n \notin S_\Gamma^c \cdot S_\Gamma^c$;
- (ii) for all $m \geq 2$

$$d(\tilde{h}_{n+1,m}, \tilde{h}_{n,m}) < 2^{-n} d(g_m, F_{r_{m-1}}),$$

$$\text{and } d(\tilde{h}_{n+1,1}, \tilde{h}_{n,1}) < 2^{-n}.$$

Such a sequence is constructed by induction on n . For the base of induction choose any $g_1 \notin S_\Gamma^c \cdot S_\Gamma^c$. For the induction step note that for each $1 \leq m \leq n$ the map

$$\xi_m(x) = g_m(g_{m+1}(\cdots(g_{n-1}(g_n \cdot x^{r_n})^{r_{n-1}})^{r_{n-2}} \cdots)^{r_{m+1}})^{r_m}.$$

is continuous and $\xi_m(e) = \tilde{h}_{n,m}$. Therefore there exists an open neighbourhood of the identity U such that for all $g \in U$ and all $2 \leq m \leq n$

$$d(\xi_m(g), \tilde{h}_{n,m}) < 2^{-n} d(g_m, F_{r_{m-1}})$$

and $d(\xi_1(g), \tilde{h}_{n,1}) < 2^{-n}$. By Lemma 3.4 the interior of $S_\Gamma^c \cdot S_\Gamma^c$ is empty and we choose $g_{n+1} \in U$ to be any element not in $S_\Gamma^c \cdot S_\Gamma^c$. This finishes the step of induction.

For each m the resulting sequence $\tilde{h}_{n,m}$ is Cauchy, so we may define $h_m = \lim_{n \rightarrow \infty} \tilde{h}_{n,m}$. Also by the definition of $\tilde{h}_{n,m}$ we have $\tilde{h}_{n,m} = g_m \tilde{h}_{n,m+1}^{r_m}$ and therefore also $h_m = g_m h_{m+1}^{r_m}$. Finally $d(\tilde{h}_{n+1,m}, \tilde{h}_{n,m}) < 2^{-n} d(g_m, F_{r_{m-1}})$ implies

$$d(h_m, g_m) < \sum_{n=m}^{\infty} 2^{-n} d(g_m, F_{r_{m-1}}) \leq d(g_m, F_{r_{m-1}}),$$

and therefore $h_m \notin F_{r_{m-1}}$, whence by item (ii) of Lemma 2.2 we see that $\|h_m^{r_{m-1}}\| \geq \|h_m\|$ holds for all $m \geq 2$.

We claim that $\|h_m^{r_{m-1}}\| \geq r_{m-1}$ for all $m \geq 2$. Indeed, if $h_m \in S_\Gamma$, then $\|h_m^{r_{m-1}}\| \geq r_{m-1}$ holds by the definition of S_Γ . If $h_m \in S_\Gamma^c$, but $h_{m+1} \in S_\Gamma$, then

$$\|h_m^{r_{m-1}}\| \geq \|h_m\| = \|g_m h_{m+1}^{r_m}\| \geq \|h_{m+1}^{r_m}\| - \|g_m\| \geq r_m - \|g_m\| = r_{m-1} + 1 > r_{m-1}.$$

Finally if $h_m \in S_\Gamma^c$ and $h_{m+1} \in S_\Gamma^c$, then by item (i) of Lemma 2.2 $h_m^{-r_m} \in S_\Gamma^c$ and therefore $g_m = h_m h_{m+1}^{-r_m} \in S_\Gamma^c \cdot S_\Gamma^c$ contradicting the choice of g_m .

We now ready to arrive at a contradiction by showing that $\|h_1\|$ is unbounded:

$$\begin{aligned} \|h_1\| &= \|g_1 h_2^{r_1}\| \geq \|h_2^{r_1}\| - \|g_1\| \geq \|h_2\| - \|g_1\| \\ &= \|g_2 h_3^{r_2}\| - \|g_1\| \geq \|h_3^{r_2}\| - \|g_1\| - \|g_2\| \geq \|h_3\| - \|g_1\| - \|g_2\| \\ &= \dots \\ &= \|g_m h_{m+1}^{r_m}\| - \sum_{l=1}^{m-1} \|g_l\| \geq \|h_{m+1}^{r_m}\| - \sum_{l=1}^m \|g_l\| \geq r_m - \sum_{l=1}^m \|g_l\| = m. \end{aligned}$$

And this is true for all m , which is absurd. □

Corollary 3.6. *Let $\Gamma = *_{a \in A} \Gamma_a$, where A is any set of cardinality at least two. Any completely metrizable group topology on Γ is discrete.*

Proof. This is immediate from the above, since $*_{a \in A} \Gamma_a = \Gamma_b * (*_{a \in A \setminus \{b\}} \Gamma_a)$. □

Corollary 3.7. *Let $\phi : G \rightarrow \Gamma$ be a surjective homomorphism from a completely metrizable topological group G onto a discrete free product $\Gamma = \ast_{a \in A} \Gamma_a$ for some index set $|A| \geq 2$. If the kernel K of ϕ is closed in G , then ϕ is continuous.*

Proof. Suppose that the kernel K is a closed normal subgroup of G and let $\pi : G \rightarrow G/K$ be the canonical projection. The surjective homomorphism ϕ induces the isomorphism $\psi : G/K \rightarrow \Gamma$. Since the group G/K being a quotient of a completely metrizable group by a closed normal subgroup is itself completely metrizable in the quotient topology (see [BK96, Theorem 5.2.2] and [Kec95, Section 8D]), Corollary 3.6 implies G/K is discrete. Therefore $\psi : G/K \rightarrow \Gamma$ is continuous, and hence so is $\phi = \psi \circ \pi$. \square

4. MORE CALCULATIONS IN THE FREE PRODUCTS

To motivate some of the calculations in this section let us discuss the strategy of proving our main result Theorem 1.5. Based on Corollary 3.7 and some additional arguments we shall find ourselves in the situation of having a surjective homomorphism $\phi : G \rightarrow \Gamma$ from a completely metrizable group into a free product that has a *dense* kernel. Density of the kernel gives a great power. Recall that in the argument of Dudley we could choose the sequence g_m from any dense set. In particular, for such a ϕ the sequence $\phi(g_m)$ can be literally any sequence we want. For instance, we shall find it convenient to have $\phi(g_m) = \gamma_0 \in \Gamma$ for a concrete $\gamma_0 \in \Gamma$. To obtain a contradiction we shall need to show that $\phi(h_m) \notin S_\Gamma^c$, where h_m is constructed as usually. Since elements h_m are obtained via the limit procedure, we shall have little control of them. The only piece of information that will be at our hands is the equality $h_m = g_m h_{m+1}^{r_m}$, and therefore also $\phi(h_m) = \gamma_0 \phi(h_{m+1})^{r_m}$. If we choose $\gamma_0 \notin S_\Gamma^c \cdot S_\Gamma^c$, then either $\phi(h_m)$ or $\phi(h_{m+1})$ has to be in S_Γ , but we need this to be the case for both of them. The somewhat tedious calculations in this section show that under some mild assumptions on γ_0 and r_m , $\gamma_0 \lambda^{r_m} \in S_\Gamma^c$ implies $\lambda \in S_\Gamma^c$ and this will be enough for our purposes.

In this section the group $\Gamma = \Gamma_1 \ast \Gamma_2$ is also assumed to have exactly two factors unless stated otherwise. The role of the groups Γ_1 and Γ_2 will not be symmetric. As we have noted earlier, elements of order two cause some troubles, and we shall use two different arguments: when both Γ_1 and Γ_2 consist “essentially” of elements of order two, and when one of Γ_1, Γ_2 has “many” elements of order bigger than two. Results of this section are needed for the latter and will be applied for Γ_1 being a “good” group in this sense.

Definition 4.1. Let γ be an element in $\Gamma_1 \ast \Gamma_2$ with the reduced form $\gamma = x_1 x_2 \cdots x_n$. If $n \geq 2$, then γ has letters from both Γ_1 and Γ_2 , so let i_1, \dots, i_m be the indices of letters from Γ_1 , i.e., depending on whether $x_1 \in \Gamma_1$ or $x_1 \in \Gamma_2$, i_k 's are either all the odd or all the even letters. The word γ is called *unbalanced* if $x_{i_k} \neq x_{i_l}^{\pm 1}$ for all $k \neq l$ and $x_{i_k} \neq x_{i_k}^{-1}$ for all k . In other words, an element is unbalanced if all of its letters from Γ_1 have order bigger than two, and the word uses neither the same letter from Γ_1 twice nor the letter from Γ_1 and its inverse.

Lemma 4.2. *An unbalanced asymmetric element of length at least 6 does not belong to $S_\Gamma^c \cdot S_\Gamma^c$.*

Proof. Let $\gamma \in \Gamma$ be such an element. By Corollary 2.5 if $\gamma \in S_\Gamma^c \cdot S_\Gamma^c$, then the reduced form of γ is of the form $\mu z_0 \mu^{-1} \nu z_1 \nu^{-1}$ for some $\mu, \nu \in \Gamma$ and some $z_0, z_1 \in X_\Gamma \setminus \{e\}$. Since γ is unbalanced, both μ and ν may consist only of letters from Γ_2 . In particular, z_0 and z_1 are the only possible letters from Γ_1 , but $\|\gamma\| \geq 6$ and so γ has at least 3 letters from Γ_1 in its reduced form. This contradiction proves the lemma. \square

Lemma 4.3. *Let $\xi \in \Gamma_1 \ast \Gamma_2$ be an unbalanced element of asymmetric type and of length $\|\xi\| \geq 6$. If $r \geq 4$ and $\lambda \in \Gamma_1 \ast \Gamma_2$ is cyclically reduced of length $\|\lambda\| \geq 2$, then $\xi \lambda^r \notin S_\Gamma^c$.*

Proof. The proof is by contradiction: suppose the statement is false and we are given elements $\xi, \lambda \in \Gamma_1 \ast \Gamma_2$ and r which comply with the assumptions of the theorem, but $\xi \lambda^r \in S_\Gamma^c$. Therefore $\xi \lambda^r = \beta z \beta^{-1}$ for some $\beta \in \Gamma_1 \ast \Gamma_2$ and $z \in \Gamma_1 \cup \Gamma_2$. A tiresome case analysis with numerous subcases will show that this is impossible. More precisely, we consider four cases depending on the relation between the types of ξ and λ and there will be subcases dealing with different possible lengths of λ . Most of the products in this proof will be reduced (except for the product $\xi \lambda^r$, the reduced form of which we aim to understand), we shall mention explicitly when a product under consideration is not reduced and a dot \cdot will denote places where reductions may occur.

In this proof we shall consistently use the following notations. Symbols x and y will denote letters in the reduced form of ξ , while symbols with the tilde, e.g. \tilde{x} and \tilde{y} , will denote letters in the reduced form of λ . Symbols u and v will denote non-trivial products of two non-trivial multipliable letters. A typical

contradiction will be obtained by showing that such a product is equal to one of its factors. For example, for $u = x \cdot \tilde{x}$ with $x, \tilde{x} \neq e$ we shall sometimes get $u = \tilde{x}$, which is clearly impossible.

Let $\xi\lambda^r = \beta z \beta^{-1}$ and so, roughly speaking, the first half of the letters in the reduced form of $\xi\lambda^r$ are just the inverses of the corresponding letters in the second half. We analyse the product $\xi\lambda^r$ and show that it cannot have the form $\beta z \beta^{-1}$. To emphasise at which parts of the reduced form of $\xi\lambda^r$ we look in the concrete argument, we under-bracket the relevant pieces. By assumption ξ has type (i, j) with $i \neq j$.

Case: ξ is of type (i, j) and λ has type (j, i) . In this case λ is of asymmetric type, and the reduced form of λ^r is just the concatenation of r copies of the reduced form of λ . Note also that the length of λ is even.

Subcase: $\|\lambda\| \geq \|\xi\| + 2$. If $\lambda = \xi^{-1}\lambda_1$ for some $\lambda_1 \in \Gamma$, then

$$\xi\lambda^r = \underbrace{\lambda_1 \xi^{-1}} \lambda_1 \xi^{-1} \lambda_1 \cdots \underbrace{\xi^{-1} \lambda_1},$$

and the product is reduced (we shall not mention this in the future). Therefore for $\xi\lambda^r$ to have the form $\beta z \beta^{-1}$ we must have $\lambda_1^{-1} = \lambda_1$ and $\xi^{-1} = \xi$, whence ξ has a symmetric type contrary to the original assumption.

If $\lambda \neq \xi^{-1}\lambda_1$, then there is a unique way to decompose ξ as $\xi_1 x \xi_2$ and λ as $\xi_2^{-1} \tilde{x} \lambda_1$ for some $\xi_1, \xi_2, \lambda_1 \in \Gamma$ and some $x, \tilde{x} \in X_\Gamma$, where $x, \tilde{x} \neq e$, x and \tilde{x} are multipliable, $\|\lambda_1\| \geq \|\xi_1\| + 2$ and $x \cdot \tilde{x} \neq e$. Note that we allow for ξ_1 and ξ_2 to be empty. So

$$\xi\lambda^r = \underbrace{\xi_1 u \lambda_1 \xi_2^{-1} \tilde{x} \lambda_1} \cdots \underbrace{\xi_2^{-1} \tilde{x} \lambda_1}, \quad u = x \cdot \tilde{x}.$$

This implies $\lambda_1 = \lambda_2 u^{-1} \xi_1^{-1}$, $\|\lambda_2\| \geq 1$ and thus

$$\xi\lambda^r = \xi_1 u \underbrace{\lambda_2 u^{-1} \xi_1^{-1} \xi_2^{-1} \tilde{x} \lambda_2 u^{-1} \xi_1^{-1}} \cdots \underbrace{\xi_2^{-1} \tilde{x} \lambda_2 u^{-1} \xi_1^{-1} \xi_2^{-1} \tilde{x} \lambda_2 u^{-1} \xi_1^{-1}}.$$

So $\lambda_2 = \lambda_2^{-1}$ and $u = \tilde{x}$. But $u = x \cdot \tilde{x}$, hence $x = e$, which is impossible.

Subcase: $\|\xi\| = \|\lambda\|$. If $\lambda = \xi^{-1}$, then $\xi\lambda^r = \lambda^{r-1}$ has asymmetric type and $\xi\lambda^r \notin S_\Gamma^c$. So we may assume that $\xi = \xi_1 x \xi_2$ and $\lambda = \xi_2^{-1} \tilde{x} \lambda_1$, x and \tilde{x} are multipliable letters, $x \cdot \tilde{x} \neq e$ and $\|\lambda_1\| = \|\xi_1\|$. Similarly to the previous subcase one sees that $\lambda_1 = \xi_1^{-1}$ and so

$$\xi\lambda^r = \xi_1 u \underbrace{\xi_1^{-1} \xi_2^{-1} \tilde{x} \xi_1^{-1}} \cdots \underbrace{\xi_2^{-1} \tilde{x} \xi_1^{-1} \xi_2^{-1} \tilde{x} \xi_1^{-1}}, \quad u = x \cdot \tilde{x}.$$

Thus $u^{-1} = \tilde{x}$ and $\xi_2 \xi_1 = \xi_1^{-1} \xi_2^{-1}$. But the latter is impossible, since $\|\xi_2 \xi_1\| = \|\xi\| - 1 \geq 5$ implies there is a letter from Γ_1 in $\xi_2 \xi_1$ that is also a letter in the reduced form of $\xi_1^{-1} \xi_2^{-1}$. This contradicts the assumption that ξ is unbalanced.

Subcase: $4 \leq \|\lambda\| \leq \|\xi\| - 2$.

Subsubcase: $\xi = \xi_1 x \lambda^{-1}$, $\lambda = \tilde{x} \lambda_1$ and $x \cdot \tilde{x} \neq e$. In this situation $\|\xi_1\| \geq 1$, $\|\lambda_1\| \geq 3$ and

$$\xi\lambda^r = \underbrace{\xi_1 u \lambda_1 \tilde{x} \lambda_1} \cdots \underbrace{\tilde{x} \lambda_1}, \quad u = x \cdot \tilde{x}.$$

If either $\|\xi_1\| \geq 2$ or $\|\xi_1\| = 1$ and $\xi_1 \in \Gamma_1$, then ξ_1 contains a letter from Γ_1 with its inverse being a letter in λ_1 , but $\xi = \xi_1 x \lambda_1^{-1} \tilde{x}^{-1}$, and in particular ξ contains two copies of the same letter from Γ_1 contradicting the assumption that ξ is unbalanced. Therefore $\|\xi_1\| = 1$, $\xi_1 \in \Gamma_2$ and thus $\lambda_1 = \lambda_2 u^{-1} \xi_1^{-1}$ with $\|\lambda_2\| \geq 1$,

$$\xi\lambda^r = \xi_1 u \underbrace{\lambda_2 u^{-1} \xi_1^{-1} \tilde{x} \lambda_2 u^{-1} \xi_1^{-1}} \cdots \underbrace{\tilde{x} \lambda_2 u^{-1} \xi_1^{-1}}.$$

Therefore $\lambda_2^{-1} = \lambda_2$ and $u = \tilde{x}$. But $u = x \cdot \tilde{x}$, hence $x = e$, which is impossible.

Subsubcase: $\xi = \xi_1 x \lambda^{-1}$, $\lambda = \tilde{x} \lambda_1$ and $x \cdot \tilde{x} = e$. Letters x and \tilde{x} cannot belong to Γ_1 , since this would contradict ξ being unbalanced, so $x = \tilde{x}^{-1} \in \Gamma_2$. Let $\xi_1 = \xi_2 y$, $\lambda = x^{-1} \tilde{y} \lambda_2$ and we must have $y \cdot \tilde{y} \neq e$ (again, because ξ is unbalanced), $\|\xi_2\| \geq 0$,

$$\xi\lambda^r = \underbrace{\xi_2 v \lambda_2 x^{-1} \tilde{y} \lambda_2} \cdots \underbrace{x^{-1} \tilde{y} \lambda_2}, \quad v = y \cdot \tilde{y}.$$

Note that $\|\lambda\| \geq 4$ implies $\|\lambda_2\| \geq 2$. If $\|\xi_2\| \geq 2$, then there exists a letter in ξ_2 from Γ_1 with its inverse being a letter in λ_2 . So $\xi = \xi_2 y x \lambda_2^{-1} \tilde{y}^{-1} x$ has two copies of a letter from Γ_1 , which is impossible. Thus $\|\xi_2\| < 2$, but $\|\xi_2\|$ is even, and therefore $\|\xi_2\| = 0$, in other words

$$\xi\lambda^r = v \underbrace{\lambda_2 x^{-1} \tilde{y} \lambda_2} \cdots \underbrace{x^{-1} \tilde{y} \lambda_2}.$$

So $\lambda_2 = \lambda_3 v^{-1}$,

$$\xi\lambda^r = v \underbrace{\lambda_3 v^{-1} x^{-1} \tilde{y} \lambda_3 v^{-1}} \cdots \underbrace{x^{-1} \tilde{y} \lambda_3 v^{-1}},$$

whence $\lambda_3^{-1} = \lambda_3$ and $v = \tilde{y}$. But $v = y \cdot \tilde{y}$, implying $y = e$, which is impossible.

Therefore the reduced form of ξ cannot be written in the form $\xi = \xi_1 x \lambda^{-1}$, and we have one subsubcase left.

Subsubcase: ξ cannot be written as $\xi_1 x \lambda^{-1}$. In this subsubcase we can write $\xi = \xi_1 x \xi_2$ and $\lambda = \xi_2^{-1} \tilde{x} \lambda_1$, for some $\xi_1, \xi_2, \lambda_1 \in \Gamma$ and some $x, \tilde{x} \in X_\Gamma \setminus \{e\}$, $x \cdot \tilde{x} \neq e$, $\|\xi_1\| \geq \|\lambda_1\| + 2$,

$$\xi \lambda^r = \xi_1 \underline{u} \lambda_1 \xi_2^{-1} \tilde{x} \lambda_1 \cdots \xi_2^{-1} \tilde{x} \lambda_1, \quad u = x \cdot \tilde{x}.$$

Therefore $\xi_1 = \lambda_1^{-1} \tilde{x}^{-1} \xi_3$, $\|\xi_3\| \geq 1$,

$$\xi \lambda^r = \lambda_1^{-1} \tilde{x}^{-1} \xi_3 \underline{u} \lambda_1 \xi_2^{-1} \tilde{x} \lambda_1 \cdots \xi_2^{-1} \tilde{x} \lambda_1 \xi_2^{-1} \tilde{x} \lambda_1.$$

Similarly to the earlier cases, since $\xi = \lambda_1^{-1} \tilde{x}^{-1} \xi_3 x \xi_2$ is unbalanced and $\|\lambda_1 \xi_2^{-1}\| \geq 3$ (because $\|\lambda\| \geq 4$) we must have $\|\xi_3\| = 1$ and $\xi_3 \in \Gamma_2$. So $\lambda_1 \xi_2^{-1} = \zeta u^{-1} \xi_3^{-1}$ for some $\zeta \in \Gamma$, $\|\zeta\| \geq 1$ and

$$\xi \lambda^r = \lambda_1^{-1} \tilde{x}^{-1} \xi_3 u \zeta u^{-1} \xi_3^{-1} \tilde{x} \cdots \tilde{x} \zeta u^{-1} \xi_3^{-1} \tilde{x} \lambda_1.$$

Finally we get $\zeta = \zeta^{-1}$ and $u = \tilde{x}$, contradicting, as before, $x \neq e$.

Subcase: $\|\lambda\| = 2$.

Subsubcase: $\xi = \xi_1 x \lambda^{-1}$, $\lambda = \tilde{x} \tilde{y}$, $x \cdot \tilde{x} \neq e$. Here $\|\xi_1\| \geq 3$ and

$$\xi \lambda^r = \xi_1 \underline{u} \tilde{y} \tilde{x} \tilde{y} \cdots \tilde{x} \tilde{y} \tilde{x} \tilde{y}, \quad u = x \cdot \tilde{x}.$$

If $\|\xi_1\| \geq 4$ or if $\tilde{y} \in \Gamma_1$, then ξ_1 contains two copies of the same letter from Γ_1 , which is impossible by assumption. So $\xi_1 = \tilde{y}^{-1} \tilde{x}^{-1} \tilde{y}^{-1}$ and $\tilde{y} \in \Gamma_2$. Then

$$\xi \lambda^r = \tilde{y}^{-1} \tilde{x}^{-1} \tilde{y}^{-1} \underline{u} \tilde{y} \tilde{x} \tilde{y} \cdots \tilde{x} \tilde{y} \tilde{x} \tilde{y},$$

and $u = \tilde{x}^{-1}$, whence $\xi = \tilde{y}^{-1} \tilde{x}^{-1} \tilde{y}^{-1} x \tilde{y}^{-1} \tilde{x}^{-1}$, $\tilde{x} \in \Gamma_1$ and such an element is not unbalanced.

Subsubcase: $\xi = \xi_1 x \lambda^{-1}$, $\lambda = \tilde{x} \tilde{y}$, $x \cdot \tilde{x} = e$. For an unbalanced ξ the equality $x \cdot \tilde{x} = e$ is only possible when $x, \tilde{x} \in \Gamma_2$ and so $\xi = \xi_2 y x \lambda^{-1}$, $\lambda = x^{-1} \tilde{y}$ and $y \cdot \tilde{y} \neq e$, because ξ is unbalanced. So

$$\xi \lambda^r = \xi_2 \underline{v} x^{-1} \tilde{y} \cdots x^{-1} \tilde{y} x^{-1} \underline{\tilde{y} x^{-1} \tilde{y}}, \quad v = y \cdot \tilde{y},$$

and $\tilde{y} \in \Gamma_1$. Similarly to the previous subsubcase we get $\|\xi_2\| = 2$, and $\xi_2 = \tilde{y}^{-1} x$, $v = \tilde{y}^{-1}$ and therefore $\xi = \tilde{y}^{-1} x y x \tilde{y}^{-1} x$, which again contradicts the assumption that ξ is unbalanced.

Subsubcase: $\xi \neq \xi_1 \lambda^{-1}$. We can write $\xi = \xi_1 x \xi_2$ and $\lambda = \xi_2^{-1} \tilde{x} \lambda_1$, where $x \cdot \tilde{x} \neq e$, $\|\xi_2\| = 0$ or $\|\xi_2\| = 1$, $\|\lambda_1\| = 1$ or $\|\lambda_1\| = 0$, and $\|\xi_1\| \geq 4$. Therefore

$$\xi \lambda^r = \xi_1 \underline{u} \lambda_1 \xi_2^{-1} \tilde{x} \lambda_1 \cdots \xi_2^{-1} \tilde{x} \lambda_1 \xi_2^{-1} \tilde{x} \lambda_1.$$

$\|\xi_1\| \geq 4$ implies that ξ starts with $\lambda^{-1} \lambda^{-1}$, which is impossible. This finishes the first case.

Case: ξ is of type (i, j) and λ is of type (i, i) . In this case there are no cancellations between ξ and λ , the length of λ is odd, and if $\lambda = \tilde{x}_1 \lambda_1 \tilde{x}_2$, then the product

$$\lambda^r = \tilde{x}_1 \lambda_1 u \lambda_1 u \cdots u \lambda_1 \tilde{x}_2, \quad u = \tilde{x}_2 \cdot \tilde{x}_1 \in \Gamma_i \setminus \{e\},$$

is reduced and $u \neq e$, because λ is cyclically reduced by assumption.

Subcase: $\|\lambda\| \geq \|\xi\| + 3$. Let $\lambda = \tilde{x}_1 \lambda_1 \tilde{x}_2$ and

$$\xi \lambda^r = \xi \tilde{x}_1 \lambda_1 u \lambda_1 u \cdots u \lambda_1 \tilde{x}_2.$$

For $\xi \lambda^r$ to be in S_Γ^c we must have $\xi = \tilde{x}_2^{-1} \xi_1$ and $\lambda_1 = \lambda_2 \xi_1^{-1}$,

$$\xi \lambda^r = \tilde{x}_2^{-1} \xi_1 \underline{\tilde{x}_1 \lambda_2} \xi_1^{-1} u \lambda_2 \xi_1^{-1} u \cdots u \lambda_2 \xi_1^{-1} \tilde{x}_2.$$

Thus $\lambda_2 = \lambda_3 \tilde{x}_1^{-1}$, $\lambda_3 = \lambda_3^{-1}$ and $u = \tilde{x}_1$. But $u = \tilde{x}_2 \cdot \tilde{x}_1$, hence $\tilde{x}_2 = e$, which is impossible.

Subcase: $\|\lambda\| = \|\xi\| + 1$. As in the previous subcase for $\lambda = \tilde{x}_1 \lambda_1 \tilde{x}_2$, $u = \tilde{x}_2 \cdot \tilde{x}_1 \neq e$ and $\xi = \tilde{x}_2^{-1} \xi_1$, $\|\xi_1\| = \|\lambda_1\|$, we have

$$\xi \lambda^r = \tilde{x}_2^{-1} \xi_1 \underline{\tilde{x}_1} \lambda_1 u \cdots u \lambda_1 \tilde{x}_2.$$

Hence $\xi_1 = \lambda_1^{-1}$, $\tilde{x}_1 = u^{-1}$ and so

$$\xi \lambda^r = \tilde{x}_2^{-1} \xi_1 \tilde{x}_1 \xi_1^{-1} \tilde{x}_1^{-1} \cdots \xi_1^{-1} \tilde{x}_1^{-1} \xi_1^{-1} \tilde{x}_2,$$

and we arrive at $\xi_1 = \xi_1^{-1}$, which is impossible since ξ is unbalanced and $\xi = \tilde{x}^{-1}\xi_1$.

Subcase: $5 \leq \|\lambda\| \leq \|\xi\| - 1$. As before $\lambda = \tilde{x}_1\lambda_1\tilde{x}_2$, $\xi = \tilde{x}_2^{-1}\xi_1$ and

$$\xi\lambda^r = \tilde{x}_2^{-1}\xi_1\tilde{x}_1\lambda_1u\cdots u\lambda_1\tilde{x}_2.$$

From $\|\lambda\| \leq \|\xi\| - 1$ we deduce $\xi_1 = \lambda_1^{-1}u^{-1}\xi_2$ and therefore

$$\xi\lambda^r = \tilde{x}_2^{-1}\lambda_1^{-1}u^{-1}\xi_2\tilde{x}_1\lambda_1u\cdots u\lambda_1u\lambda_1\tilde{x}_2.$$

Since $\|\lambda_1\| \geq 3$, if $\|\xi_2\| \geq 2$ or if $\|\xi_2\| = 1$ and $\xi_2 \in \Gamma_1$, then the word ξ is not unbalanced; so $\xi_2 \in \Gamma_2$ and $\lambda_1 = \lambda_2\tilde{x}_1^{-1}\xi_2^{-1}$ and

$$\xi\lambda^r = \tilde{x}_2^{-1}\xi_2\tilde{x}_1\lambda_2^{-1}u^{-1}\xi_2\tilde{x}_1\lambda_2\tilde{x}_1^{-1}\xi_2^{-1}u\cdots u\lambda_2\tilde{x}_1^{-1}\xi_2^{-1}u\lambda_2\tilde{x}_1^{-1}\xi_2^{-1}\tilde{x}_2.$$

This implies $\lambda_2 = \lambda_2^{-1}$ and $u = \tilde{x}_1$. Contradiction.

Subcase: $\|\lambda\| = 3$. Let $\lambda = \tilde{x}_1\tilde{y}\tilde{x}_2$, then

$$\xi\lambda^r = \xi\tilde{x}_1\tilde{y}u\tilde{y}u\cdots u\tilde{y}u\tilde{y}\tilde{x}_2.$$

Therefore the reduced form of ξ start with $\tilde{x}_2^{-1}\tilde{y}^{-1}u^{-1}\tilde{y}^{-1}u^{-1}$, and therefore cannot be unbalanced.

Case: ξ is of type (i, j) and λ is of type (j, j) . In this case there has to be a “full cancellation”: either the reduced form of λ^r starts with ξ^{-1} or the reduced form of ξ ends in the reduced form of λ^{-r} . For if this were not the case, then $\xi = \xi_1x\xi_2$ and $\lambda^r = \xi_2^{-1}\tilde{x}\lambda_1$ with $x \cdot \tilde{x} \neq e$, hence $\xi\lambda^r = \xi_1u\lambda_1$ is reduced with $u = x \cdot \tilde{x}$, and therefore $\xi\lambda^r$ has type (i, j) , and in particular it is not in S_{Γ}^c .

Subcase: $\|\lambda\| \geq \|\xi\| + 3$. Let $\xi = \xi_1x$, $\lambda = x^{-1}\xi_1^{-1}\lambda_1\tilde{x}_2$, where $x, \tilde{x}_2 \neq e$ and

$$\xi\lambda^r = \lambda_1u\xi_1^{-1}\lambda_1u\cdots\xi_1^{-1}\lambda_1\tilde{x}_2, \quad u = \tilde{x}_2 \cdot x^{-1} \in \Gamma_j \setminus \{e\}.$$

One sees that $\lambda_1 = \tilde{x}_2^{-1}\lambda_2$, $\lambda_2^{-1} = \lambda_2$, $u = \tilde{x}_2$, and, as usually, this implies $x^{-1} = e$.

Subcase: $\|\lambda\| = \|\xi\| + 1$. In this subcase $\xi = \xi_1x$ and $\lambda = x^{-1}\xi_1^{-1}\tilde{x}_2$ and

$$\xi\lambda^r = u\xi_1^{-1}u\xi_1^{-1}u\cdots u\xi_1^{-1}\tilde{x}_2.$$

Hence $u = \tilde{x}_2^{-1}$ and $\xi_1^{-1} = \xi_1$, which is a contradiction.

Subcase: $5 \leq \|\lambda\| \leq \|\xi\| - 1$. Let $\lambda = \tilde{x}_1\tilde{y}\lambda_1\tilde{x}_2$ and $\xi = \xi_1yu^{-1}\lambda_1^{-1}\tilde{y}^{-1}\tilde{x}_1^{-1}$, where $u = \tilde{x}_2 \cdot \tilde{x}_1$. Note that $\|\lambda_1\| \geq 2$. As we argued at the beginning of the case, there has to a full cancellation, and we must have $y \cdot \tilde{y} = e$, and therefore also $y \in \Gamma_2$ (otherwise ξ is not unbalanced). So

$$\xi\lambda^r = \xi_1 \cdot \lambda_1uy^{-1}\lambda_1u\cdots uy^{-1}\lambda_1\tilde{x}_2,$$

where the product is not necessarily reduced as the dot indicates. If $\|\xi_1\| > 0$, then the last letter of ξ_1 is in Γ_1 (since $y \in \Gamma_2$); this letter has to cancel with the first letter of λ_1 , but this implies that ξ is not unbalanced. Therefore $\|\xi_1\| = 0$ and

$$\xi\lambda^r = \lambda_1u\tilde{y}\lambda_1u\cdots u\tilde{y}\lambda_1\tilde{x}_2,$$

which implies $\lambda_1 = \tilde{x}_2^{-1}\lambda_2$, $\lambda_2 = \lambda_2^{-1}$, $u = \tilde{x}_2$ and again $\tilde{x}_1 = e$, which is impossible.

Subcase: $\|\lambda\| = 3$. Let $\lambda = \tilde{x}_1\tilde{y}\tilde{x}_2$, and $\xi = \xi_1z_0yu^{-1}\tilde{y}^{-1}\tilde{x}_1^{-1}$. Since ξ is unbalanced, either $y \cdot \tilde{y} \neq e$ or $z_0 \cdot u \neq e$, and therefore $\xi\lambda^r$ has type (i, j) and is not in S_{Γ}^c .

Case: both ξ and λ are of type (i, j) . In this case the product $\xi\lambda^r$ is reduced and has type (i, j) , and so is not in S_{Γ}^c . \square

Lemma 4.4. *Let $\xi \in \Gamma_1 * \Gamma_2$ be an unbalanced element of asymmetric type, $\|\xi\| \geq 6$. If $\gamma \notin S_{\Gamma}^c$ and $r \geq 4$, then $\xi\gamma^r \notin S_{\Gamma}^c$.*

Proof. The proof is again by contradiction: assume that we do have ξ, γ and r such that the assumptions hold and $\xi\gamma^r \in S_\Gamma^c$. Let $\gamma = \alpha\lambda\alpha^{-1}$ with λ being cyclically reduced, $\|\lambda\| \geq 2$, and $\xi\gamma^r = \beta z\beta^{-1}$ for some $z \in X_\Gamma$. First of all we note that it cannot happen that $\xi = \xi_1 x \xi_2$ and $\alpha = \xi_2^{-1} \tilde{x} \alpha_1$ for some $\xi_1, \xi_2, \alpha_1 \in \Gamma$ and some non-trivial multipliable $x, \tilde{x} \in X_\Gamma$ with $x \cdot \tilde{x} \neq e$. Indeed, if this were the case, then the product

$$\xi\gamma^r = \xi_1 u \alpha_1 \lambda \alpha_1^{-1} \tilde{x}^{-1} \xi_2, \quad u = x \cdot \tilde{x}$$

is reduced and therefore $\xi\gamma^r$ has asymmetric type, and cannot be an element in S_Γ^c . So either the product $\xi\alpha$ is reduced, or one of ξ, α completely cancels the other one. More formally, we have the following cases.

Case: $\xi\gamma^r$ is reduced. In this case the product $\xi\alpha\lambda^r\alpha^{-1}$ is reduced (here λ^r is understood as a single element, not as a product).

Subcase: $\|\xi\| \geq \|\alpha\|$. For $\xi\alpha\lambda^r\alpha^{-1}$ to be of the form $\beta z\beta^{-1}$ we must have $\xi = \alpha\xi_1$ and therefore $\xi\gamma^r = \alpha\xi_1\alpha\lambda^r\alpha^{-1} = \alpha(\xi_1\alpha\lambda^r)\alpha^{-1}$. Note that $\xi = \alpha\xi_1$ is asymmetric and unbalanced if and on if so is $\xi_1\alpha$. We therefore apply Lemma 4.3 to $\xi_1\alpha$ and λ .

Subcase: $\|\xi\| < \|\alpha\|$. Let $K \geq 0$ be the largest such that $\alpha = \xi^K \alpha_1$. Then $\xi\gamma^r = \xi \xi^K \alpha_1 \lambda^r \alpha_1^{-1} \xi^{-K} = \xi^K (\xi \alpha_1 \lambda^r \alpha_1^{-1}) \xi^{-K}$. The set S_Γ^c is conjugation invariant, therefore we need to consider $\xi \alpha_1 \lambda^r \alpha_1^{-1}$ only. We claim that $\|\alpha_1\| < \|\xi\|$. If this were not the case and $\|\alpha_1\| \geq \|\xi\|$, then for $\xi \alpha_1 \lambda^r \alpha_1^{-1}$ to be of the form $\beta z\beta^{-1}$ we must have $\alpha_1 = \xi \alpha_2$, contradicting the maximality of K . So $\|\alpha_1\| < \|\xi\|$ and we are under the conditions of the previous subcase with α_1 in the place of α .

Case: Full reduction.

Subcase: $\|\xi\| \geq \|\alpha\|$ and therefore $\xi = \xi_1 \alpha^{-1}$. In this subcase $\xi\gamma^r = \xi_1 \cdot \lambda^r \alpha^{-1}$, where the product is not necessarily reduced, and therefore $\alpha^{-1} \xi \gamma^r \alpha = \alpha^{-1} \xi_1 \cdot \lambda^r$, where $\alpha^{-1} \xi_1$ is asymmetric and unbalanced, because so is $\xi = \xi_1 \alpha^{-1}$. It remains to apply Lemma 4.3.

Subcase: $\|\xi\| < \|\alpha\|$ and therefore $\alpha = \xi^{-1} \alpha_1$. Let $K \geq 0$ be maximal such that $\alpha = \xi^{-K} \alpha_2$, then $\xi\gamma^r = \xi^{-K+1} \alpha_2 \lambda^r \alpha_2^{-1} \xi^K$. It is enough to show that $\xi^{-1} \alpha_2 \lambda^r \alpha_2^{-1} \notin S_\Gamma^c$. Since $\xi\gamma^r$ is assumed to be of the form $\beta z\beta^{-1}$ and by maximality of K we conclude that $\|\alpha_2\| \leq \|\xi\|$ and therefore can apply the previous subcase with α_2 in the role of α . \square

5. AUTOMATIC CONTINUITY FOR HOMOMORPHISMS INTO FREE PRODUCTS

In this section we shall finally prove Theorem 1.5. As we have mentioned earlier, the core of the argument is to show that there are no *surjective* homomorphisms $\phi : G \rightarrow \Gamma$ with *dense kernels*. Depending on the structure of Γ , we shall use somewhat different modifications of the Dudley's argument. One of the special cases will be $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2$. This case is, in fact, already known. More precisely, Rosendal [Ros11, Theorem 3.2] in a somewhat different terminology has proved

Theorem 5.1 (Rosendal). *For a completely metrizable topological group G and any homomorphism $\phi : G \rightarrow \mathbb{Z}_2 * \mathbb{Z}_2$ one of the two possibilities hold: either ϕ is continuous, or $\phi(G)$ is contained in one of the factors.*

In particular this implies that there does not exist a surjective homomorphism from a completely metrizable group onto $\mathbb{Z}_2 * \mathbb{Z}_2$ with a dense kernel. We shall need this corollary and for the purpose of completeness we give a proof of it using the setting of this paper. This argument is, in fact, a reformulation of Rosendal's original proof.

In the rest of the section G denotes a completely metrizable topological group.

Lemma 5.2. *Let $\phi : G \rightarrow \mathbb{Z}_2 * \mathbb{Z}_2$ be a surjective homomorphism. The kernel of ϕ cannot be dense in G .*

Proof. Suppose the kernel of ϕ were dense. By surjectivity of ϕ there is some $g \in G$ such that $\phi(g) = xy$, where x and y are the generators of the two factors in $\mathbb{Z}_2 * \mathbb{Z}_2$. Let K be the kernel of ϕ , and so gK is dense in G . We construct a sequence $(g_m)_{m=1}^\infty$ of elements in G such that $\phi(g_m) = xy$, and the limit

$$h_m = \lim_{n \rightarrow \infty} g_m (g_{m+1} (\cdots (g_{n-1} (g_n)^{r_{n-1}})^{r_{n-2}} \cdots)^{r_{m+1}})^{r_m}$$

exists, where $r_m = 2^m$. We claim that $\|\phi(h_m)^{r_{m-1}}\| \geq r_{m-1}$ for any $m \geq 2$ and $\phi(h_m) \in S_{\mathbb{Z}_2 * \mathbb{Z}_2}$. By continuity $h_m = g_m h_{m+1}^{r_m}$ and therefore $\phi(h_m) = xy \phi(h_{m+1})^{r_m}$. If $\phi(h_{m+1}) \in S_{\mathbb{Z}_2 * \mathbb{Z}_2}^c$, then $\phi(h_{m+1})$ has order 2 and therefore $\phi(h_{m+1})^{r_m} = e$. In this case $\|\phi(h_m)^{r_{m-1}}\| = (xy)^{r_{m-1}} = 2r_{m-1}$. If $\phi(h_{m+1}) \in S_{\mathbb{Z}_2 * \mathbb{Z}_2}$, then $\phi(h_{m+1}) = (xy)^q$ for some non-zero integer q , and therefore $\|\phi(h_m)^{r_{m-1}}\| = \|(xy)^{r_{m-1}(q^{r_m+1})}\| \geq r_{m-1}$. This proves the claim and therefore $\|\phi(h_1)\| \geq m$ for any m by the same argument as the end of Theorem 3.5. This contradiction proves the lemma. \square

Lemma 5.3. *Let $\phi : G \rightarrow \Gamma$, where $\Gamma = \Gamma_1 * \Gamma_2$, be a surjective homomorphism and assume that $\Gamma_1 * \Gamma_2 \neq \mathbb{Z}_2 * \mathbb{Z}_2$ and that $\Gamma_1 * \Gamma_2$ does not have elements of order p for a prime p . The kernel of ϕ cannot be dense in G .*

Proof. Suppose the kernel of ϕ were dense. By Lemma 2.6 and surjectivity of ϕ there is $g \in G$ such that $\phi(g) \notin S_\Gamma^c * S_\Gamma^c$. Using the density of the kernel, and hence the density of any coset, we construct a sequence $(g_m)_{m=1}^\infty$ in G such that $\phi(g_m) = \phi(g) \notin S_\Gamma^c \cdot S_\Gamma^c$ and the limit

$$h_m = \lim_{n \rightarrow \infty} g_m(g_{m+1}(\cdots(g_{n-1}(g_n)^{r_{n-1}})^{r_{n-2}} \cdots)^{r_{m+1}})^{r_m}$$

exists, where $r_m = p^{m+m|\phi(g)|}$. We claim that $\|\phi(h_m)^{r_{m-1}}\| \geq r_{m-1}$ for all $m \geq 2$. Indeed, if $\phi(h_{m-1}) \in S_\Gamma$, then this is true by the definition of the set S_Γ . If $\phi(h_m) \in S_\Gamma^c$, then $\phi(g_m) \notin S_\Gamma^c \cdot S_\Gamma^c$ implies $\phi(h_{m+1}) \in S_\Gamma$ and therefore $\|\phi(h_m)\| = \|\phi(g_m)\phi(h_{m+1})^{r_m}\| \geq r_m - \phi(g) \geq r_{m-1}$. Finally, r_{m-1} is a power of p and therefore $\|\phi(h_m)^{r_{m-1}}\| \geq \|\phi(h_m)\|$ by item (ii) of Lemma 2.2, which proves the claim. This inequalities imply $\|\phi(h_1)\| \geq m$ as before and the lemma follows. \square

Lemma 5.4. *Let $\phi : G \rightarrow \Gamma$, where $\Gamma = \Gamma_1 * \Gamma_2$, be a surjective homomorphism and assume there is an unbalanced $\xi \in \Gamma_1 * \Gamma_2$ of asymmetric type and length $\|\xi\| \geq 6$. The kernel of ϕ cannot be dense in G .*

Proof. Suppose the kernel of ϕ were dense. Let $\xi \in \Gamma_1 * \Gamma_2$ be an unbalanced element of asymmetric type and of length at least 6. By surjectivity of ϕ we can pick $g \in G$ such that $\phi(g) = \xi$. Using the density of the kernel construct a sequence $(g_m)_{m=1}^\infty$ such that $\phi(g_m) = \xi$ and the limit

$$h_m = \lim_{n \rightarrow \infty} g_m(g_{m+1}(\cdots(g_{n-1}(g_n)^{r_{n-1}})^{r_{n-2}} \cdots)^{r_{m+1}})^{r_m}$$

exists, where $r_m = m + m\|\xi\|$. Since ξ is of asymmetric type, unbalanced and $\|\xi\| \geq 6$, Lemma 4.2 implies $\xi \notin S_\Gamma^c \cdot S_\Gamma^c$. Since $\phi(h_m) = \xi\phi(h_{m+1})^{r_m}$, Lemma 4.4 then implies $\phi(h_m) \in S_\Gamma$ for all $m \geq 2$, and we again arrive at an absurd inequality $\|\phi(h_1)\| \geq m$ for all m . \square

Before the final spurt we would like to recall the classical Kurosh's subgroup theorem:

Theorem 5.5 (Kurosh). *Any subgroup Γ' of a free product $\Gamma = *_{a \in A} \Gamma_a$ can be written as $F(Z) * (*_{b \in B} \Gamma'_b)$, where $Z \subseteq \Gamma$ generates a free group $F(Z)$ and for each $b \in B$ there are $a \in A$ and $\gamma_b \in \Gamma$ such that $\Gamma'_b = \gamma_b \Lambda_b \gamma_b^{-1}$ and $\Lambda_b < \Gamma_a$.*

Theorem 5.6. *If $\phi : G \rightarrow \Gamma$ is a homomorphism from a completely metrizable group Γ into a discrete group $\Gamma = *_{a \in A} \Gamma_a$, then either ϕ is continuous or the image of ϕ is contained in one of the factors of Γ .*

Proof. Let K denote the kernel of ϕ . The closure of the kernel \bar{K} is also a completely metrizable group and we may consider the restriction of ϕ onto \bar{K} . If $\bar{K} = K$, then Theorem 3.7 implies ϕ is continuous and we assume that $\bar{K} \neq K$. In this case the group $\Gamma' = \phi(\bar{K})$ is non-trivial and we can apply Kurosh's subgroup theorem to write $\Gamma' = F(Z) * (*_{b \in B} \Gamma'_b)$. The homomorphism $\phi : \bar{K} \rightarrow \Gamma'$ is surjective and has a dense kernel. We claim that $\phi(\bar{K})$ is contained in a factor of Γ' ; in other words we claim that Γ' has only one factor. Suppose this is false. Pick two factors of Γ' and let π be a canonical projection from Γ' onto the free product of these two factors. The map $\pi \circ \phi : \bar{K} \rightarrow \pi(\Gamma')$ is surjective, has a dense kernel and its image is a free product of two nontrivial groups. By Lemma 5.2 the group $\pi(\Gamma')$ is not the infinite dihedral group and by Lemma 5.4 there are no unbalanced asymmetric words of length at least 6 in $\pi(\Gamma')$. But then there can be at most 4 elements of order bigger than 2 in each of the factors of $\pi(\Gamma')$, for if there were 5 such elements in say Γ_1 , then one could select three of them x_1, x_2, x_3 such that $x_i \neq x_j^{\pm 1}$ for $i \neq j$ and so $x_1 y x_2 y x_3 y$ is an asymmetric unbalanced element of length 6 for any non-trivial $y \in \Gamma_2$. Hence there must be a prime p such that neither Γ_1 nor Γ_2 have elements of order p and therefore Lemma 5.3 applies and gives a contradiction.

Thus $\phi(\bar{K})$ is contained in a factor of Γ' , which by Dudley's Theorem 1.1 cannot be the free factor, therefore $\phi(\bar{K})$ is contained in a factor of Γ , i.e., $\phi(\bar{K}) \subseteq \gamma \Gamma_{a_0} \gamma^{-1}$ for some $\gamma \in \Gamma$ and some $a_0 \in A$. We can write $\Gamma = *_{a \in A} \tilde{\Gamma}_a$, where $\tilde{\Gamma}_a = \gamma \Gamma_a \gamma^{-1}$, so there is no loss in generality to assume that $\gamma = e$.

Assume first that $\phi : G \rightarrow \Gamma$ is surjective. We claim that in this case $\Gamma = \Gamma_{a_0}$. Indeed, if $|A| \geq 2$ and there is $b \in A$ such that $b \neq a_0$, then pick $g \in G$ such that $\phi(g) \in \Gamma_b \setminus \{e\}$ and choose an $h \in \bar{K} \setminus K$. Since \bar{K} is a normal subgroup of G , we must have $\phi(ghg^{-1}) \in \Gamma_a$, but $\phi(g)\phi(h)\phi(g)^{-1} = x_b x_{a_0} x_b^{-1}$ for non-trivial $x_{a_0} \in \Gamma_{a_0}$ and $x_b \in \Gamma_b$. This is a contradiction.

So the theorem is proved under the assumption that ϕ is surjective. If ϕ is not surjective, let Γ' be its image. Using Kurosh's subgroup theorem and applying what we have proved for the now surjective $\phi : G \rightarrow \Gamma'$ we

derive that either ϕ is continuous (in which case it is also continuous viewed as a map $\phi : G \rightarrow \Gamma$) or the image of ϕ is contained in a factor of Γ' . But any factor of Γ' is contained up to a conjugation in a factor of Γ , except possibly for the free group factor $F(Z)$. But if the image of ϕ is a free group, then ϕ is continuous by Dudley's Theorem 1.1. \square

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