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Countable Borel equivalence relations

Chapter 1

First contact

1.1 Hi, my name is Cber.

Definition 1.1.1. An *equivalence relation* on a set X is a set $\mathsf{E} \subseteq X \times X$ such that for all $x, y, z \in X$

- $(x, x) \in \mathsf{E}$;
- $(x, y) \in \mathsf{E} \implies (y, x) \in \mathsf{E}$;
- $(x, y) \in \mathsf{E}$ and $(y, z) \in \mathsf{E} \implies (x, z) \in \mathsf{E}$.

When X is a standard Borel space, we say that E is Borel (resp. analytic) if it is a Borel (resp. analytic) subset of the product space $\mathsf{E} \subseteq X \times X$. Two points $x, y \in X$ are E-equivalent if $(x, y) \in \mathsf{E}$; we shall denote this by $x \mathsf{E} y$. An E-*equivalence class* of $x \in X$ is denoted by $[x]_{\mathsf{E}}$ and consists of all the points $y \in X$ that are equivalent to x:

$$[x]_{\mathsf{E}} = \{y \in X : x \mathsf{E} y\}.$$

More generally, for a subset $A \subseteq X$, $[A]_{\mathsf{E}}$ denotes the *saturation* of A:

$$[A]_{\mathsf{E}} = \{y \in X : x \mathsf{E} y \text{ for some } x \in A\}.$$

In this notation $[x]_{\mathsf{E}} = [\{x\}]_{\mathsf{E}}$.

We say that an equivalence relation E is *countable* if each E-equivalence class is countable; we say E is *finite* if so is any E-class. Countable Borel equivalence relations form the object of this notes, so we adopt an abbreviation *cber* to denote them.

Here are some examples of equivalence relations.

- Identity relation $\Delta \subseteq X \times X$, $\Delta = \{(x, x) : x \in X\}$ is the trivial example of an equivalence relation.
- Important class of equivalence relation comes from actions of Polish groups; these are called *orbit equivalence relations.* Let G be a Polish group acting in a Borel way on a standard Borel space X. Points $x, y \in X$ are orbit equivalent if they belong to the same orbit of the action:

  $$\mathsf{E}_X^G = \{(x, y) \in X \times X : Gx = Gy\}.$$

  Such an equivalence relation is always analytic. When G is countable, E_X^G is a cber.
- With any countable group G comes a particularly important action — the Bernoulli shift: $G \curvearrowright 2^G$ by $(gx)(f) = x(g^{-1}f)$ for all $g, f \in G$. In the case $G = \mathbb{Z}$ this action is the left shift on bi-infinite binary sequences.

- E_0 is a cber on $2^{\mathbb{N}}$ where two sequences are E_0-equivalent whenever they agree from some point on: $x \mathsf{E}_0 y$ if and only if there is $n \in \mathbb{N}$ such that $x(m) = y(m)$ for all $m \geq n$. An important homeomorphism of the Cantor space, called the *odometer*, is associated with E_0. Odometer is a map $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by "adding 1 to the sequence" in the following sense. If x starts with m ones and then comes a zero, $x = 1^m 0*$, then $\sigma(x)$ flips the first m ones to zeroes, the first 0 to 1, and agrees with x on the rest of indices, $\sigma(x) = 0^m 1*$. This rule defines σ on all of $2^{\mathbb{N}}$ except for the sequence of all ones. If $x = 1^\infty$, then $\sigma(x)$ is defined to be the sequence of all zeroes, $\sigma(1^\infty) = 0^\infty$. Exercise 1.1 encourages you to check that σ is indeed a homeomorphism of the Cantor space.

  The odometer, being a homeomorphism of $2^{\mathbb{N}}$, is a Borel automorphism of the Cantor space, and thus generates an action of $\mathbb{Z}$, so one may consider the orbit equivalence relation $\mathsf{E}^{\mathbb{Z}}_{2^{\mathbb{N}}}$ given by this action. It turns out that $\mathsf{E}^{\mathbb{Z}}_{2^{\mathbb{N}}}$ is "almost" equal to E_0; the only difference between $\mathsf{E}^{\mathbb{Z}}_{2^{\mathbb{N}}}$ and E_0 is that $\mathsf{E}^{\mathbb{Z}}_{2^{\mathbb{N}}}$ glues the E_0-class of 0^∞ and the E_0-class of 1^∞ into a single $\mathsf{E}^{\mathbb{Z}}_{2^{\mathbb{N}}}$-class. On the rest of the space they coincide. Exercise 1.2 offers you to check this statement.

- A slight variation of the previous example leads to the *tail equivalence relation* E_{t} on $2^{\mathbb{N}}$, where $x \mathsf{E}_{\mathrm{t}} y$ whenever x and y have the same "tail" — there exist $k_1, k_2 \in \mathbb{N}$ such that $x(k_1 + n) = y(k_2 + n)$ for all $n \in \mathbb{N}$. There is no canonical group action that realizes E_{t} (though as we shall see soon enough there is some group action, and in fact an action of $\mathbb{Z}$, that realizes E_{t} as an orbit equivalence relation), but E_{t} is an orbit equivalence relation of a *semigroup* action. Let $s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the left shift: $(sx)(n) = x(n + 1)$ for all $n \in \mathbb{N}$. The tail equivalence relation can then be described by noting that $x \mathsf{E}_{\mathrm{t}} y$ holds if and only if $s^{k_1}(x) = s^{k_2}(y)$ for some $k_1, k_2 \in \mathbb{N}$. Despite superficial similarities in their definitions, E_0 and E_{t} are quite different in some important aspects.

- *Turing equivalence relation* $\equiv_{\mathrm{T}}$ on $2^{\mathbb{N}}$ is defined by setting $x \equiv_{\mathrm{T}} y$ if x and y are Turing reducible to each other. Informally speaking, $x \in 2^{\mathbb{N}}$ is Turing reducible to $y \in 2^{\mathbb{N}}$ is there is a Turing machine (i.e., a computer program) that computes x if it is provided with an oracle y. Since there are only countably many computer programs (each program, after all, is just a finite string in a finite alphabet), Turing relation $\equiv_{\mathrm{T}}$ is countable. It is, in fact, a cber on $2^{\mathbb{N}}$.

- While E_0 and E_{t} are cbers, we would like to conclude this list with an example of an uncountable Borel equivalence relation E_1 defined on $\mathbb{R}^{\mathbb{N}}$. Its definition copies that of E_0 on $2^{\mathbb{N}}$. Two sequences of reals $x, y \in \mathbb{R}^{\mathbb{N}}$ are E_1-equivalent whenever there is $n \in \mathbb{N}$ such that $x(m) = y(m)$ for all $m \geq n$. Importance of E_1 lies in the fact that it cannot be realized as an orbit equivalence of a Polish group action (and, moreover, it cannot be even reduced to such a relation). This result is due to A. S. Kechris and A. Louveau. An interested reader is referred to [Hjo00, Theorem 8.2].

1.2 Feldman–Moore's Theorem

The goal of this section is to prove Feldman–Moore's Theorem, which claims that every cber arises as an orbit equivalence relation of a Borel action of a countable group. We begin by recalling Luzin–Novikov's Theorem, the proof of which can be found, for instance, in [Kec95, Theorem 18.10].

Theorem 1.2.1 (Luzin–Novikov)**.** *Let $P \subseteq X \times Y$ be a Borel subset of the product of standard Borel spaces. Suppose every section P_x, $x \in X$, is countable. Projection $\mathrm{proj}_X P$ is Borel and, moreover, P can be written as a union $\bigcup_{n \in \mathbb{N}} P_n$, where each P_n is a graph of a Borel function.*

An easy corollary of the above is that every countable-to-one Borel function admits a Borel inverse.

Corollary 1.2.2. *Let $f : X \to Y$ be a Borel countable-to-one function between standard Borel spaces. The image $f(X)$ is Borel in Y, and there exists a Borel function $g : f(X) \to X$ such that $f \circ g(y) = y$ for all $y \in f(X)$.*

Proof. Exercise 1.3. □

Theorem 1.2.3 (Feldman–Moore [FM77]). *Let $\mathsf{E} \subseteq X \times X$ be a cber on a standard Borel space X. There exists a Borel action of a countable group $H \curvearrowright X$ such that $\mathsf{E} = \mathsf{E}_X^H$. Moreover, one can additionally assume that H is generated by elements $\{h_i : i \in \mathbb{N}\}$ such that*

(i) $h_i^2 = \mathrm{id}$ *for all* $i \in \mathbb{N}$;

(ii) $x \mathsf{E} y$ *if and only if* $x = y$ *or* $h_i x = y$ *for some* $i \in \mathbb{N}$.

Proof. Since $\mathsf{E} \subseteq X \times X$ has countable sections, Luzin–Novikov's Theorem 1.2.1 applies, and we may write $\mathsf{E} = \bigcup_n P_n$, where each $P_n \subseteq X \times X$ is a graph of a Borel function: if $(x, y) \in P_n$ and $(x, y') \in P_n$, then $y = y'$. We use the notation P_n^{-1} to denote the set

$$\big\{(x, y) \in X \times X : (y, x) \in P_n\big\}.$$

Since E is symmetric, $\mathsf{E} = \bigcup_n P_n^{-1}$. Let $P_{m,n} = P_m \cap P_n^{-1}$, and note that $\mathsf{E} = \bigcup_{m,n} P_{m,n}$. As X is assumed to be a standard Borel space, there is no loss in generality to assume that $X = [0, 1]$. Let $I, J \subseteq [0, 1]$ be a pair of disjoint closed intervals with rational endpoints; note that $I \times J \subseteq (X \times X) \setminus \Delta$. For $m, n \in \mathbb{N}$ consider the set $Z = Z(m, n, I, J)$ given by

$$Z = \mathrm{proj}_1\big\{(x, y) \in P_{m,n} : x \in I \text{ and } y \in J\big\}.$$

With each such Z we associate a map $h = h(m, n, I, J) : Z \to X$ whose graph (see Figure 1.1) is the set $\{(x, y) \in P_{m,n} : x \in Z\}$.

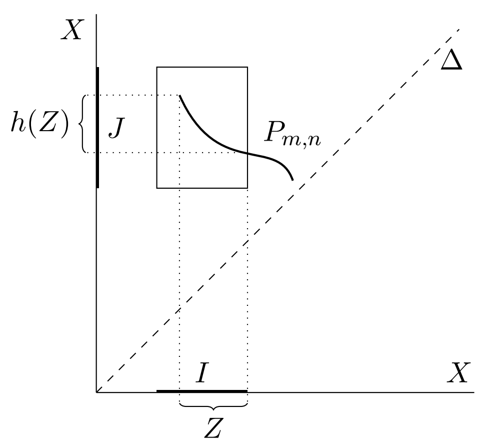

Figure 1.1: Definition of the function $h(I, J, m, n)$.

Note that

- $h(Z) \cap Z = \varnothing$, because $Z \subseteq I$ and $h(Z) \subseteq J$;
- h is injective, because if $(x, y) \in Z$ and $(x', y) \in Z$, then $(y, x), (y, x') \in P_n$, and so $x = x'$ since P_n is a graph of a function;

- $\big(x, h(x)\big) \in \mathsf{E}$ for all $x \in Z$, since $P_{m,n} \subseteq \mathsf{E}$ and $\big(x, h(x)\big) \in P_{m,n}$.

One may therefore extend h first to a Borel bijection $h : Z \cup h(Z) \to Z \cup h(Z)$ by setting $h\big(h(x)\big) = x$ for all $x \in Z$, and then to an automorphism $h : X \to X$ by declaring $h(x) = x$ for all $x \in X \setminus \big(Z \cup h(Z)\big)$.

The function $h = h(m, n, I, J)$ depends on four parameters. Since I and J are assumed to have rational endpoints, there are only countably many such automorphisms h, and we may thus enumerate them as $\{h_i : i \in \mathbb{N}\}$. Let $H \leq \operatorname{Aut} X$ be the group generated by $\{h_i\}_{i\in\mathbb{N}}$. We claim that H satisfies the conclusion of the theorem. We need to check that $x \mathsf{E} y$ implies $x = y$ or $h_i x = y$ for some $i \in \mathbb{N}$. Let $x, y \in X$ be such that $x \mathsf{E} y$ and $x \neq y$. Since $\mathsf{E} = \bigcup_{m,n} P_{m,n}$, there is some $m, n \in \mathbb{N}$ such that $(x, y) \in P_{m,n}$. The assumption $x \neq y$ allows us to pick disjoint $I, J \subseteq [0, 1]$ with rational endpoints such that $x \in I$ and $y \in J$. By definition of $h = h(m, n, I, J)$ one has $hx = y$. We are therefore done, as $h(m, n, I, J) = h_i$ for some $i \in \mathbb{N}$. □

Corollary 1.2.4. *An immediate corollary of Feldman-Moore's Theorem is that a saturation of a Borel set is always Borel, since* $[A]_{\mathsf{E}} = \bigcup_{h\in H} hA$*, where the action* $H \curvearrowright X$ *realizes* E.

An interesting question is whether any E can be realized as an orbit equivalence relation E_X^H for a *free* action $H \curvearrowright X$. A crude obstruction to this is to have equivalence classes of different cardinalities, e.g., if E has a finite and an infinite class, then E obviously cannot be realized by a free action. But even if every E-class is infinite, it need not admit a realization as an orbit equivalence of a free action. The following example is due to Scott Adams [Ada88]. For a countable group H let $\mathrm{Free}(2^H)$ denote the "free part" of the Bernoulli shift:

$$\mathrm{Free}\big(2^H\big) = \big\{x \in 2^H : hx \neq x \text{ for all } h \in H\big\}.$$

Let F_2 be the free group on two generators and let E be an equivalence relation on the disjoint union $\mathrm{Free}(2^{\mathbb{Z}}) \sqcup \mathrm{Free}(2^{F_2})$ given by $x \mathsf{E} y$ if and only if either $x, y \in \mathrm{Free}(2^{\mathbb{Z}})$ and $x \mathsf{E}^{\mathbb{Z}}_{2^{\mathbb{Z}}} y$, or $x, y \in \mathrm{Free}(2^{F_2})$ and $x \mathsf{E}^{F_2}_{2^{F_2}} y$. Adams [Ada88] showed that E is not given by a free action of any countable group.

1.3 Smooth equivalence relations

Definition 1.3.1. We say that a Borel equivalence relation $\mathsf{E} \subseteq X \times X$ is *smooth* if there exists a Borel function $f : X \to Y$ into some standard Borel space Y such that $x \mathsf{E} y$ holds if and only if $f(x) = f(y)$. In terms of reducibility of Borel equivalence relations, E is smooth if it reduces to the equality relation on Y.

A Borel set $A \subseteq X$ is said to be a Borel *transversal* for an equivalence relation E if A intersects every E-class in exactly one point: $\big|[x]_{\mathsf{E}} \cap A\big| = 1$ for all $x \in X$. A Borel function $s : X \to X$ is said to be a Borel *selector* for E if $x \mathsf{E} s(x)$ for all $x \in X$ and $x \mathsf{E} y$ implies $s(x) = s(y)$ for all $x, y \in X$. In other words, a selector is a function that assigns to every $x \in X$ a distinguished element from its E-class.

Proposition 1.3.2. *Let* E *be a cber on* X*. The following are equivalent:*

(i) E *is smooth;*

(ii) E *admits a Borel transversal;*

(iii) E *admits a Borel selector.*

Proof. (i) ⇒ (ii) Let $f : X \to Y$ be a reduction of E to the equality on Y. The function f is countable-to-one, thus by Corollary 1.2.2 it admits a Borel inverse $g : f(X) \to X$. The set $g \circ f(X)$ is a Borel transversal for E.

(ii) ⇒ (iii) Let $A \subseteq X$ be a Borel transversal for X. Set $s : X \to X$ by defining its graph to be

$$\mathrm{graph}(s) = \big\{(x, y) \in \mathsf{E} : y \in A\big\}.$$

Since $\operatorname{graph}(s)$ is Borel, so is the function s itself, which is easily seen to be a selector.

(iii) $\Rightarrow$ (i) A Borel selector $s : X \to X$ witnesses smoothness as $x \mathsf{E} y$ if and only if $s(x) = s(y)$. □

Equivalence between (ii) and (iii) is valid for all (not necessarily countable) Borel equivalence relation. Indeed, the implication (ii) $\Rightarrow$ (iii) did not use countability of E, and to see (iii) $\Rightarrow$ (ii) note that $\{x : s(x) = x\}$ is a Borel transversal for E whenever $s : X \to X$ is a Borel selector. But in general smoothness is a strictly weaker condition than admitting a Borel selector. Here is an example. Let $X \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be a Borel set such that $\operatorname{proj}_1 X = \mathbb{N}^{\mathbb{N}}$ and yet X does not have a Borel uniformization, i.e., there is no Borel $P \subseteq X$ which is a graph of a function such that $\operatorname{proj}_1 P = \mathbb{N}^{\mathbb{N}}$; such a set X exists by [Kec95, Exercise 18.17]. We define E on X by declaring $x \mathsf{E} y$ whenever $\operatorname{proj}_1(x) = \operatorname{proj}_1(y)$. Obviously, $\operatorname{proj}_1 : X \to \mathbb{N}^{\mathbb{N}}$ witnesses smoothness of E, but E does not admit a Borel selector, for that would mean that X has a Borel uniformization.

The simplest example of a smooth cber is provided by the following proposition.

Proposition 1.3.3. *If E is a Borel equivalence relation on X with only countably many E-classes, then E is smooth.*

Proof. Pick a representative from each E-class and let T be the set of these representatives. Since T is countable, it is a Borel transversal for E, hence E is smooth by Proposition 1.3.2 (note that only implication (i) $\Rightarrow$ (ii) used that E is countable). □

Example 1.3.4. Let $\mathbb{Z} \curvearrowright \mathbb{R}$ be the action generated by the shift $x \mapsto x + 1$. This actions generates a smooth equivalence relations, and the unit interval $[0, 1)$ is a Borel transversal for the action.

A more interesting example of a Borel equivalence relation admitting a Borel selector is as follows. Let G be a Polish group and let $H \leq G$ be a closed subgroup. Consider the natural action $H \curvearrowright G$ by multiplication from the left. E_G^H-equivalence classes are precisely the right cosets Hg. The orbit equivalence relation given by this action is Borel (why?). A theorem of Jacques Dixmier (see [Kec95, Theorem 12.17]) states that E_G^H admits a Borel selector.

Definition 1.3.5. Let E be a cber on X. A Borel subset $A \subseteq X$ is *smooth* if the restriction $\mathsf{E} \cap A \times A$ of the equivalence relation E onto A is smooth. We let $\mathscr{W}$ to denote the family of all smooth Borel subsets of X; $\mathscr{W}$ is called the *wandering ideal*.

The following proposition shows that $\mathscr{W}$ is indeed a σ-ideal of Borel sets.

Proposition 1.3.6. *Let A, B, and A_n, $n \in \mathbb{N}$, be Borel subsets of X.*

(i) If $A \in \mathscr{W}$ and $B \subseteq A$, then $B \in \mathscr{W}$.

(ii) If $A \in \mathscr{W}$, then $[A]_{\mathsf{E}} \in \mathscr{W}$.

(iii) If $A_n \in \mathscr{W}$ for all $n \in \mathbb{N}$, then $\bigcup_n A_n \in \mathscr{W}$.

Proof. (i) Let $f : A \to Y$ be a map such that $x \mathsf{E} y \iff f(x) = f(y)$ for all $x, y \in A$. The restriction $f|_B$ witnesses smoothness of B.

(ii) Let $T \subseteq A$ be a Borel transversal for $\mathsf{E} \cap A \times A$. The same set T is also a Borel transversal for $\mathsf{E} \cap [A]_{\mathsf{E}} \times [A]_{\mathsf{E}}$. Thus $[A]_{\mathsf{E}}$ is smooth by Proposition 1.3.2(ii).

(iii) By item (i) it is enough to show that $\left[\bigcup_n A_n\right]_{\mathsf{E}}$ is smooth. Since $\left[\bigcup_n A_n\right]_{\mathsf{E}} = \bigcup_n [A_n]_{\mathsf{E}}$, and because of item (ii), we may assume without loss of generality that each A_n is E-invariant. Let $\tilde{A}_n$ be the "disjointification" of A_n: $\tilde{A}_0 = A_0$ and $\tilde{A}_n = A_n \setminus \bigcup_{k<n} A_k$ for all $n \in \mathbb{N}$. Note that $\tilde{A}_n$ are also E-invariant, and

$$\bigsqcup_n \tilde{A}_n = \bigcup_n A_n.$$

Since $\tilde{A}_n \subseteq A_n$, each $\tilde{A}_n \in \mathscr{W}$ by item (i). Let $f_n : \tilde{A}_n \to Y_n$ witness smoothness of $\mathsf{E} \cap \tilde{A}_n \times \tilde{A}_n$, where Y_n is a standard Borel space. Let $f : \bigsqcup_n \tilde{A}_n \to \bigsqcup_n Y_n$ be defined by $f(x) = f_n(x)$ whenever $x \in \tilde{A}_n$; here $\bigsqcup_n Y_n$ is endowed with the union Borel structure. Since $\tilde{A}_n$ are E-invariant and pairwise disjoint, it is evident that $x \,\mathsf{E}\, y \iff f(x) = f(y)$ for all $x, y \in \bigsqcup_n \tilde{A}_n = \bigcup_n A_n$. □

The wandering ideal will play a role similar to the ideal of null sets in measure theory — once we are able to prove the desired result modulo a smooth set, it will usually be easy to modify the argument to work everywhere.

1.4 Decomposition into a finite and aperiodic parts

Definition 1.4.1. A subset $A \subseteq X$ is said to be E-invariant if it is equal to its own saturation: $A = [A]_{\mathsf{E}}$. In other words, A is E-invariant if $x \in A$ and $x \,\mathsf{E}\, y$ imply $y \in A$.

Proposition 1.4.2. *Let E be a cber on X. There is a partition of X into E-invariant Borel pieces*

$$X = X_\infty \sqcup \bigsqcup_{n=1}^{\infty} X_n$$

such that X_n, $n \in \mathbb{N} \cup \{\infty\}$, consists of all the classes of cardinality n: if $x \in X_n$, then $\big|[x]_{\mathsf{E}}\big| = n$. Such a decomposition is unique.

Proof. Uniqueness of the decomposition is evident, we just need to check that sets X_n are necessarily Borel. By Feldman–Moore's Theorem 1.2.3 we may pick an action $H \curvearrowright X$ of a countable group such that $\mathsf{E} = \mathsf{E}_X^H$; let $H = \{h_i : i \in \mathbb{N}\}$. The set X_n, $n \in \mathbb{N}$, is then given by

$$X_n = \big\{x \in X : \exists k_1, \ldots, k_n \in \mathbb{N} \text{ such that } h_{k_i}x \neq h_{k_j}x \text{ for all } 1 \leq i, j \leq n,\ i \neq j, \text{ and}$$
$$\text{for any } l \in \mathbb{N} \text{ there exists } i \leq n \text{ for which } h_l x = h_{k_i}x\big\}.$$

Sets X_n, $n \in \mathbb{N}$, are therefore Borel, and hence so is $X_\infty = X \setminus \bigcup_{n=1}^{\infty} X_n$. □

Definition 1.4.3. Recall that a countable equivalence relation E is finite if each E-class is finite, i.e., if $X_\infty = \varnothing$ in the decomposition above. We say that E is *aperiodic* if each E-class is infinite, i.e., $X_\infty = X$.

For most of the questions we are interested in these notes, finite equivalence relations will be trivial for the following reason.

Proposition 1.4.4. *Any finite Borel equivalence relation is smooth.*

Proof. Let E be a finite Borel equivalence relation on a standard Borel space X. We may assume that $X = [0, 1]$. Pick an action $H \curvearrowright X$ of a countable group such that $\mathsf{E} = \mathsf{E}_X^H$, $H = \{h_i : i \in \mathbb{N}\}$. Consider the set $A \subseteq X$ given by

$$A = \{x \in X : x \leq h_n x \text{ for all } n \in \mathbb{N}\}.$$

Since each E-class is finite, every E-class has a minimal element, and so A is a Borel transversal for E, hence E is smooth by Proposition 1.3.2. □

1.5 Full groups

Two important algebraic objects attached to every equivalence relation are its full and partial full groups.

Definition 1.5.1. Let E be a cber on X. A *full group* of E is denoted by $[\mathsf{E}]$ and consists of all Borel automorphisms of X that preserve E:

$$[\mathsf{E}] = \big\{ f : X \to X \;\big|\; f \text{ is a Borel bijection and } x \,\mathsf{E}\, f(x) \text{ for all } x \in X \big\}.$$

A *partial full group* of E, denoted by $[\![\mathsf{E}]\!]$, consists of bijections between Borel subsets of X which preserve E:

$$[\![\mathsf{E}]\!] = \big\{ f : A \to B \;\big|\; A, B \subseteq X \text{ are Borel, } f \text{ is a bijection, and } x \,\mathsf{E}\, f(x) \text{ for all } x \in A \big\}.$$

For an element $f \in [\![\mathsf{E}]\!]$, $f : A \to B$, we use $\mathrm{dom}(f) = A$ to denote the domain of f, and $\mathrm{ran}(f) = B$ denotes its range.

Definition 1.5.2. Let E be a cber on X. Given two Borel sets $A, B \subseteq X$, we say that A and B are *equidecomposable* if there exists $f \in [\![\mathsf{E}]\!]$ such that $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) = B$. We denote equidecomposability by $A \underset{\mathsf{E}}{\sim} B$, or just by $A \sim B$.

Exercise 1.4 offers you to check that $\sim$ is an equivalence relation. The following proposition explains the choice of the term "equidecomposable."

Proposition 1.5.3. *Suppose that $\mathsf{E} = \mathsf{E}_X^H$ is realized as an orbit equivalence of a Borel action of a countable group $H \curvearrowright X$. Let $H = \{h_n : n \in \mathbb{N}\}$ be an enumeration of H. Borel sets $A, B \subseteq X$ are equidecomposable if and only if there are partitions $A = \bigsqcup_{n=0}^{\infty} A_n$ and $B = \bigsqcup_{n=0}^{\infty} B_n$ into Borel pieces (some of which may be empty) such that $h_n(A_n) = B_n$ for all $n \in \mathbb{N}$.*

Proof. One direction is immediate: if $A = \bigsqcup A_n$ and $B = \bigsqcup B_n$ are decomposed as above, we may set $f : A \to B$ to be given by $f(x) = h_n x$ whenever $x \in A_n$; this map witnesses $A \sim B$. We now prove the other direction.

Suppose $A \sim B$, let $f : A \to B$ be a function in $[\![\mathsf{E}]\!]$ witnessing this. Since $x \,\mathsf{E}\, f(x)$ for all $x \in X$ and $\mathsf{E} = \mathsf{E}_X^H$, for each $x \in X$ there exists some $n \in \mathbb{N}$ such that $f(x) = h_n x$. Let $N : A \to \mathbb{N}$ be the function that chooses minimal such index:

$$N(x) = \min\{n \in \mathbb{N} : f(x) = h_n x\}.$$

It is easy to see that N is Borel, and we may set $A_n = N^{-1}(n)$. □

When equivalence relation is smooth, we have a simple necessary and sufficient condition for sets A and B to be equidecomposable.

Proposition 1.5.4. *Let E be a smooth cber on X, and let $A, B \subseteq X$ be Borel sets. A and B are equidecomposable if and only if*

$$\big|[x]_{\mathsf{E}} \cap A\big| = \big|[x]_{\mathsf{E}} \cap B\big| \quad \text{for all } x \in X. \tag{1.1}$$

Proof. Necessity of the condition is obvious and is valid regardless of whether E is smooth as any $f : A \to B$ witnessing equidecomposability gives a bijection between $[x]_{\mathsf{E}} \cap A \to [x]_{\mathsf{E}} \cap B$ for all $x \in X$. We prove sufficiency of this condition.

Let $A, B \subseteq X$ be Borel sets satisfying (1.1). Let $T \subseteq X$ be a Borel transversal for E, pick a realization $\mathsf{E} = \mathsf{E}_X^H$ and an enumeration $H = \{h_n : n \in \mathbb{N}\}$. Consider the function $N : X \to \mathbb{N}$ which assigns to x the first $n \in \mathbb{N}$ such that $h_n x \in T$:

$$N(x) = \min\{n \in \mathbb{N} : h_n x \in T\}.$$

The function N is Borel. Let now $M_A : A \to \mathbb{N}$ be given by

$$M_A(x) = \bigl|\bigl\{h_k^{-1} \circ h_{N(x)}(x) : k \leq N(x)\bigr\} \cap A\bigr|.$$

Here is a less cryptic explanation of this formula. The function M_A enumerates points of A within each E-class; in other words, $M_A : [x]_{\mathsf{E}} \to \mathbb{N}$ is an injection with its image being an initial segment of $\mathbb{N}$; in particular, when $A \cap [x]_{\mathsf{E}}$ is infinite, $M_A : [x]_{\mathsf{E}} \to \mathbb{N}$ is, in fact, a bijection.

The function $M_B : B \to \mathbb{N}$ can be defined in a similar way, and the condition on sets A and B ensures that $M_A\bigl([x]_{\mathsf{E}} \cap A\bigr) = M_B\bigl([x]_{\mathsf{E}} \cap B\bigr)$ for any $x \in X$. We are now ready to define $f : A \to B$, $f \in [\![\mathsf{E}]\!]$, by declaring

$$f(x) = y \text{ whenever } x \mathsf{E} y \text{ and } M_A(x) = M_B(y).$$ □

The converse to Proposition 1.5.4 is also true: If $A \sim B$ holds for all $A, B \subseteq X$ satisfying (1.1), then E must necessarily be smooth. This result is due to Achim Ditzen, Alexandr S. Kechris, Sławomir Solecki, and Stevo Todorcevic [KST99, Theorem 1.1]. The proof is currently beyond our techniques, but we shall soon develop the necessary tools.

While any cber is generated by an action of a countable group, smooth relations are generated by dynamically very simple actions of $\mathbb{Z}$. We shall later introduce the notion of a hyperfinite relation, and the following proposition will imply that any smooth countable relation must be hyperfinite.

Proposition 1.5.5. *Let E be a cber on X, and suppose that all E-classes have the same cardinality. Let T be a Borel transversal for E.*

(i) If $\bigl|[x]_{\mathsf{E}}\bigr| = \infty$ for all $x \in X$, then there exists $f \in [\mathsf{E}]$ such that

$$X = \bigsqcup_{n \in \mathbb{Z}} f^n(T).$$

(ii) If $\bigl|[x]_{\mathsf{E}}\bigr| = n$, $n \in \mathbb{N}$, for all $x \in X$, then there exists $f \in [\mathsf{E}]$ such that $f^n = \mathrm{id}$ and

$$X = \bigsqcup_{k=0}^{n-1} f^k(T).$$

Proof. (i) Let $\mathsf{E} = \mathsf{E}_X^H$ be given by an action of a countable group, $H = \{h_n : n \in \mathbb{N}\}$. Set $T_0 = T$ and $f_0 : T \to T_0$ to be the identity map. We construct inductively Borel sets T_n and Borel bijections $f_n : T \to T_n$, $f_n \in [\![\mathsf{E}]\!]$ as follows. Suppose T_i, $0 \leq i \leq n$, have been constructed. Let $N : T \to \mathbb{N}$ be given by

$$N(x) = \min\Bigl\{n \in \mathbb{N} : h_n x \not\in \bigcup_{i=0}^{n} T_i\Bigr\}.$$

Note that the set, of which minimum is taken, is non-empty as $\bigl|[x]_{\mathsf{E}}\bigr| = \infty$ by assumption. The function N is Borel, and we set

$$T_{n+1} = \bigl\{h_{N(x)}x : x \in T\bigr\} \quad \text{and} \quad f_{n+1}(x) = h_{N(x)}x.$$

It is straightforward to check that T_{n+1} is Borel, $T \sim T_{n+1}$ via f_{n+1}, and $X = \bigsqcup_{n \in \mathbb{N}} T_n$. By reenumerating T_n and f_n with $\mathbb{Z}$ being the index set, we may assume that we have a partition

$$X = \bigsqcup_{n \in \mathbb{Z}} \tilde{T}_n \quad \text{and bijections} \quad \tilde{f}_n : T \to \tilde{T}_n,\ \tilde{f}_n \in [\![\mathsf{E}]\!], n \in \mathbb{Z},$$

such that $\tilde{T}_0 = T$ and $\tilde{f}_0 = \mathrm{id}$. One may now define the desired automorphism $f : X \to X$ by setting

$$f(x) = \tilde{f}_{n+1} \circ \tilde{f}_n^{-1}(x) \quad \text{whenever } x \in \tilde{T}_n.$$

(ii) The proof of this item is similar and is requested in Exercise 1.5. □

Corollary 1.5.6. *If* E *is a smooth cber on* X*, then there exists a Borel action* $\mathbb{Z} \curvearrowright X$ *such that* $\mathsf{E} = \mathsf{E}_X^{\mathbb{Z}}$.

Proof. By Proposition 1.4.2 we may decompose $X = \bigsqcup_{n\in\mathbb{N}\cup\{\infty\}} X_n$ into Borel pieces, such that all classes in $\mathsf{E}|_{X_n}$ consist of n-elements. We may now apply Proposition 1.5.5 to each $\mathsf{E}|_{X_n}$ separately and get $f_n \in \big[\mathsf{E}|_{X_n}\big]$ such that for all $x, y \in X_n$ one has $x \mathsf{E} y$ if and only if $f_n^k(x) = y$ for some $k \in \mathbb{Z}$. Define $f : X \to X$ by setting $f(x) = f_n(x)$ whenever $x \in X_n$. Evidently $x \mathsf{E} y \iff f^k(x) = y$ for some $k \in \mathbb{Z}$ folds for all $x, y \in X$. Thus $\mathsf{E} = \mathsf{E}_X^{\mathbb{Z}}$, where the action $\mathbb{Z} \curvearrowright X$ is determined by the generator $f \in [\mathsf{E}]$. □

1.6 Invariant measures

Definition 1.6.1. Let E be a cber on X. A measure μ on X is said to be E-*invariant* if $\mu(A) = \mu(B)$ for all equidecomposable $A \sim B$. If H is a countable group acting on X, we say that μ is *H-invariant* if $\mu(hA) = \mu(A)$ for all Borel $A \subseteq X$ and all $h \in H$. A measure is *ergodic* if for any E-invariant Borel subset $A \subseteq X$ one has either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

We say that a measure μ is *finite* if $\mu(X) < \infty$, and μ is a *probability* measure if $\mu(X) = 1$.

Proposition 1.6.2. *Let* E *be realized as* E_X^H *for a Borel action* $H \curvearrowright X$ *of a countable group, and let* μ *be a measure on* X*. The measure* μ *is* E*-invariant if and only if it is* H*-invariant.*

Proof. Suppose first that μ is E-invariant. Pick a Borel set $A \subseteq X$, and $h \in H$. Since $h : A \to hA$ witnesses $A \sim hA$, we get $\mu\big(hA\big) = A$, and so μ is H-invariant.

Let now μ be H-invariant, and pick Borel sets $A, B \subseteq X$ such that $A \sim B$. By Proposition 1.5.3 we may decompose $A = \bigsqcup_{n\in\mathbb{N}} A_n$ and $B = \bigsqcup_{n\in\mathbb{N}} B_n$ into Borel pieces such hat $h_n A_n = B_n$, where $H = \{h_n : n \in \mathbb{N}\}$ is an enumeration of H. By H-invariance of μ, $\mu(A_n) = \mu(B_n)$ for all $n \in \mathbb{N}$, whence $\mu(A) = \mu(B)$ by σ-additivity. So μ is E-invariant. □

Example 1.6.3. Consider the orbit equivalence relation $\mathsf{E}_{2^{\mathbb{N}}}^{\mathbb{Z}}$ given by the odometer map $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Let μ_0 be the measure on $\{0, 1\}$ given by $\mu_0(0) = \mu_0(1) = 1/2$, and let μ be the *Bernoulli measure* on $2^{\mathbb{N}}$, that is the product of measures μ_0 on each copy of $\{0, 1\}$. By Proposition 1.6.2, to show that μ is $\mathsf{E}_{2^{\mathbb{N}}}^{\mathbb{Z}}$-invariant is equivalent to checking that it is invariant under the action of $\mathbb{Z}$, which, of course, is enough to check on the generator. Indeed, μ is invariant under the odometer, and the details are requested in Exercise 1.6.

Since E_0 differs from $\mathsf{E}_{2^{\mathbb{N}}}^{\mathbb{Z}}$ only on the set of μ-measure zero (Exercise 1.2), μ is also E_0-invariant.

Proposition 1.6.4. *Smooth aperiodic cbers don't have finite invariant measures.*

Proof. Suppose towards a contradiction that $\mathsf{E} \subseteq X \times X$ is smooth and aperiodic, and that μ is a finite E-invariant measure on X. Pick a Borel transversal T for E and apply Proposition 1.5.5(i) to get $f \in [\mathsf{E}]$ such that $X = \bigsqcup_{n\in\mathbb{Z}} f^n(T)$. We thus have

$$\infty > \mu(X) = \sum_{n\in\mathbb{Z}} \mu\big(f^n(T)\big) = \sum_{n\in\mathbb{Z}} \mu(T) = \infty \cdot \mu(T) \implies \mu(T) = 0.$$

But $\mu(T) = 0$ implies $\mu(X) = 0$. □

In fact, one may immediately strengthen the above proposition as follows.

Corollary 1.6.5. *Let* E *be a cber on* X*. If* $A \in \mathscr{W}$*, then* $\mu(A) = 0$ *for any finite* E*-invariant measure* μ *on* X*.*

Proof. It is enough to show that $\mu\big([A]_{\mathsf{E}}\big) = 0$ for any smooth $A \subseteq X$. By Proposition 1.3.6, the set $[A]_{\mathsf{E}}$ is smooth. Suppose μ is a finite E-invariant measure on X such that $\mu\big([A]_{\mathsf{E}}\big) \neq 0$. The restriction of μ onto $[A]_{\mathsf{E}}$ is then an $\mathsf{E}|_{[A]_{\mathsf{E}}}$-invariant finite measure, contradicting Proposition 1.6.4. □

Corollary 1.6.6. *Since we have shown in Example 1.6.3 that* E_0 *admits a finite invariant measure, we may conclude that* E_0 *is not smooth.*

1.7 Spaces of invariant measures

Theorem 1.7.1. *Let* H *be a countable group. There exists a compact metric space* U *and a continuous action* $H \curvearrowright U$ *such that for any Borel action* $H \curvearrowright X$ *on a standard Borel space there exists a Borel equivariant injection* $\xi : X \to U$*.*

Proof. Let U be the unit ball of $\ell^\infty(H)$ endowed with the weak$*$ topology when ℓ^∞ is viewed as the dual of $\ell^1(H)$. By Alaoglu's Theorem U is a compact Polish space. We let H act on U by permuting the coordinates: $(hx)(g) = x(h^{-1}g)$ for all $h, g \in H$ and $x \in U$. It is easy to see that this action is continuous.

Let now $H \curvearrowright X$ be a Borel action of H on a standard Borel space. Without loss of generality we may assume that $X = [0,1]$. Let $\xi : X \to U$ be given by $\big(\xi(x)\big)(g) = g^{-1}x \in [0,1]$. This map is Borel and for all $g, h \in H$ one has

$$\big(h\xi(x)\big)(g) = \big(\xi(x)\big)(h^{-1}g) = \big(h^{-1}g\big)^{-1}x = g^{-1}hx = \big(\xi(hx)\big)(g).$$

Thus ξ is a Borel embedding of $H \curvearrowright X$ into $H \curvearrowright U$. □

Let X be a compact Polish space, and let $\mathtt{MEAS}(X)$ denote the set of Borel probability measures on X. It is a standard fact in functional analysis that $\mathtt{MEAS}(X)$ is a convex compact subset in the weak$*$ topology of the dual to the space of continuous functions on X (see Appendix A).

Suppose now that we have a countable group H acting continuously on a compact metrizable X. A neighborhood of $\mu \in \mathtt{MEAS}(X)$ in the weak$*$ topology is parametrized by $\epsilon > 0$ and a finite family of functions $f_1, \ldots, f_n \in C(X)$, and is given by

$$U(\mu; \epsilon, f_1, \ldots, f_n) = \Big\{\nu \in \mathtt{MEAS}(X) : \Big|\int f_i \, d\mu - \int f_i \, d\nu\Big| < \epsilon \text{ for all } i \leq n\Big\}.$$

Let $\mathtt{INV} = \mathtt{INV}(H \curvearrowright X) \subseteq \mathtt{MEAS}(X)$ denote the set of H-invariant probability measures on X,

$$\mathtt{INV} = \big\{\mu \in \mathtt{MEAS}(X) : \mu(hA) = \mu(A) \text{ for all Borel } A \subseteq X \text{ and all } h \in H\big\}.$$

Since the actions is assumed to be continuous, $\mathtt{INV}$ is closed in the weak$*$ topology. It is easy to check that $\mathtt{INV}$ is a convex subset of $\mathtt{MEAS}(X)$.

Finally, let $\mathtt{EINV} = \mathtt{EINV}(H \curvearrowright X) \subseteq \mathtt{INV}(H \curvearrowright X)$ denote the set of ergodic H-invariant probability measures on X, i.e., $\mu \in \mathtt{EINV}$ if and only if $\mu \in \mathtt{INV}$ and $\mu(A) \in \{0,1\}$ for any Borel H-invariant set $A \subseteq X$.

Proposition 1.7.2. *Let* X *be a compact metrizable space, and let* $H \curvearrowright X$ *be a continuous action of a countable group. Ergodic* H*-invariant measures are precisely the extreme points of the set of all* H*-invariant measures:* $\operatorname{ext} \mathtt{INV} = \mathtt{EINV}$.

Proof. Let μ be an ergodic measure on X, and assume towards a contradiction that μ is not an extreme point in $\mathtt{INV}$. We may therefore represent $\mu = p\mu_1 + (1-p)\mu_2$ for some $\mu_1, \mu_2 \in \mathtt{INV}$ and some $p \in (0,1)$. We may further decompose $\mu_2 = \nu_1 + \nu_2$, $\nu_1, \nu_2 \in \mathtt{MEAS}(X)$, into an absolutely continuous part $\nu_1 \ll \mu_1$, and an orthogonal part $\nu_2 \perp \mu_1$. Since such a decomposition is unique, it is easy to see that ν_i are H-invariant (but typically not probability measures). One may now decompose $X = X_1 \sqcup X_2$ into H-invariant Borel pieces such that $\nu_i(X_i) = \nu_i(X)$. Since

$$\mu(X_1) = p\mu_1(X_1) + (1-p)\nu_1(X_1) \quad \text{and} \quad \mu(X_2) = (1-p)\nu_2(X),$$

and since $\mu_1(X_1) = \mu_1(X) \neq 0$, for μ to be ergodic we need to have $\nu_2 = 0$, whence $\mu = p\mu_1 + (1-p)\nu_1$, where $\nu_1 \ll \mu_1$. By performing the same argument with roles of μ_1 and ν_1 interchanged we get that $\mu_1 \sim \nu_1$, so there is a strictly positive function $f \in L^1(X, \mu_1)$ such that for all $A \subseteq X$

$$\nu_1(A) = \int_A f \, d\mu_1$$

and the function f is moreover H-invariant. If f is not essentially constant, there is $r \in \mathbb{R}^{>0}$ such that for sets

$$X_{\leq r} = \{x \in X : f(x) \leq r\} \quad \text{and} \quad X_{>r} = \{x \in X : f(x) > r\}$$

we have $\mu_1(X_{\geq r}) \neq 0 \neq \mu_1(X_{>r})$. Note that both $X_{\geq r}$ and $X_{<r}$ are H-invariant, which contradicts ergodicity of μ.

For the other direction, if μ is an extreme point of $\mathtt{INV}$, but not ergodic, then there is a decomposition $X = X_1 \sqcup X_2$ into H-invariant pieces such that $\mu(X_1) \cdot \mu(X_2) \neq 0$. Set $\mu_i(A) = \mu(A \cap X_i)/\mu(X_i)$, $i = 1, 2$, and note that $\mu = p\mu_1 + (1-p)\mu_2$, where $p = \mu(X_1)$, so μ is not an extreme point. □

Since the set of extreme points in a compact metrizable convex set is necessarily a G_δ subset, we may conclude that $\mathtt{EINV}$ is a G_δ subset of $\mathtt{INV}$. We may summarize all the above into the following statement.

Theorem 1.7.3. *If $H \curvearrowright X$ is a continuous action of a countable group on a compact metric space, then the topology on the set $\mathtt{INV}$ of H-invariant measures on X generated by neighborhoods of the form*

$$U(\mu; \epsilon, f_1, \ldots, f_n) = \left\{\nu \in \mathtt{INV} : \left|\int f_i \, d\nu - \int f_i \, d\mu\right| < \epsilon \text{ for all } i \leq n\right\}$$

is a compact Polish topology. The Borel structure on $\mathtt{INV}$ is the smallest σ-algebra which makes measurable all maps of the form

$$\mathtt{INV} \ni \mu \mapsto \mu(A) \in \mathbb{R},$$

where A is a Borel subset of X. The set $\mathtt{EINV}$ of ergodic H-invariant measures is a G_δ subset of $\mathtt{INV}$.

Proof. We need to explain only the statement about the Borel structure on $\mathtt{INV}$. For this see [Kec95, Theorem 17.24]. □

Definition 1.7.4. For a cber E on X we let $\mathtt{INV}(\mathsf{E})$ to denote the set of all E-invariant probability measures on X, and $\mathtt{EINV}(\mathsf{E})$ will denote the set of ergodic E-invariant probability measures.

Corollary 1.7.5. *Let E be a cber on a standard Borel space X. Endow $\mathtt{INV}(\mathsf{E})$ with the σ-algebra generated by maps $\mathtt{INV}(\mathsf{E}) \ni \mu \mapsto \mu(A)$, $A \subseteq X$ is Borel. The space $\mathtt{INV}(\mathsf{E})$ with this σ-algebra is a standard Borel space and the set $\mathtt{EINV}(\mathsf{E})$ of E-ergodic measures is a Borel subset of $\mathtt{INV}(\mathsf{E})$.*

Proof. Let E be generated by a Borel action $H \curvearrowright X$. In view of Proposition 1.6.2 we have $\mathtt{INV}(\mathsf{E}) = \mathtt{INV}(H \curvearrowright X)$ and $\mathtt{EINV}(\mathsf{E}) = \mathtt{EINV}(H \curvearrowright X)$. By Theorem 1.7.1, there is a universal continuous action $H \curvearrowright U$ on a compact space U, so the action $H \curvearrowright X$ admits a Borel embedding into $H \curvearrowright U$. We may assume for notational simplicity that $X \subseteq U$. By Theorem 1.7.3, $\mathtt{INV}(H \curvearrowright U)$ is the standard Borel space with the Borel algebra generated by maps $\mu \mapsto \mu(A)$. In particular, the set

$$Z = \big\{\mu \in \mathtt{INV}(H \curvearrowright U) : \mu(X) = 1\big\} \text{ is Borel.}$$

But clearly $\mathtt{INV}(\mathsf{E}) = \mathtt{INV}(H \curvearrowright U) \cap Z$ and $\mathtt{EINV}(\mathsf{E}) = \mathtt{EINV}(H \curvearrowright U) \cap Z$, and the corollary follows. □

1.8 Vanishing Marker Sequence

Definition 1.8.1. Let E be a cber on X. A set $A \subseteq X$ is said to be a *complete section* if A intersects each E-class: $[A]_{\mathsf{E}} = X$. A sequence $(S_n)_{n\in\mathbb{N}}$ of subsets of X is a *vanishing sequence of markers* if each S_n is a complete section, $S_n \supseteq S_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcap_n S_n = \varnothing$.

Lemma 1.8.2. *Let* E *be an aperiodic smooth cber. There exists a vanishing sequence of markers for* E.

Proof. Exercise 1.9. □

Proposition 1.8.3 (Slaman–Steel [SS88])**.** *Every aperiodic cber admits a vanishing sequence of markers.*

Proof. Let E be an aperiodic cber on X, which we may assume to be the Cantor set $X = 2^{\mathbb{N}}$. Pick a Borel action $H \curvearrowright X$ which realizes E: $\mathsf{E}_X^H = \mathsf{E}$. Consider the map $\zeta : X \to X$ that assigns to $x \in X$ the minimal element of $\overline{[x]_{\mathsf{E}}}$ in the lexicographical ordering (note that lexicographical ordering coincides with the ordering inherited from $[0,1]$, when $2^{\mathbb{N}}$ is realized as the “middle third” Cantor subset of the unit interval; therefore any closed subset has a minimal element). Somewhat more formally, ζ can be defined as follows. Let $\{z_n : n \in \mathbb{N}\}$ be a countable dense set in $2^{\mathbb{N}}$. The function ζ is defined by

$$\zeta(x) = y \iff \forall n\, h_n x \geq y \text{ and } \forall n\, (z_n > y \implies \exists m\, h_m x < z_n).$$

Clearly ζ is Borel. Note that the set $T = \{x : \zeta(x) = x\}$ intersects every E-class in at most one point, so we may partition $X = X_0 \sqcup X_1$, where $X_0 = [T]_{\mathsf{E}}$, $X_1 = X \setminus X_0$, and the restriction of E on X_0 is smooth. Lemma 1.8.2 guarantees that we may construct a vanishing sequence of markers (S_n^0) for the restriction of E onto X_0. If we construct a marker sequence (S_n^1) for E on X_1, then $(S_n^0 \cup S_n^1)$ will be a vanishing marker sequence for the whole E. So there is no loss in generality to assume that $X_1 = X$, or, in other words, that $\zeta(x) \neq x$ for all $x \in X$, and therefore sets

$$S_n = \{x \in X : x(i) = \zeta(x)(i) \text{ for all } i \leq n\}$$

have empty intersection. They are also nested, and each S_n is evidently a complete section. □

Corollary 1.8.4. *Let* E *be an aperiodic cber on* X*. There exists a partition of* $X = A \sqcup B$ *into two Borel complete sections.*

Proof. Let $(S_n)_{n=0}^{\infty}$ be a vanishing sequence of markers for E. Note that by Exercise 1.8, $\big|[x]_{\mathsf{E}} \cap S_n\big| = \infty$ for any $x \in X$ and all $n \in \mathbb{N}$. Consider the function $N : X \to \mathbb{N}$ given by

$$N(x) = \min\big\{n : [x]_{\mathsf{E}} \cap S_n \text{ is a proper subset of } [x]_{\mathsf{E}}\big\}.$$

We may therefore put $A = \{x : x \in S_{N(x)}\}$ and $B = X \setminus A$. □

In fact, one can do better than this as it is always possible to partition the phase space of an aperiodic cber into two equidecomposable parts.

Proposition 1.8.5. *For any aperiodic* E *on* X *there exists a Borel partition of* $X = A \sqcup B$ *into equidecomposable pieces* $A \sim B$.

Proof. By Feldman–Moore's Theorem 1.2.3, we may take a group action $H \curvearrowright X$ such that $\mathsf{E} = \mathsf{E}_X^H$ and moreover there are $h_n \in H$ such that $h_n^2 = \mathrm{id}$ for all $n \in \mathbb{N}$ and $x \mathsf{E} y$ if and only if $x = y$ or $h_n x = y$ for some $n \in \mathbb{N}$. Let $A_n \subseteq X$ be such that $h_n(A_n) \cap A_n = \varnothing$ and $h_n x = x$ for all $x \in X \setminus \big(A_n \cup h_n A_n\big)$. Define sets $\tilde{A}_n \subseteq A_n$ by induction as follows. Set $\tilde{A}_0 = A_0$ and let

$$\tilde{A}_{n+1} = \Big\{x \in A_{n+1} : x, h_{n+1}x \notin \bigcup_{i \leq n} \big(\tilde{A}_n \cup h_n \tilde{A}_n\big)\Big\}.$$

Evidently sets $\tilde{A}_n$ are pairwise disjoint. Set $A = \bigsqcup_n \tilde{A}_n$, and define $f : A \to X$ by putting $f(x) = h_n x$ for $x \in \tilde{A}_n$.

First we claim that f is injective. Pick distinct $x, y \in A$. If $x, y \in \tilde{A}_n$ for some n, then clearly $f(x) \neq f(y)$, so assume that $x \in \tilde{A}_m$, $y \in \tilde{A}_n$ and let us suppose for definiteness that $m < n$. By definition of $\tilde{A}_n$, $h_n y \notin h_m \tilde{A}_m$, hence $f(y) = h_n y \neq h_m x = f(x)$.

Next we note that $f(A) \cap A = \varnothing$. To see that pick some $x, y \in A$, $x \in \tilde{A}_m$ and $y \in \tilde{A}_n$ for some $m \neq n$. If $m < n$, then $h_n y \notin \tilde{A}_m$ by the definition of $\tilde{A}_n$; if $n < m$, then $x \notin h_n \tilde{A}_n$ by the definition of $\tilde{A}_m$. In either case $x \neq f(y)$.

Since $f \in [\![\mathsf{E}]\!]$, $A \sim f(A)$. We finally claim that $\big|[x]_{\mathsf{E}} \setminus (A \cup f(A))\big| \leq 1$ for all $x \in X$, i.e., we assert that $A \cup f(A)$ omits at most one point from each E-class. Suppose towards a contradiction that we have $x, y \in X$ such that $x \mathsf{E} y$ and $x, y \notin A \cup f(A)$. By the choice of h_n, one has $x \in A_n$ and $h_n x = y$ for some $n \in \mathbb{N}$. Clearly $x, h_n x \notin \bigcup_{i<n}(\tilde{A}_i \cup h_i \tilde{A}_i)$, thus $x \in \tilde{A}_n$, hence $x \in A$; contradiction.

The set $T = X \setminus \big(A \cup f(A)\big)$ is therefore a Borel transversal for the restriction of E onto $[T]_{\mathsf{E}}$. Let $X_1 = X \setminus [T]_{\mathsf{E}}$, and set $A' = A \cap X_1$, $B' = f(A) \cap X_1$. The partition $X_1 = A' \sqcup B'$ satisfies the conclusion of the proposition for the restriction of E onto X_1. Since $\mathsf{E}|_{[T]_{\mathsf{E}}}$ is smooth and aperiodic, it is easy to find $A'' \subseteq [T]_{\mathsf{E}}$ and $B'' \subseteq [T]_{\mathsf{E}}$ such that $A'' \sim B''$ and $[T]_{\mathsf{E}} = A'' \sqcup B''$. Finally, the partition $X = (A' \cup A'') \sqcup (B' \cup B'')$ is as desired. □

Exercises

Exercise 1.1. Check that the odometer map $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined in Section 1.1 is a homeomorphism. Show that σ is *minimal*, i.e., show that every orbit of σ is dense in $2^{\mathbb{N}}$.

Exercise 1.2. Let $\mathsf{E}_{2^{\mathbb{N}}}^{\mathbb{Z}}$ be an orbit equivalence relation on $2^{\mathbb{N}}$ given by the odometer map $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Show that

$$x \,\mathsf{E}_{2^{\mathbb{N}}}^{\mathbb{Z}}\, y \iff (x \,\mathsf{E}_0\, y) \text{ or } \big(x \,\mathsf{E}_0\, 0^\infty \text{ and } y \,\mathsf{E}_0\, 1^\infty\big) \text{ or } \big(x \,\mathsf{E}_0\, 1^\infty \text{ and } y \,\mathsf{E}_0\, 0^\infty\big).$$

In plain words, show that $\mathsf{E}_{2^{\mathbb{N}}}^{\mathbb{Z}}$ glues two E_0-classes, namely those of 0^∞ and 1^∞, into a single $\mathsf{E}_{2^{\mathbb{N}}}^{\mathbb{Z}}$-class, and is otherwise identical to E_0.

Exercise 1.3. Prove Corollary 1.2.2.

Exercise 1.4. Check that for any cber E equidecomposability $\underset{\mathsf{E}}{\sim}$ is an equivalence relation.

Exercise 1.5. Prove item (ii) of Proposition 1.5.5.

Exercise 1.6. Show that the Bernoulli measure on $2^{\mathbb{N}}$ is invariant under the odometer map.

▸ *Exercise* 1.7. Show that the Bernoulli measure is the *unique* probability invariant measure for the odometer on $2^{\mathbb{N}}$.

Exercise 1.8. Let $(S_n)_{n=0}^{\infty}$ be a vanishing sequence of markers for an aperiodic cber on X. Show that $\big|[x]_{\mathsf{E}} \cap S_n\big| = \infty$ for all $x \in X$ and all $n \in \mathbb{N}$.

Exercise 1.9. Using Proposition 1.5.5, show that every aperiodic smooth cber admits a vanishing sequence of markers.

Intermezzo I

Glimm–Effros dichotomy

We would like now to prove a very important result in the theory of countable Borel equivalence relations. It turns out that E_0 is, in a certain sense, the simplest non-smooth cber.

Definition I.1. Let E and E' be cbers on standard Borel spaces X and X' respectively. We say that E *embeds* into E', and denote this by $\mathsf{E} \sqsubseteq \mathsf{E}'$, if there exists a Borel injection $\zeta : X \to X'$ such that

$$x \,\mathsf{E}\, y \iff \zeta(x) \;\mathsf{E}'\; \zeta(y).$$

When X and X' are topological spaces we say that E *continuously embeds* into E', denoted by $\mathsf{E} \sqsubseteq_c \mathsf{E}'$, if the map ζ above can be chosen to be continuous.

The following is a Theorem 3.4.5 in [BK96].

Theorem I.2. *Let $H \curvearrowright X$ be a continuous action of a countable group on a Polish space; put $\mathsf{E} = \mathsf{E}_X^H$. If there is a dense orbit and $\mathsf{E} \subseteq X \times X$ is meager, then $\mathsf{E}_0 \sqsubseteq_c \mathsf{E}$.*

Proof. Since E is meager in X, we may find a countable family of open sets $O_n \subseteq X$ such that for each $n \in \mathbb{N}$

- O_n is dense in $X \times X$;
- $O_n \supseteq O_{n+1}$;
- $O_0 = X \setminus \Delta$, where $\Delta = \{(x, x) : x \in X\}$;
- $\mathsf{E} \subseteq X \setminus \bigcap_n O_n$, i.e., if $(x, y) \in \bigcap O_n$, then $\neg(x \,\mathsf{E}\, y)$.

Since E is symmetric, we may assume that each O_n is symmetric as well. We pick a complete metric d on X and construct a scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ of open subsets of X and elements $h_n \in H$, $n \geq 1$, such that for all $s, t \in 2^{<\mathbb{N}}$

1. $\overline{U}_{s^\frown i} \subseteq U_s$ for $i = 0, 1$;
2. $\operatorname{diam} U_s \leq 2^{-|s|}$, for $s \neq \varnothing$;
3. $U_{s^\frown 0} \cap U_{s^\frown 1} = \varnothing$;
4. $U_s \times U_t \subseteq O_n$ whenever $|s| = n = |t|$ and $s(n-1) \neq t(n-1)$;

5. $\zeta^s(U_{0^n}) = U_s$, where $|s| = n$ and $\zeta^s = \zeta_1^s \circ \cdots \circ \zeta_n^s$,

$$\zeta_j^s = \begin{cases} \mathrm{id} & \text{if } s(j) = 0, \\ h_j & \text{if } s(j) = 1. \end{cases}$$

To clarify item (5), for $n = 2$ it says that $U_{01} = h_2 U_{00}$, $U_{10} = h_1 U_{00}$, and $U_{11} = h_1 \circ h_2 U_{00}$ (see Figure I.1). The order in which h_i's are applied is important as generally $h_1 \circ h_2 \neq h_2 \circ h_1$.

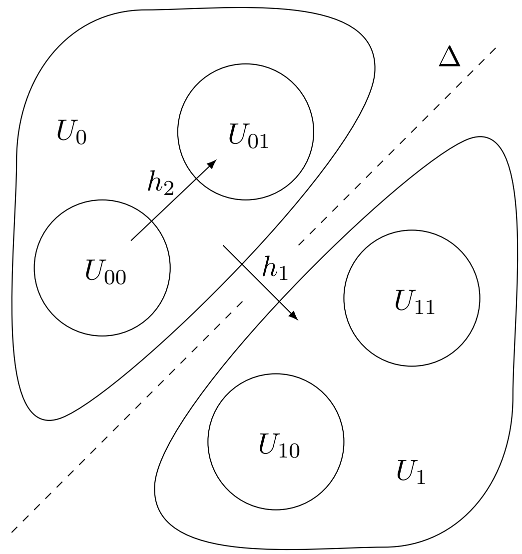

Figure I.1: Constructing sets U_s, $s \in 2^{<\mathbb{N}}$.

First let us finish the proof under the assumption that such a scheme has been constructed. Items (1-2) ensure that for each $x \in 2^{\mathbb{N}}$ the intersection $\bigcap_n U_{x|_n}$ consists of exactly one point, so we may define a map $\xi : 2^{\mathbb{N}} \to X$ by setting $\xi(x)$ to be such that $\bigcap_n U_{x|_n} = \{\xi(x)\}$. The function ξ is continuous, and it is injective by (3). We claim that it witnesses $\mathsf{E}_0 \sqsubseteq_c \mathsf{E}$. Indeed, if $x, y \in 2^{\mathbb{N}}$ are not E_0-equivalent, then there are infinitely many n such that $x(n) \neq y(n)$, hence by (4) one has $\big(\xi(x), \xi(y)\big) \in O_n$ for all n such that $x(n-1) \neq y(n-1)$, but $O_n \supseteq O_{n+1}$, so $\big(\xi(x), \xi(y)\big) \in \bigcap_n O_n$; thus $\big(\xi(x), \xi(y)\big) \notin \mathsf{E}$. So $\neg(x \,\mathsf{E}_0\, y) \implies \neg\big(\xi(x) \,\mathsf{E}\, \xi(y)\big)$.

For the other direction, suppose $x \,\mathsf{E}_0\, y$ and let n_0 be such that $x(k) = y(k)$ for all $k > n_0$. By item (5) for each n we have elements $\zeta^{x|_n} \in H$ and $\zeta^{y|_n} \in H$ such that $U_{x|_n} = \zeta^{x|_n}(U_{0^n})$ and $U_{y|_n} = \zeta^{y|_n}(U_{0^n})$. Put $\alpha_n = \zeta^{y|_n} \circ (\zeta^{x|_n})^{-1}$. The definition of ζ^s and the fact that $x(n) = y(n)$ for all $n > n_0$ implies that $\alpha_n = \alpha_{n_0}$ and $\alpha_{n_0}(U_{x|_n}) = U_{y|_n}$ for all $n \geq n_0$. We therefore have

$$\big\{\alpha_{n_0}\xi(x)\big\} = \alpha_{n_0} \bigcap_n U_{x|_n} = \bigcap_{n \geq n_0} \alpha_{n_0} U_{x|_n} = \bigcap_{n \geq n_0} U_{y|_n} = \big\{\xi(y)\big\}.$$

So $\alpha_{n_0}\xi(x) = \xi(y)$, which proves that $\xi(x) \,\mathsf{E}\, \xi(y)$. We have thus shown that $x \,\mathsf{E}_0\, y \iff \xi(x) \,\mathsf{E}\, \xi(y)$, as claimed.

It remains to construct sets $(U_s)_{s \in 2^{\mathbb{N}}}$ and elements $h_n \in H$. For $U_\varnothing$ we may take $X \setminus \Delta$. By assumption there is $z \in X$ such that $[z]_{\mathsf{E}}$ is dense in X. We may therefore find distinct $x_0, x_1 \in [z]_{\mathsf{E}}$ such that $(x_0, x_1) \in O_1$. Let $h_1 \in H$ be such that $h_1 x_0 = x_1$ and let U_0 and U_1 be small enough neighborhoods of x_0 and x_1

such that $h_1 U_0 = U_1$ and $U_0 \times U_1 \subseteq O_1$. By further shrinking U_0 and U_1 if necessary we may assume that $\overline{U}_0 \subseteq U_\varnothing$, $\overline{U}_1 \subseteq U_\varnothing$, and $\operatorname{diam} U_i < 1/2$.

At the next step we want to find distinct $x_{00}, x_{01} \in U_0 \cap [z]_{\mathsf{E}}$ such that for $x_{10} = h_1 x_{00}$ and $x_{11} = h_1 x_{01}$ one has $(x_s, x_t) \in O_2$, whenever $|s| = 2 = |t|$ and $s(1) \neq t(1)$. If no such x_{00}, x_{01} exist, then

$$\begin{aligned}([z]_{\mathsf{E}} \cap U_0)^2 \subseteq \Delta \cup (\mathrm{id} \times \mathrm{id})(X \setminus O_2) \cup (h_1^{-1} \times \mathrm{id})(X \setminus O_2)\\ \cup (\mathrm{id} \times h_1^{-1})(X \setminus O_2) \cup (h_1^{-1} \times h_1^{-1})(X \setminus O_2).\end{aligned}$$

Since the right hand side of this inclusion is closed, we may add closure to the left hand side, which violates the assumption that $X \setminus O_2$ is nowhere dense. Once x_{00}, x_{01} are picked, we may find $h_2 \in H$ such that $x_{01} = h_2 x_{00}$, and set $x_{10} = h_1 x_{00}$ and $x_{11} = h_1 x_{01} = h_1 \circ h_2 x_{00}$. Since each pair $(x_s, x_t) \in O_2$, when $s(1) \neq t(1)$, we can find neighborhood $U_{00} \subseteq U_0$ around x_{00}, such that $U_s \times U_t \subseteq O_2$, $|s| = 2 = |t|$ and $s(1) \neq t(1)$, where $U_s = \zeta^s(U_s)$. By shrinking U_{00} further if necessary we may assume that items (1), (2), and (3) are satisfied. This finishes the second step of the construction, which can be continued in a similar fashion. □

Before we prove the main result of this chapter, we need one more definition. Let $H \curvearrowright X$ be a continuous action of a countable group on a Polish space. A point $x \in X$ is said to be *recurrent* if there are h_n, $n \in \mathbb{N}$, such that $h_n x \to x$ and $h_n x \neq x$ for all $n \in \mathbb{N}$. We may now derive following [Nad98, 9.10] what is called the Glimm–Effros Dichotomy for cbers.

Theorem I.3 (Glimm–Effros Dichotomy)**.** *For any cber* E *exactly one of the following two possibilities holds.*

1. E *is smooth.*
2. $\mathsf{E}_0 \sqsubseteq \mathsf{E}$.

Proof. Let E be realized by a Borel action $H \curvearrowright X$. Since the periodic part of E is always smooth, we may assume without loss of generality that E is aperiodic. We may also find a Polish topology on X such that the action $H \curvearrowright X$ is continuous.

Our first claim is that if there is a recurrent point $x \in X$, then $\mathsf{E}_0 \sqsubseteq \mathsf{E}$. Let $x_0 \in X$ be recurrent, set $Y = \overline{[x_0]_{\mathsf{E}}}$, note that Y is an E-invariant Borel set and consider the restriction of E onto Y. The orbit of x_0 is clearly dense in Y. So by Theorem I.2 above, if we can show that $\mathsf{E}|_Y \subseteq Y \times Y$ is meager, then $\mathsf{E}_0 \sqsubseteq \mathsf{E}|_Y \sqsubseteq \mathsf{E}$, and the claim will be proved. Suppose towards a contradiction that $\mathsf{E}|_Y$ is not meager in $Y \times Y$, hence it must be comeager in some non-empty open subset of $Y \times Y$, i.e., there are non-empty open sets $U_1, U_2 \subseteq Y$ such that

$$\forall^*(y_1, y_2) \in U_1 \times U_2 \quad (y_1, y_2) \in \mathsf{E}|_Y.$$

By Kuratowski–Ulam this is equivalent to

$$\forall^* y_1 \in U_1 \, \forall^* y_2 \in U_2 \quad (y_1, y_2) \in \mathsf{E}|_Y.$$

In particular, there is some $y_1 \in U_1$ such that $\forall^* y_2 \in U_2$ one has $(y_1, y_2) \in \mathsf{E}|_Y$. By the set $\{y_2 \in Y : (y_1, y_2) \in \mathsf{E}|_Y\}$ is countable, so we have a countable set that is comeager in U_2, whence there must be an isolated point $z \in U_2$. In other words, there is an open subset $V \subseteq X$ such that $V \cap Y = \{z\}$. Since $[x_0]_{\mathsf{E}}$ is dense in Y, there is $h \in H$ such that $hx_0 = z$, hence $x_0 = h^{-1}z$ is also an isolated point in Y. But an isolated point cannot be recurrent, for if $h_n x_0 \to x_0$ and $h_n x_0 \neq x_0$ for all n, then $h_n x_0 \notin h^{-1}V$, but $x_0 \in h^{-1}V$. This contradiction shows that $\mathsf{E}|_Y$ is meager in $Y \times Y$ and the claim is proved.

So, we may assume that no point in X is recurrent and we shall prove that in this case E is smooth. Pick a compatible metric d on X and set

$$F_n = \bigcap_{h \in H} \big\{x \in X : hx = x \text{ or } d(hx, x) \geq 1/n\big\}.$$

In words the set F_n consists of those points $x \in X$ such that each $h \in H$ either fixes x or moves it by at least $1/n$. We claim that $X = \bigcup_n F_n$. More precisely, any $x \in X \setminus \bigcup_n F_n$ would be recurrent, as $x \notin F_n$ allows us to pick $h_n \in H$ such that $h_n x \neq x$ and $d(x, h_n x) < 1/n$, and therefore $h_n x \to x$ showing that x is recurrent.

Let now $A \subseteq X$ be a subset of diameter $\operatorname{diam} A < 1/n$. The set $A \cap F_n$ intersects any E-class in at most one point. Indeed, if $x, y \in A \cap F_n$ are E-equivalent, then there is $h \in H$ such that $hx = y$. Since $\operatorname{diam} A < 1/n$, we have $d(x, y) < 1/n$, but $d(x, hx) \geq 1/n$ unless $hx = x$, so we are forced to conclude that $x = y$. Since X is Polish, we may partition each $F_n = \bigsqcup_{k=0}^{\infty} A_k^n$ into Borel sets of diameter at most $1/n$. Thus each A_k^n is a smooth set, which shows that so is $X = \bigcup_{k,n} A_k^n$. □

The Glimm–Effros dichotomy is valid, in fact, for all Borel equivalence relations. This deep result is due to Leo Harrington, Alexander Kechris, and Alain Louveau[HKL90]. The original argument relied on the methods of effective descriptive set theory. A classical proof has since been found by Benjamin Miller [Mil12].

A measure μ on X is said to be E-*quasi-invariant* if equidecomposability preserves the null sets: $\mu(A) = 0 \iff \mu(B) = 0$ for all Borel $A, B \subseteq X$ such that $A \sim B$. A measure μ on X is called *non-atomic* if it does not have any point masses: $\mu(\{x\}) = 0$ for all $x \in X$. Recall also that μ is E-ergodic if for any E-invariant subset $Y \subseteq X$ one has either $\mu(Y) = 0$ or $\mu(X \setminus Y) = 0$. For a measure μ on X we let $\mathscr{N}_\mu$ to denote the ideal of μ-null sets on X.

Fix a cber E on a standard Borel space X. Let $\mathtt{QE}$ denote the set of all quasi-invariant, ergodic, non-atomic, probability measures on X. We remind that $\mathscr{W}$ denotes the ideal of smooth sets on X. The following characterization of the wandering ideal is due to Saharon Shelah and Benjamin Weiss [SW82].

Theorem I.4. *For any cber* E *the wandering ideal is the intersection of* $\mathscr{N}_\mu$ *over all* $\mu \in \mathtt{QE}$*:* $\mathscr{W} = \bigcap_{\mu \in \mathtt{QE}} \mathscr{N}_\mu$.

Proof. We begin by showing the inclusion $\mathscr{W} \subseteq \bigcap_{\mu \in \mathtt{QE}} \mathscr{N}_\mu$. Pick a smooth set $A \subseteq X$ and a transversal $T \subseteq A$ for $\mathsf{E}|_A$. Let $H \curvearrowright X$ be a countable group action generating E. Similarly to the proof of Proposition 1.6.2, one shows that μ is E-quasi-invariant if and only if $\mu(A) = 0 \iff \forall\, h \in H\ \mu(hA) = 0$. Since $A \subseteq [T]_{\mathsf{E}} = \bigcup_{h \in H} hT$, it is enough to check that $\mu(T) = 0$ for all $\mu \in \mathtt{QE}$. Pick some μ and assume that $\mu(T) \neq 0$. Since μ is non-atomic, the restriction of μ onto T is isomorphic to the (re-normalized) Lebesgue measure on $[0, 1]$. We may therefore partition $T = T_1 \sqcup T_2$ into two Borel sets of positive measure: $\mu(T_1) \cdot \mu(T_2) > 0$. Since T was a transversal, $[T_1]_{\mathsf{E}}$ and $[T_2]_{\mathsf{E}}$ are two disjoint Borel E-invariant sets of positive measure. Therefore μ is not ergodic, implying that $\mu(T) = 0$ for all $\mu \in \mathtt{QE}$ as claimed.

For the reverse inclusion we are going to show that for any non-smooth $A \subseteq X$ there exists $\mu \in \mathtt{QE}$ such that $\mu(A) > 0$. By Glimm–Effros dichotomy Theorem I.3, we may find a Borel embedding $\xi : 2^{\mathbb{N}} \to A$ such that $x \mathrel{\mathsf{E}_0} y \iff \xi(x) \mathrel{\mathsf{E}} \xi(y)$. Let $B = \xi\bigl(2^{\mathbb{N}}\bigr)$; note that B is Borel, as ξ is one-to-one, and $\mathsf{E}|_B$ is isomorphic to E_0. Let as before $H \curvearrowright X$ generate E, and fix an enumeration $H = \{h_n : n \in \mathbb{N}\}$; it is convenient to assume that $h_0 = \mathrm{id}$. Define the measure μ on X by setting

$$\mu(C) = \sum_{n=0}^{\infty} 2^{-n-1} \nu(h_n C \cap B),$$

where ν is the measure on B obtained by pushing forward via ξ the Bernoulli measure on $2^{\mathbb{N}}$. We claim that $\mu \in \mathtt{QE}$ and $\mu(A) > 0$. The measure μ is a probability measure, since $\nu(h_n X \cap B) = \nu(B) = 1$ for all n, so $\mu(X) = \sum_{n=0}^{\infty} 2^{-n-1} = 1$. Also $\mu(A) \geq \nu(h_0 A \cap B)/2 = \nu(B)/2 = 1/2 > 0$, and it clearly non-atomic as the Bernoulli measure is non-atomic. To show ergodicity, note that for any E-invariant $Z \subseteq X$ either $\nu(Z \cap B) = 0$ or $\nu\bigl((X \setminus Z) \cap B\bigr) = \nu(B \setminus Z) = 0$, because ν is $\mathsf{E}|_B$-invariant. Since for any E-invariant Z we have $h_n Z = Z$ for all $n \in \mathbb{N}$ and thus

$$\mu(Z) = \sum_{n=0}^{\infty} 2^{-n-1} \nu(h_n Z \cap B) = \nu(Z \cap B),$$

the measure μ is seen to be ergodic.

It remains to check that μ is E-quasi-invariant. To this end we show $\mu(C) = 0$ implies $\mu([C]_{\mathsf{E}}) = 0$. This implies quasi-invariance, as $C \sim D$ forces $[C]_{\mathsf{E}} = [D]_{\mathsf{E}}$. By definition of μ, $\mu(C) = 0$ yields $\nu(h_n C_n \cap B) = 0$ for all $n \in \mathbb{N}$. Using E-invariance of $[C]_{\mathsf{E}}$, we therefore have

$$0 = \mu(C) = \sum_{n=0}^{\infty} \nu(h_n C \cap B) \geq \nu\Big(\bigcup_n h_n C \cap B\Big) = \nu\big([C]_{\mathsf{E}} \cap B\big) =$$
$$= \nu\big([C]_{\mathsf{E}} \cap B\big) \sum_{n=0}^{\infty} 2^{-n-1} = \sum_{n=0}^{\infty} 2^{-n-1} \nu(h_n [C]_{\mathsf{E}} \cap B) = \mu([C]_{\mathsf{E}}).$$

Thus $\mu([C]_{\mathsf{E}}) = 0$, and μ is quasi-invariant. □

Chapter 2

Compressible equivalence relations

2.1 When do we have an invariant measure?

We have seen in Proposition 1.6.4 that smoothness is an obstruction for an aperiodic cber to have a finite invariant measure. A natural question is whether this is the only obstruction. Recall that a tail equivalence relation E_t on $2^{\mathbb{N}}$ was defined by declaring $x \mathsf{E}_t y$ whenever there exist $k_1, k_2 \in \mathbb{N}$ such that $x(k_1 + n) = y(k_2 + n)$ for all $n \in \mathbb{N}$.

Proposition 2.1.1. *The tail equivalence relation* E_t *is not smooth, yet it does not admit a finite invariant measure.*

Proof. Suppose E_t is smooth. By Proposition 1.3.2 there is a Borel selector $s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. Since s is a Borel function, there must be a dense G_δ subset $Z \subseteq X$ such that $s|_Z : Z \to X$ is continuous (see [Kec95, Theorem 8.38]). By considering $\bigcap_{n \in \mathbb{Z}} \sigma^n(Z)$ instead of Z, we may assume without loss of generality that Z is invariant under the odometer map. Since Z must be uncountable, we may pick $x, y \in Z$ such that $\neg(x \mathsf{E}_t y)$. Let $x_n \in 2^{\mathbb{N}}$ be defined by changing the first n digits of x to the corresponding digits of y:

$$x_n(i) = \begin{cases} y(i) & \text{if } i < n, \\ x(i) & \text{if } i \geq n. \end{cases}$$

Note that for each n there is $m_n \in \mathbb{Z}$ such that $\sigma^{m_n}(x) = x_n$. Since Z is assumed to be σ invariant, $x_n \in Z$ for all $n \in \mathbb{N}$. Since obviously $x_n \to y$, continuity of s guarantees that $s(x_n) \to s(y)$, but $x_n \mathsf{E}_t x$ for all $n \in \mathbb{N}$, hence $s(x_n) = s(x)$. So $s(x) \to s(y)$, i.e., $s(x) = s(y)$, implying that $x \mathsf{E}_t y$, contradicting the choice of $x, y \in Z$. Thus E_t is not smooth.

Now to the existence of an invariant measure. Suppose towards a contradiction that there is a finite E_t-invariant measure μ on $2^{\mathbb{N}}$. Let $A \subset 2^{\mathbb{N}}$ be the family of all sequences that start with zero:

$$A = \{x \in 2^{\mathbb{N}} : x(0) = 0\}.$$

Let $f : X \to A$ be the right shift map which adds a leading zero:

$$(fx)(n) = \begin{cases} 0 & \text{if } n = 0, \\ x(n-1) & \text{otherwise.} \end{cases}$$

Note that $f : X \to A$ is a bijection which preserves E_t, and so $X \underset{\mathsf{E}_t}{\sim} A$. Thus it must be the case that $\mu(A) = \mu(X)$, but $X \setminus A$ is a complete section for E_t. So we have $\mu(X \setminus A) = 0$ and $[X \setminus A]_{\mathsf{E}_t} = X$, which forces us to conclude that $\mu(X) = 0$. $\square$

To summarize, being smooth is not the only obstruction for having a finite invariant measure. Scrutinizing the argument in Proposition 2.1.1, one comes up with the following phenomenon, which prevents E_{t} from having an invariant measure.

Definition 2.1.2. A cber E on X is said to be a *compressible* equivalence relation if there exists a set $A \subseteq X$ such that $X \sim A$ and $X \setminus A$ is a complete section. In a more verbose fashion, E is compressible if X is equidecomposable with a proper subset which omits at least one point from each E-class.

The proof of Proposition 2.1.1 shows that E_{t} is compressible, and also that no compressible cber admits a finite invariant measure. It turns out that compressibility is the precise obstruction for having an invariant measure. This is the content of Nadkarni's Theorem, which we shall prove at the end of this chapter.

Remark 2.1.3. Let us note that if E has a finite equivalence class, then E cannot be compressible, because if $X \sim A$, then

$$\big|[x]_{\mathsf{E}}\big| = \big|[x]_{\mathsf{E}} \cap A\big| \quad \text{for all } x \in X.$$

So if $\big|[x]_{\mathsf{E}}\big| < \infty$, then $[x]_{\mathsf{E}} = [x]_{\mathsf{E}} \cap A$, implying that $[X \setminus A]_{\mathsf{E}} \neq X$.

2.2 Properties of compressible relations

Compressibility can be reformulated in a number of equivalent ways, some of which look significantly stronger.

Proposition 2.2.1. *Let E be a cber on X. The following are equivalent*

(i) E *is compressible.*

(ii) There exist pairwise disjoint Borel sets $B_n \subseteq X$, $n \in \mathbb{N}$, such that each B_n is a complete section and $B_n \sim B_m$ for all $m, n \in \mathbb{N}$.

(iii) There are pairwise disjoint Borel sets $A_n \subseteq X$, $n \in \mathbb{N}$, such that $X \sim A_n$ for all $n \in \mathbb{N}$.

Proof. (i) ⇒ (ii) Let $A \subseteq X$ be such that $X \sim A$ and $B := X \setminus A$ is an E-complete section; let also $f : X \to A$ be an element of $[\![\mathsf{E}]\!]$ witnessing $X \sim A$. Set $B_n = f^n(B)$, and note that $B_0 \sim B_n$ via f^n and each B_n is an E-complete section.

(ii) ⇒ (iii) Let $H \curvearrowright X$ be an action that realizes E, enumerate $H = \{h_n : n \in \mathbb{N}\}$, and consider a Borel function $N : X \to \mathbb{N}$

$$N(x) = \min\{n \in \mathbb{N} : h_n x \in B_0\}.$$

Pick a countable family of injective maps $\tau_n : \mathbb{N} \to \mathbb{N}$ with disjoint images $\tau_n(\mathbb{N}) \cap \tau_m(\mathbb{N}) = \varnothing$ for $m \neq n$, and let $f_n : B_0 \to B_n$ witness $B_0 \sim B_n$. Functions $g_n : X \to X$ are defined by

$$g_n(x) = f_{\tau_n\left(N(x)\right)} \circ h_{N(x)} x,$$

and are easily checked to be injective, thus $X \sim g_n(X)$ for all $n \in \mathbb{N}$. Since $g_n(X) \subseteq \bigcup_{i \in \mathbb{N}} B_{\tau_n(i)}$, we get $g_n(X) \cap g_m(X) = \varnothing$, $m \neq n$, by the choice of functions τ_n.

(iii) ⇒ (i) This implication is obvious, as $X \sim A_0$, and $A_1 \subseteq X \setminus A_0$, so $X \setminus A_0$ is an E-complete section. □

Definition 2.2.2. Let E be a cber on X and $A, B \subseteq X$ be Borel sets. We use the notation $A \preceq B$ to denote existence of a Borel subset $B' \subseteq B$ such that $A \sim B'$. When $B' \subseteq B$ can be found such that $A \sim B'$ and moreover $[B \setminus B']_{\mathsf{E}} = [B]_{\mathsf{E}}$, then a strict notation $A \prec B$ is used.

A standard Schröder–Bernstein argument is available for the relation $\preceq$.

Proposition 2.2.3. *If $A \preceq B$ and $B \preceq A$, then $A \sim B$.*

Proof. The most standard proof of Schröder–Bernstein Theorem works. □

Definition 2.2.4. We have defined the notion of a compressible equivalence relation, and it is convenient now to define what it means for a set to be compressible. Let E be a cber on X and let $A \subseteq X$ be a Borel set. We say that A is compressible if there exists a subset $B \subseteq A$ such that $A \sim B$ and $[A \setminus B]_{\mathsf{E}} = [A]_{\mathsf{E}}$. In other words, A is compressible if the restriction of E onto $A \times A$ is a compressible equivalence relation.

Note that a subset of a compressible set may not be compressible. Indeed, if $A \cap [x]_{\mathsf{E}}$ is finite for some $x \in X$, then A cannot be compressible; in particular, no compressible set is finite. But any *E-invariant* subset of a compressible set is compressible (see Exercise 2.1). We let $\mathscr{H}$ (or $\mathscr{H}_{\mathsf{E}}$ if we want to emphasize dependence on E) to denote the family of all Borel subsets of X whose saturation is a compressible set:

$$\mathscr{H} = \big\{A \subseteq X : A \text{ is Borel and} [A]_{\mathsf{E}} \text{ is compressible}\big\}$$

We call $\mathscr{H}$ the *Hopf ideal* of E (Exercise 2.2 suggests checking that $\mathscr{H}$ is indeed a σ-ideal of Borel sets).

Remark 2.2.5. Note that relations $\preceq$ and $\prec$ are transitive. In fact, if $A \prec B$ and $B \preceq C$, then $A \prec C$. In particular, if $A \prec B$ and $B \preceq A$, then $A \prec A$, which is just another way of saying that A is compressible.

Proposition 2.2.6. *Let E be a cber on X and let $A \subseteq X$ be Borel. If A is compressible, then $A \sim [A]_{\mathsf{E}}$. In particular, $[A]_{\mathsf{E}}$ is also compressible.*

Proof. Since the identity map shows $A \preceq [A]_{\mathsf{E}}$ and since we have the Schröder–Bernstein argument (see Proposition 2.2.3), it is enough to show that $[A]_{\mathsf{E}} \preceq A$. We may apply Proposition 2.2.1(iii) to get pairwise disjoint Borel subsets $A_n \subseteq A$, $n \in \mathbb{N}$, and bijections $f_n : A \to A_n$, $f_n \in [\![\mathsf{E}]\!]$. Let $H = \{h_n : n \in \mathbb{N}\}$ be a countable group acting on X and realizing E. Set $N : [A]_{\mathsf{E}} \to A$ to be given by

$$N(x) = \min\{n \in \mathbb{N} : h_n x \in A\},$$

and set $g : [A]_{\mathsf{E}} \to A$ to be $g(x) = f_{N(x)} \circ h_{N(x)} x$. This map shows that $[A]_{\mathsf{E}} \preceq A$, and so $[A]_{\mathsf{E}} \sim A$. □

2.3 Nadkarni's Theorem

As we have already anticipated, compressibility is the only obstruction for a cber to admit a finite invariant measure. This result for cbers that are realized by an action of $\mathbb{Z}$ is due to Mahendra Nadkarni [Nad90], and Howard Becker and Alexander Kechris [BK96, Section 4] supplied the necessary modifications to make the argument work for a general cber. The proof of the theorem also benefits from ideas of Eberhard Hopf [Hop32]. The rest of this chapter follows closely the presentation in [Nad90] and [BK96].

Theorem 2.3.1 (Nadkarni)**.** *A cber does not have any finite invariant measures if and only if it is compressible.*

The proof of the theorem will take us a while, and will gradually emerge by the end of this chapter. We would like to start with a discussion of the following question: How one may start constructing an invariant measure? Let us say we have got a cber E on X and a set $A \subseteq X$. How should we decide what the measure of A should be?

The key observation is that E-equidecomposable sets must necessarily have the same measure. Here is how it can be used. Let $F \subseteq X$ be a "sampling set" — we shall try to measure sets in the "units of F". Let us say we have an E-invariant measure on X, call it μ. If one can partition A into pieces $A_1 \sqcup \cdots \sqcup A_n \sqcup R$

where each $A_i \sim F$ is equidecomposable with F, then measure of A will be at least n times the measure of F. If moreover in this decomposition $R \preceq F$, then we also have an upper estimate $\mu(A) \leq (n+1)\mu(F)$. Suppose we have a similar estimate for the whole phase space X with respect to the same sampling set F:

$$m\mu(F) \leq \mu(X) \leq (m+1)\mu(F).$$

Combining these two estimates one gets that

$$\frac{n}{m+1} \leq \frac{\mu(A)}{\mu(X)} \leq \frac{n+1}{m}. \tag{2.1}$$

The normalization of measure is, of course, immaterial, so we may as well assume $\mu(X) = 1$, which yields an estimate on the measure of A. As m and n grow, that is as we take "smaller" sampling sets F, this estimate improves its precision, and bounds converge to $\mu(A)$.

We have argued so far under the assumption that we have an invariant measure μ, but the estimate we arrived at depends on purely descriptive parameters — the number of times a sampling set F fits into A. So here is a roadmap for constructing an invariant measure. Pick a "vanishing" sequence $(F_i)_{i=0}^{\infty}$ for E and show that for any Borel set A bounds in equation 2.1 when computed with respect to F_i converge as $i \to \infty$.

As usually, the reality is more rugged than the roadmap, and we shall have to introduce some technical complications into our plan to make it work, but the above discussion hopefully demystifies the origin of an invariant measure. We begin by developing tools to compare possible measures of two given Borel sets.

Lemma 2.3.2. *Let E be a cber on X and let $A, B \subseteq X$ be Borel sets; let $Z = [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$. There is a partition of $Z = P \sqcup Q$ into E-invariant pieces such that*

$$A \cap P \prec B \cap P \quad \text{and} \quad B \cap Q \preceq A \cap Q.$$

Moreover, P and Q are unique modulo the Hopf ideal $\mathscr{H}$ in the sense that if $Z = P' \sqcup Q'$ is another such partition, then $P \mathbin{\triangle} P'$ and $Q \mathbin{\triangle} Q'$ belong to $\mathscr{H}$.

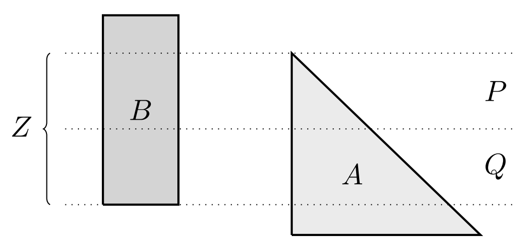

Figure 2.1: Decomposing $Z = [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$ into P and Q.

Proof. Let $H \curvearrowright X$, $H = \{h_n : n \in \mathbb{N}\}$, be an action of a countable group realizing E. Set inductively

$$A_n = \Big\{x \in A \setminus \bigcup_{i<n} A_i : h_n x \in B \setminus \bigcup_{i<n} B_i\Big\}$$
$$B_n = h_n(A_n).$$

Note that sets A_n are pairwise disjoint, and so are sets B_n. Note also that $A_n \ni x \mapsto h_n x \in B_n$ is a bijection witnessing $A_n \sim B_n$. Therefore $\tilde{A} \sim \tilde{B}$, where $\tilde{A} = \bigsqcup_n A_n$ and $\tilde{B} = \bigsqcup_n B_n$. One may now set $P = Z \cap [B \setminus \tilde{B}]_{\mathsf{E}}$ and $Q = Z \setminus P$.

To show uniqueness of such a decomposition, suppose $Z = P' \sqcup Q'$ is another such partition. To show that $P \vartriangle P'$ and $Q \vartriangle Q'$ are in $\mathscr{H}$, it is enough to show that $P' \cap Q$ and $Q' \cap P$ are compressible. Set $S = P' \cap Q$. Since $A \cap P' \prec B \cap P'$, and since S is an E-invariant subset of P', we have $A \cap S \prec B \cap S$. Similarly, $B \cap Q \preceq A \cap Q$ implies $B \cap S \preceq A \cap S$, thus $A \cap S \prec A \cap S$, so $A \cap S$ is compressible (by Exercise 2.4), hence so is $S = [A \cap S]_{\mathsf{E}}$ via Proposition 2.2.6. The argument for $Q' \cap P$ is similar. □

Given Borel sets $A, B \subseteq X$ and $n \in \mathbb{N}$, we shall use the following notation.

- $A \preceq nB$ means that one can represent A as $\bigcup_{i=1}^{n} A_i$ in such a way that $A_i \preceq B$ for each $1 \leq i \leq n$. Note that $A \preceq 1B$ is equivalent to $A \preceq B$.

- $A \prec nB$ means that moreover in the representation $A = \bigcup_{i=1}^{n} A_i$ as above we can have $A_i \prec B$ for at least one $i \leq n$. It is worth making a few comments about this notion. First of all, this definition is equivalent to a seemingly weaker one. Suppose the set A admits a representation $A = \bigcup_{i=1}^{n} A_i$ such that $f_i : A_i \to B$ witness $A_i \preceq B$ and $\bigcup_{i=1} \bigl[B \setminus f_i(A_i)\bigr]_{\mathsf{E}} = [B]_{\mathsf{E}}$. We claim that in this case we necessarily have $A \prec nB$. Indeed, set $X_1 = \bigl[B \setminus f_1(A_1)\bigr]_{\mathsf{E}}$ and define for $k < n$
$$X_{k+1} = \bigl[B \setminus f_{k+1}(A_{k+1})\bigr]_{\mathsf{E}} \setminus \bigcup_{i \leq k} X_k.$$
Evidently each X_k is E-invariant and by assumption $[B]_{\mathsf{E}} = \bigsqcup_{k=1}^{n} X_k$. Now set
$$\begin{aligned} A_1' &= \bigsqcup_{i=1}^{n} (A_i \cap X_i) \\ A_k' &= \bigl(A_k \cap (X \setminus X_k)\bigr) \sqcup (A_1 \cap X_k), \quad \text{for } k > 1. \end{aligned}$$
The maps $f_k' : A_k' \to B$ defined by
$$f_1'(x) = f_k(x) \text{ whenever } x \in A_k \cap X_i$$
$$f_k'(x) = \begin{cases} f_k(x) & \text{if } x \notin X_k \\ f_1(x) & \text{otherwise} \end{cases} \quad \text{for } k > 1$$
witness $A_k' \preceq B$ and $A_1' \prec B$.

  We note that $A \prec 1B$ is the same as $A \prec B$ defined earlier.

- $A \succeq nB$ denotes existence of pairwise disjoint subsets $B_i \subseteq A$, $1 \leq i \leq n$, such that $B_i \sim B$. We may say in this case that A contains at least n copies of B.

- $A \succeq \infty B$ will similarly denote existence of an infinite pairwise disjoint family $B_i \subseteq A$, $i \in \mathbb{N}$, such that $B_i \sim B$ for all $i \in \mathbb{N}$. Note that $A \succeq \infty B$ implies A is compressible by Proposition 2.2.1(ii).

- Finally, $A \approx nB$ will signify the possibility to decompose $A = \bigsqcup_{i=1}^{n} B_i \sqcup R$ into Borel pieces such that $B_i \sim B$ and $R \prec B$. In particular, $A \approx 0B$ is another way of denoting $A \prec B$. Note that $A \approx nB$ implies that $A \succeq nB$ and $A \prec (n+1)B$.

Proposition 2.3.3. *If $A \succeq nB$ and $C \preceq mB$ for some $m \leq n$, $m, n \in \mathbb{N}$, then $C \preceq A$. If moreover $C \prec mB$, then $C \prec A$. In particular,*

a) if $A \approx nB$ and $C \approx mB$ for some $m < n$, then $C \prec A$;

b) if $A \approx nB$ and $A \approx mB$ for some $m \neq n$, then A is compressible

Proof. Suppose we have pairwise disjoint sets $B_i \subseteq A$, $1 \le i \le n$, together with maps $f_i : B_i \to B$ witnessing $B_i \sim B$, and suppose also that C is written as $\bigcup_{j=1}^m C_j$, where each $C_j \preceq B$. By considering $C_j' = C_j \setminus \bigcup_{k<j} C_k$ instead of C_j, we may assume that C_j are pairwise disjoint; thus $C = \bigsqcup_{j=1}^m C_j$. For the moreover part we as lo assume that $C_m \prec B$. Pick maps $g_j : C_j \to B$, which show that $C_j \preceq B$.

Consider a function $\xi : C \to A$ defined by the formula

$$\xi(x) = f_j^{-1} \circ g_j(x) \text{ if } x \in C_j.$$

It is easy to check that $\xi : C \to A$ is an injection and $\xi \in [\![\mathsf{E}]\!]$. For the moreover part we have $\big[B \setminus g_m(C_m)\big]_{\mathsf{E}} = [B]_{\mathsf{E}}$ and since $[C]_{\mathsf{E}} \subseteq [B]_{\mathsf{E}} \subseteq [A]_{\mathsf{E}}$, one may conclude that

$$\begin{aligned}\big[A \setminus \xi(A)\big]_{\mathsf{E}} &\supseteq \big([A]_{\mathsf{E}} \setminus [B]_{\mathsf{E}}\big) \cup \big[B_m \setminus f_m^{-1} \circ g_m(C_m)\big]_{\mathsf{E}} = \big([A]_{\mathsf{E}} \setminus [B]_{\mathsf{E}}\big) \cup \big[B \setminus g_m(C_m)\big]_{\mathsf{E}} \\ &= \big([A]_{\mathsf{E}} \setminus [B]_{\mathsf{E}}\big) \cup [B]_{\mathsf{E}} = [A]_{\mathsf{E}}.\end{aligned}$$

Thus $C \prec A$ as claimed. □

Proposition 2.3.4. *Let E be a cber on X, let $A, B \subseteq X$ be Borel sets, and let $Z = [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$. There exists a partition $Z = Q_\infty \sqcup \bigsqcup_{n=0}^\infty Q_n$ of Z into E-invariant Borel pieces such that $A \cap Q_n \approx n(B \cap Q_n)$ for all $n \in \mathbb{N}$, and $A \cap Q_\infty \succeq \infty(B \cap Q_\infty)$.*

Moreover, such a decomposition is unique up to a compressible perturbation, i.e., if

$$Z = Q'_\infty \sqcup \bigsqcup_{n=0}^{\infty} Q'_n \text{ is another such partition,}$$

then $Q_n \triangle Q'_n \in \mathscr{H}$ for all $n \in \mathbb{N} \cup \{\infty\}$.

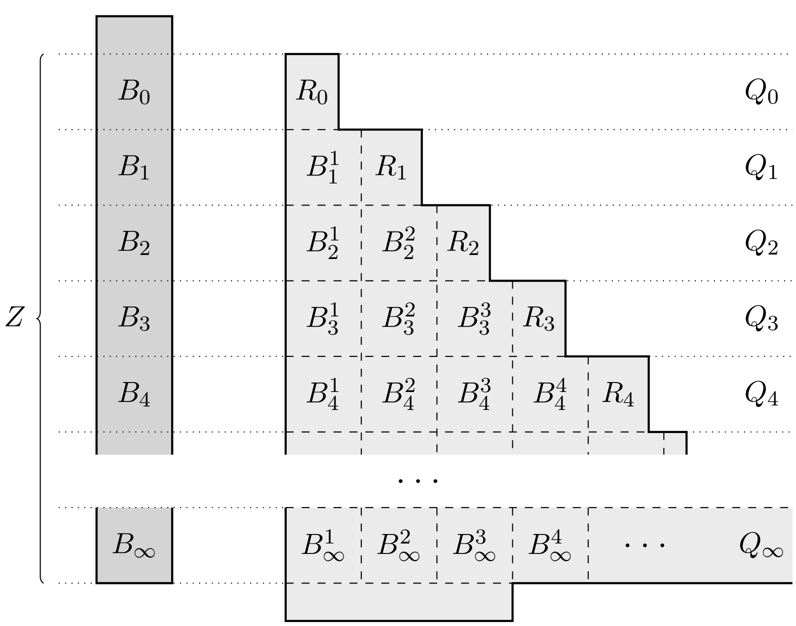

Figure 2.2: Partition of $Z = [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$ into sets Q_n. The set B is in darker gray to the left, and A is in light gray to the right.

Proof. Let us first provide a little more details to the statement and explain the illustration in Figure 2.2. If we set $B_n = B \cap Q_n$, then the proposition asserts that $A \cap Q_n \approx nB_n$, i.e., $A \cap Q_n$ can be partitioned into Borel pieces

$$A \cap Q_n = B_n^1 \sqcup B_n^2 \sqcup \cdots \sqcup B_n^n \sqcup R_n$$

such that $B_n \sim B_n^i$ for all $i \le n$, and $R_n \prec B_n$.

The decomposition depicted in Figure 2.2 is constructed by induction. For the base we apply Lemma 2.3.2 to A and B and get a partition $Z = \tilde{P}_0 \sqcup \tilde{Q}_0$ into invariant Borel pieces such that $A \cap \tilde{P}_0 \prec B \cap \tilde{P}_0$ and $B \cap \tilde{Q}_0 \preceq A \cap \tilde{Q}_0$. We set $Q_0 = \tilde{P}_0$, $B_0 = B \cap Q_0$, and $R_0 = A \cap Q_0$. Since $B \cap \tilde{Q}_0 \preceq A \cap \tilde{Q}_0$, we may find a Borel subset $\tilde{B}^1 \subseteq A \cap \tilde{Q}_0$ such that $B \cap \tilde{Q}_0 \sim \tilde{B}^1$.

To build the next layer of decomposition we apply Lemma 2.3.2 to sets $B \cap \tilde{Q}_0$ and $A_1 = (A \cap \tilde{Q}_0) \setminus \tilde{B}^1$ yielding a partition of $\big[B \cap \tilde{Q}_0\big]_{\mathsf{E}} \cap [A_1]_{\mathsf{E}} = [A_1]_{\mathsf{E}}$ into invariant sets $\tilde{P}_1 \cup \tilde{Q}_1$ such that

$$A_1 \cap \tilde{P}_1 \prec B \cap \tilde{Q}_0 \cap \tilde{P}_1 = B \cap \tilde{P}_1 \quad \text{and} \quad B \cap \tilde{Q}_0 \cap \tilde{Q}_1 = B \cap \tilde{Q}_1 \preceq A_1 \cap \tilde{Q}_1.$$

We set $Q_1 = \tilde{Q}_0 \setminus \tilde{Q}_1$, $B_1 = B \cap Q_1$, $B_1^1 = \tilde{B}^1 \cap Q_1$, and $R_1 = (A \cap Q_1) \setminus B_1^1$. Since $B \cap \tilde{Q}_1 \preceq A_1 \cap \tilde{Q}_1$, we may find $\tilde{B}^2 \subseteq A_1 \cap \tilde{Q}_1$ such that $B \cap \tilde{Q}_1 \sim \tilde{B}^2$. Note that $\tilde{B}^2$ is necessarily disjoint from $\tilde{B}^1$.

The process continues by applying Lemma 2.3.2 to sets $B \cap \tilde{Q}_1$ and $A_2 = (A \cap \tilde{Q}_1) \setminus \tilde{B}^2$. As a result, we construct sets Q_n, B_n, B_n^i, $1 \le i \le n$, and R_n for all $n \in \mathbb{N}$, which satisfy all the conclusions of the lemma. The sets Q_n, $n \in \mathbb{N}$, may not cover all of Z, so we set $Q_\infty = Z \setminus \bigsqcup_n Q_n$ and show that $Q_\infty \in \mathscr{H}$.

During the run of the construction above, we also construct disjoint Borel sets $\tilde{B}^{n+1} \subseteq A$ such that $B \cap \tilde{Q}_n \sim \tilde{B}^{n+1} \cap \tilde{Q}_n$ for all n. Since Q_∞ is a subset of $\tilde{Q}_n$ for all n, we may set $B_\infty^{n+1} = \tilde{B}^{n+1} \cap Q_\infty$, and get infinitely many disjoint Borel subsets of Q_∞ such that $B \cap Q_\infty \sim B_\infty^n$ for every $n \ge 1$. In particular, $B_\infty^n \sim B_\infty^m$ for all $m, n \ge 1$, and by Proposition 2.2.1(ii) Q_∞ is a compressible set, since $[B \cap Q_\infty]_{\mathsf{E}} = Q_\infty$.

It remains to check uniqueness of such a decomposition. Suppose $Z = Q'_\infty \sqcup \bigsqcup Q'_n$ is a different partition of Z with the same list of properties. Since Q'_∞ and Q_∞ are compressible, to show $Q_n \bigtriangleup Q'_n \in \mathscr{H}$ for all $n \in \mathbb{N} \cup \{\infty\}$, it is enough to check that $Q_n \cap Q'_m \in \mathscr{H}$ for all $m \ne n$ in $\mathbb{N}$. Let $S = Q_n \cap Q'_m$. We have $A \cap Q_n \approx n(B \cap Q_n)$ and also $A \cap Q'_m \approx m(B \cap Q'_m)$. Since S is an invariant subset of both Q_n and Q'_m, we also have $A \cap S \approx n(B \cap S)$ and $A \cap S \approx m(B \cap S)$. Thus Proposition 2.3.3 applies and shows that $A \cap S$ is compressible. By Proposition 2.2.6, the set $[A \cap S]_{\mathsf{E}} = S$ is also compressible, and the uniqueness follows. □

2.4 The fraction function

This section as well as Sections 2.6 and 2.8 closely follow the material from [Nad90] together with remarks suggested in [BK96].

For any pair of Borel sets $A, B \subseteq X$ we fix a decomposition of $Z = [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$ into sets $Q_\infty \sqcup \bigsqcup_n Q_n$ as in Proposition 2.3.4 and associate with it a *fraction function* $[A/B] : X \to \mathbb{N}$ defined by

$$\left[\frac{A}{B}\right](x) = \begin{cases} n & \text{if } x \in Q_n \text{ for some } n \in \mathbb{N}, \\ \infty & \text{if } x \in Q_\infty, \\ 0 & \text{otherwise.} \end{cases}$$

The function $[A/B]$ does depend on the choice of the partition of Z, but in a very mild way: if $[A/B]'$ is defined with respect to another way of decomposing $Z = Q'_\infty \sqcup \bigsqcup_n Q'_n$, then

$$\big\{x \in X : [A/B](x) \ne [A/B]'(x)\big\} \text{ is in the Hopf's ideal.}$$

Given functions $\xi, \zeta : X \to \mathbb{R}$, we shall use notations like $\zeta = \xi \bmod \mathscr{H}$, $\zeta \le \xi \bmod \mathscr{H}$, etc. to denote that the set of $x \in X$ such that $\zeta(x) \ne \xi(x)$, $\zeta(x) \not\le \xi(x)$, etc. belongs to $\mathscr{H}$. Since set Q_∞ in the definition of the fraction function is compressible, if we are interested in the behavior of $[A/B]$ only mod $\mathscr{H}$, then we may safely disregard points x in Q_∞. Here is a rather long list of properties of the fraction function, most of which are very natural to expect based on its definition.

Proposition 2.4.1. *The fraction functions possess the following properties for all Borel sets $A, B, C, D \subseteq X$.*

(i) If $x \mathsf{E} y$, then $[A/B](x) = [A/B](y)$.

(ii) If $A \sim C$, then $[A/B] = [C/B]$ mod $\mathscr{H}$.

(iii) If $B \sim D$, then $[A/B] = [A/D]$ mod $\mathscr{H}$.

(iv) If $A \preceq C$, then $[A/B] \leq [C/B]$ mod $\mathscr{H}$.

(v) If $B \preceq D$, then $[A/B] \geq [A/D]$ mod $\mathscr{H}$.

(vi) If S is E-invariant, then $[A/B]\big|_S = \big[(A \cap S)/B\big]\big|_S$ mod $\mathscr{H}$, i.e.,

$$\big\{x \in S : [A/B](x) \neq [A \cap S/B](x)\big\} \in \mathscr{H}.$$

(vii) The set $Y = \big\{x \in X : [A/B](x) < [C/B](x)\big\}$ is E-invariant and $Y \cap A \preceq Y \cap C$.

(viii) If B is an E-complete section, then $[A/B][B/C] \leq [A/C] < \big([A/B] + 1\big)\big([B/C] + 1\big)$ mod $\mathscr{H}$.

(ix) If A and C are disjoint, then $[A/B] + [C/B] \leq \big[(A \cup C)/B\big] \leq [A/B] + 1 + [C/B] + 1$ mod $\mathscr{H}$.

Proof. Item (i) is obvious, since sets Q_n in the definition of the fraction function are E-invariant. Items (ii) and (iii) will follow from (iv) and (v) respectively, because $A \sim C$ is equivalent to $A \preceq C$ and $C \preceq A$.

(iv) Let Q_n, $n \in \mathbb{N} \cup \{\infty\}$, be the decomposition of $[A]_\mathsf{E} \cap [B]_\mathsf{E}$ associated with $[A/B]$, and let Q'_n, $n \in \mathbb{N} \cup \{\infty\}$, be the decomposition for $[C/B]$. Since $A \preceq C$, we have $[A]_\mathsf{E} \cap [B]_\mathsf{E} \subseteq [C]_\mathsf{E} \cap [B]_\mathsf{E}$, so it is enough to show that for all $m, n \in \mathbb{N}$, $m < n$, the set $Q_n \cap Q'_m$ is compressible. Set $S = Q_n \cap Q'_m$ and note that by the conditions on Q_n and Q'_m we have $A \cap S \approx n(B \cap S)$ and $C \cap S \approx m(B \cap S)$. Proposition 2.3.3 implies $C \cap S \prec A \cap S$. Since by assumption $A \cap S \preceq C \cap S$, we conclude that $A \cap S \prec A \cap S$, hence $S = [A \cap S]_\mathsf{E}$ is compressible.

Item (v) is proved similarly to the previous one, and we omit the argument.

(vi) If Q_n, $n \in \mathbb{N} \cup \{\infty\}$, is a decomposition of $[A]_\mathsf{E} \cap [B]_\mathsf{E}$ associated with $[A/B]$, then

$$(Q_\infty \cap S) \sqcup \bigsqcup_n s(Q_n \cap S)$$

is a partition of $[A \cap S]_\mathsf{E} \cap [B]_\mathsf{E}$, which satisfies the conclusion of Proposition 2.3.4. Since we have shown that such a partition is unique up to a compressible perturbation, we get $[A/B]\big|_S = \big[(A \cap S)/B\big]\big|_S$ mod $\mathscr{H}$.

(vii) Let Q_n, $n \in \mathbb{N} \cup \{\infty\}$, be the decomposition associated with $[A/B]$. Note that sets $Y \cap (X \setminus [B]_\mathsf{E})$ and $Y \cap Q_\infty$ are empty, so the set $Y = \big\{x \in X : [A/B](x) < [C/B](x)\big\}$ can be split into two pieces:

$$Y_1 = Y \cap ([B]_\mathsf{E} \setminus [A]_\mathsf{E}),$$
$$Y_2 = \bigcup_{n=0}^{\infty} (Q_n \cap Y) = [A]_\mathsf{E} \cap Y.$$

Note also that $A \cap Y_1 = \varnothing$, so evidently $A \cap Y_1 \preceq C \cap Y_1$. It remains to check that $A \cap Y_2 \preceq C \cap Y_2$. If Q'_n is the decomposition associated with $[C/B]$, then we need to show that for any $m \in \mathbb{N}$ and any $n \in \mathbb{N} \cup \{\infty\}$, $m < n$, we have $A \cap Q_m \cap Q'_n \preceq C \cap Q_m \cap Q'_n$. This follows from Proposition 2.3.3.

(viii) If $x \notin [A]_{\mathsf{E}}$ or $x \notin [C]_{\mathsf{E}}$, then $[A/B](x)[B/C](x) = 0 = [A/C](x)$. Since the item is claimed to hold mod $\mathscr{H}$, it remains to consider the following situation. Let Q_n, Q'_n, and Q''_n be the decompositions associated with $[A/B]$, $[B/C]$, and $[A/C]$, respectively. We show that the inequality is true mod $\mathscr{H}$ on each $S = Q_k \cap Q'_l \cap Q''_m$ for $k, l, m \in \mathbb{N}$. We have

$$A \cap S \approx k(B \cap S) \quad \text{and} \quad B \cap S \approx l(C \cap S) \implies (A \cap S) \succeq kl(C \cap S).$$

Since also $(A \cap S) \approx m(C \cap S)$, if $kl > m$, then $A \cap S$ is compressible by Proposition 2.3.3. Now for the other direction, suppose $m \geq (k+1)(l+1)$. Then $A \cap S$ admits at least $(k+1)(l+1)$-many copies of $C \cap S$. But each set of $(l+1)$-many copies of $C \cap S$ admits a copy of $B \cap S$ (because $B \cap S \approx l(C \cap S)$), thus $A \cap S$ contains at least $(k+1)$-many copies of $B \cap S$. Since we also have $A \cap S \approx k(B \cap S)$, $A \cap S$ is compressible, and the inequality is proved mod $\mathscr{H}$.

(ix) The argument is left for Exercise 2.7. □

2.5 Subsets of uniform proportion

Lemma 2.5.1. *For any aperiodic cber E on X there exists a decreasing sequence of Borel sets $(F_n)_{n=0}^{\infty}$ such that $F_0 = X$ and $F_{n+1} \sim (F_n \setminus F_{n+1})$ for all $n \in \mathbb{N}$.*

Proof. We start by setting $F_0 = X$ and employing Proposition 1.8.5 to partition $X = F_1 \sqcup Y_1$ into equidecomposable pieces, $F_1 \sim Y_1$. Note that the restriction of E onto F_1 must be aperiodic, so we may partition F_1 into equidecomposable $F_2 \sqcup Y_2$, and continue the process in the same fashion. The sequence F_n is as desired. □

Remark 2.5.2. We call a sequence $(F_n)_{n\in\mathbb{N}}$ as in Lemma 2.5.1 a *fundamental sequence* for E. It is worth noting that while each F_n in a fundamental sequence is necessarily a complete section, the sequence may not vanish, but the saturation of its intersection $S = [F_\infty]_{\mathsf{E}}$, $F_\infty = \bigcap_n F_n$, must be a compressible set. Indeed, if $f_n : F_n \to \big(F_{n-1} \setminus F_n\big)$, $n \geq 1$, are bijections from $[\![\mathsf{E}]\!]$, then $F_\infty \sim f_n(F_\infty)$ for all $n \geq 1$, and sets $f_n(F_\infty)$ are pairwise disjoint, because so are sets $F_{n-1} \setminus F_n$. Thus by Propositions 2.2.1, S is compressible.

For the rest of this section we pick a fundamental sequence $(F_n)_{n=0}^{\infty}$ for E. Note that we necessarily have $[F_n/F_{n+1}] = 2 \bmod \mathscr{H}$ for all $n \in \mathbb{N}$, and, moreover, $[F_n/F_{n+m}] = 2^m \bmod \mathscr{H}$ for all $n, m \in \mathbb{N}$. It is convenient to choose the partition of X guaranteed by Proposition 2.3.4 in such a way that

$$[F_n/F_{n+m}](x) = 2^m \text{ holds for all } x \in X.$$

Also, for any invariant Borel subset $Y \subseteq X$ we agree to choose the partition which arises from intersection with Y of the partition associated with $[X/F_n]$, i.e.,

$$[Y/F_n](x) = \begin{cases} 2^n & \text{if } x \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.5.3. *Let E be an aperiodic cber on X. For any $A \subseteq X$ there exists a subset $B \subseteq A$ such that $\vartheta(B) = \vartheta(A)/2$ for all E-invariant probability measures ϑ on X.*

Proof. Pick some $m \in \mathbb{N}$, and let Q_n, $n \in \mathbb{N} \cup \{\infty\}$, be the partition of $[A]_{\mathsf{E}}$ associated with $[A/F_m]$. We may ignore Q_∞ as $\vartheta(Q_\infty)$ is always zero. Each $A \cap Q_n$ can be partitioned as

$$A \cap Q_n = A_1^n \sqcup \cdots \sqcup A_n^n \sqcup R_n,$$

where $A_j^n \sim F_m \cap Q_n$ for all $1 \leq j \leq n$. Set

$$Z_m = \bigsqcup_n \Big(A_1^n \sqcup \cdots A_{\lfloor n/2 \rfloor}^n \Big) \text{ and } Z_m' = \bigsqcup_n \Big(A_{\lfloor n/2 \rfloor + 1}^n \sqcup \cdots \sqcup A_{2\lfloor n/2 \rfloor}^n \Big).$$

Note that $Z_m \sim Z_m'$, $Z_m \cap Z_m' = \varnothing$, and $(A \cap Q_n) \setminus (Z_m \sqcup Z_m') \subseteq A_n^n \sqcup R_n$, so

$$0 \leq \vartheta(A \cap Q_n) - \vartheta(Z_m \cap Q_n) - \vartheta(Z_m' \cap Q_n) \leq \vartheta(A_n^n) + \vartheta(R_n) \leq 2\vartheta(F_m \cap Q_n).$$

Summing over all n we get that

$$0 \leq \vartheta(A) - \vartheta(Z_m) - \vartheta(Z_m') \leq 2\vartheta\big(F_m \cap [A]_{\mathsf{E}}\big) \leq 2\vartheta(F_m) = 2^{-m+1}. \tag{2.2}$$

We are ready to construct sets B_n and B_n' by induction as follows. For the base, apply the above for to A and F_1 to get subsets $Z_1, Z_1' \subseteq A$. Set $B_1 = Z_1$ and $B_1' = Z_1'$. If B_n and B_n' have been constructed, apply the above procedure to $A \setminus (B_n \sqcup B_n')$ and $m = n+1$ yielding sets Z_{n+1}, Z_{n+1}'. Set $B_{n+1} = B_n \sqcup Z_{n+1}$ and $B_{n+1}' = Z_{n+1}'$. Finally, set

$$B = \bigcup_n B_n = \bigsqcup_n Z_n \quad \text{and} \quad B' = \bigcup_n B_n' = \bigsqcup_n Z_n'.$$

It is easy to see that $B \sim B'$ and (2.2) implies that $\vartheta(A) = \vartheta(B) + \vartheta(B')$, thus $\vartheta(B) = \vartheta(A)/2$ for any E-invariant Borel probability measure ϑ. □

Corollary 2.5.4. *Let E be an aperiodic cber on X. For any $A \subseteq X$ and any $a \in [0,1]$ there exists a subset $B \subseteq A$ such that $\vartheta(B) = a\vartheta(A)$ for all E-invariant probability measures ϑ on X.*

Proof. Using Proposition 2.5.3 we may find $A_1 \subseteq A$ such that $\vartheta(A_1) = \vartheta(A)/2$; set $A_1' = A \setminus A_1$. Using the same proposition for A_1 we can find $A_2 \subseteq A_1$ such that $\vartheta(A_1) = 2\vartheta(A_2)$; set $A_2' = A_1 \setminus A_2$. Continuing in the same fashion, we may construct a decreasing sequence $A_n \supseteq A_{n+1}$ and pairwise disjoint A_n' such that $\vartheta(A_n') = 2^{-n}\vartheta(A)$ for all $n \geq 1$ and all E-invariant probability measures ϑ. Take the parameter a and consider its dyadic representation $a = \sum_{k=1}^\infty \epsilon_k 2^{-k}$, $\epsilon_k \in \{0,1\}$. Set

$$B = \bigsqcup_{\substack{n \geq 1 \\ \epsilon_n = 1}} A_n'.$$

One has

$$\vartheta(B) = \sum_k \epsilon_k \vartheta(A_k') = \sum_k \epsilon_k 2^{-k} \vartheta(A) = a\vartheta(A).$$ □

An interesting observation following from the existence of fundamental sequences and Proposition 2.3.4 is that invariant measures are uniquely determined by their values on invariant sets.

Theorem 2.5.5. *Let E be an aperiodic cber, and let μ and ν be E-invariant Borel probability measures on X. If $\mu(Z) = \nu(Z)$ for all E-invariant Borel sets $Z \subseteq X$, then $\mu = \nu$.*

Proof. Pick a Borel set $A \subseteq X$ and an $\epsilon > 0$. We are going to show that $\bigl|\mu(A) - \nu(A)\bigr| < \epsilon$. Pick m_0 so large that $2^{-m_0} < \epsilon$ and consider the partition $[A]_{\mathsf{E}} = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} Q_n$ associated with $[A/F_{m_0}]$. Note that X can be partitioned into 2^{m_0} many Borel pieces each equidecomposable with F_{m_0}, which implies that

$$\mu(F_{m_0} \cap Z) = 2^{-m_0}\mu(Z) = 2^{-m_0}\nu(Z) = \nu(F_{m_0} \cap Z)$$

for any invariant Borel $Z \subseteq X$. Note also that $\mu(Q_\infty) = 0 = \nu(Q_\infty)$, as Q_∞ is compressible.

On Q_n the set $A \cap Q_n$ can be partitioned as $A \cap Q_n = \bigsqcup_{i=1}^n A_i^n \sqcup R_n$, where $A_i^n \sim F_{m_0} \cap Q_n$ and $R_n \prec F_{m_0} \cap Q_n$. This implies that

$$\mu(A \cap Q_n) \in \bigl[n\mu(F_{m_0} \cap Q_n), (n+1)\mu(F_{m_0} \cap Q_n)\bigr] = \bigl[n2^{-m_0}\mu(Q_n), (n+1)2^{-m_0}\mu(Q_n)\bigr].$$

A similar estimate is valid for ν as well. We therefore have

$$\begin{aligned}\bigl|\mu(A) - \nu(A)\bigr| &= \Bigl|\sum_{n=0}^{\infty}\mu(A \cap Q_n) - \sum_{n=0}^{\infty}\nu(A \cap Q_n)\Bigr| \leq \sum_{n=0}^{\infty}\bigl|\mu(A \cap Q_n) - \nu(A \cap Q_n)\bigr| \\ &\leq \sum_{n=0}^{\infty}\bigl|(n+1)2^{-m_0}\mu(Q_n) - n2^{-m_0}\mu(Q_n)\bigr| \\ &= 2^{-m_0}\sum_{n=0}^{\infty}\mu(Q_n) = 2^{-m_0}\mu(A) \leq 2^{-m_0} < \epsilon.\end{aligned}$$

□

2.6 Local measures

The following lemma describes the behavior of functions $[A/F_n]$ as $n \to \infty$.

Lemma 2.6.1. *Let $A, B \subseteq X$ be Borel.*

1. *The limit $\lim_{n\to\infty}[A/F_n](x)$ exists mod $\mathscr{H}$; it is equal to zero on $X \setminus [A]_{\mathsf{E}}$ and it is equal to ∞ on $[A]_{\mathsf{E}}$ mod $\mathscr{H}$.*
2. *The limit $\lim_{n\to\infty}\frac{[A/F_n](x)}{[B/F_n](x)}$ exists[1] mod $\mathscr{H}$; it assumes a non-zero finite value on $[A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$ modulo the Hopf's ideal.*

Proof. (1) It is clear that $\lim_{n\to\infty}[A/F_n](x) = 0$ for each $x \in X \setminus [A]_{\mathsf{E}}$, since $[A/F_n](x) = 0$ for all such x and all $n \in \mathbb{N}$. We show that $\lim_{n\to\infty}[A/F_n](x) = \infty$ mod $\mathscr{H}$ for $x \in [A]_{\mathsf{E}}$. By Proposition 2.4.1(viii),

$$[A/F_{n+m}] \geq [A/F_n][F_n/F_{n+m}] = [A/F_n] \cdot 2^m \mod \mathscr{H}.$$

If $[A/F_n](x) \neq 0$ for some n, then $[A/F_{n+m}](x) \to \infty$ mod $\mathscr{H}$ as $m \to \infty$. It is therefore enough to show that the set

$$Y = \bigl\{x \in [A]_{\mathsf{E}} : [A/F_n](x) = 0 \text{ for all } n \in \mathbb{N}\bigr\} \text{ is compressible.}$$

Let Q_i^n, $i \in \mathbb{N} \cup \{\infty\}$, be the decomposition associated with $[A/F_n]$. By the definition of the fraction function, $Y = \bigcap_n Q_0^n$, i.e., $A \cap Y \preceq F_n \cap Y$ for all $n \in \mathbb{N}$. This means that $\bigl[F_n/(A \cap Y)\bigr]\big|_Y \geq 1$ mod $\mathscr{H}$ for all $n \in \mathbb{N}$. Since by Proposition 2.4.1(viii) for $x \in Y$ we have

$$\left[\frac{F_0}{A}\right] \geq \left[\frac{F_0}{F_n}\right]\left[\frac{F_n}{A}\right] \geq 2^n\left[\frac{F_n}{A}\right] \geq 2^n \mod \mathscr{H},$$

[1] We assign value 0 to fraction if the numerator is 0 even if the denominator is also zero, and we assign the value ∞ if the numerator is infinite. The latter is less important though, as the behavior of the fraction is studied only up to compressible perturbations, and the set of points where the fraction function is infinite is always compressible.

we conclude that $[F_0/A]\big|_Y = \infty \bmod \mathscr{H}$, which by the definition of the fraction function and Proposition 2.3.4 implies that Y is compressible.

(2) Because of the way we defined the value of a fraction when either the numerator or the denominator is zero, the statement is obvious for $x \notin [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$. So we show that for $x \in [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$ the limit $\lim_{n\to\infty} \frac{[A/F_n](x)}{[B/F_n](x)}$ exists mod $\mathscr{H}$ and attains a finite non-zero value mod $\mathscr{H}$. The key to this is again Proposition 2.4.1(viii), which gives for all $n, m \in \mathbb{N}$

$$\begin{aligned}
&[A/F_{n+m}] \le \big([A/F_n]+1\big)\big([F_n/F_{n+m}]+1\big) = \big([A/F_n]+1\big)(2^m+1) \mod \mathscr{H},\\
&[B/F_{n+m}] \ge [B/F_n][F_n/F_{n+m}] = [B/F_n]\cdot 2^m \mod \mathscr{H}, \quad \text{whence}\\
&\frac{[A/F_{n+m}]}{[B/F_{n+m}]} \le \frac{[A/F_n]+1}{[B/F_n]}(1+2^{-m}) \mod \mathscr{H}.
\end{aligned}$$

We may thus conclude that

$$\limsup_{m\to\infty} \frac{[A/F_{n+m}]}{[B/F_{n+m}]} \le \frac{[A/F_n]+1}{[B/F_n]} \mod \mathscr{H} \quad \text{for all } n \in \mathbb{N}.$$

Note that by item (1), $\lim [B/F_n]\big|_{[B]_{\mathsf{E}}} \to \infty \bmod \mathscr{H}$, so the limsup in the formula above is finite mod $\mathscr{H}$. Also, since the limsup in the left hand side does not depend on $n \in \mathbb{N}$, and since the inequality is true for all $n \in \mathbb{N}$, we get for $x \in [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$

$$\limsup_{n\to\infty} \frac{[A/F_n]}{[B/F_n]} = \limsup_{m\to\infty} \frac{[A/F_{n+m}]}{[B/F_{n+m}]} \le \liminf_{n\to\infty} \frac{[A/F_n]+1}{[B/F_n]} = \liminf_{n\to\infty} \frac{[A/F_n]}{[B/F_n]} \mod \mathscr{H}.$$

This shows that for $x \in [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$ the limit $\lim_{n\to\infty} \frac{[A/F_n](x)}{[B/F_n](x)}$ exists mod $\mathscr{H}$ and is finite. To show that it is non-zero mod $\mathscr{H}$ we use similar inequalities with roles of A and B interchanged (see Exercise 2.8). □

The previous lemma allows us to define the *local measure function* by setting

$$\mathfrak{m}(A, x) = \lim_{n\to\infty} \frac{[A/F_n](x)}{[X/F_n](x)} = \lim_{n\to\infty} \frac{[A/F_n](x)}{2^n}$$

whenever the limit exists, and default the value to 0, whenever the limit does not exist.

Proposition 2.6.2. *The local measure function satisfies the following properties for all Borel $A, B \subseteq X$.*

(i) $\mathfrak{m}(A, \cdot\,) : X \to \mathbb{R}^{\ge 0}$ *is a Borel function.*

(ii) $\mathfrak{m}(X, x) = 1$ *and* $\mathfrak{m}(\varnothing, x) = 0$ *for all* $x \in X$.

(iii) If $A \sim B$, *then* $\mathfrak{m}(A, x) = \mathfrak{m}(B, x)$ *mod* $\mathscr{H}$.

(iv) If $x \,\mathsf{E}\, y$, *then* $\mathfrak{m}(A, x) = \mathfrak{m}(A, y)$.

(v) $\mathfrak{m}(A, x) = 0$ *mod* $\mathscr{H}$ *if and only if* $A \in \mathscr{H}$.

(vi) $\mathfrak{m}(A, x) > 0$ *mod* $\mathscr{H}$ *for* $x \in [A]_{\mathsf{E}}$.

(vii) If $Y = \{x \in X : \mathfrak{m}(A, x) < \mathfrak{m}(B, x)\}$, *then* $A \cap Y \preceq B \cap Y$ *mod* $\mathscr{H}$, *i.e., there exists* $Y' \subseteq Y$ *such that* $Y \setminus Y' \in \mathscr{H}$ *and* $A \cap Y' \preceq B \cap Y'$.

(viii) If $A_n \subseteq X$ *are pairwise disjoint, then* $\mathfrak{m}\big(\bigcup_n A_n, x\big) = \sum_n \mathfrak{m}(A_n, x)$ *mod* $\mathscr{H}$.

(ix) If $S \subseteq X$ is E-invariant, then $\mathfrak{m}(A,x)\big|_S = \mathfrak{m}(A\cap S,x)\big|_S$ mod $\mathscr{H}$ in the sense that the set

$$\big\{x \in S : \mathfrak{m}(A,x) \neq \mathfrak{m}(A\cap S,x)\big\} \in \mathscr{H}.$$

Proof. Item (i) is obvious, since $[A/F_n]$ are Borel, and a pointwise limit of Borel functions is Borel; (ii) is evident from the definition of $\mathfrak{m}$. Items (iii) and (iv) follow from Proposition 2.4.1(ii) and Proposition 2.4.1(i) respectively. Items (v) and (vi) form the content of Lemma 2.6.1(2). Also (ix) is evident from Proposition 2.4.1(vi). So it remains to prove (vii) and (viii).

(vii) Let

$$\tilde{Y}_n = \big\{x \in X : [A/F_n](x) < [B/F_n](x)\big\},$$

and set $Y' = Y \cap \bigcup_n \tilde{Y}_n$; we may partition $Y' = \bigsqcup_{n\in\mathbb{N}} Y_n$, where $Y_n = (Y' \cap \tilde{Y}_n) \setminus \bigcup_{i<n} Y_i$. Note that each Y_n is E-invariant; using $Y_n \subseteq \tilde{Y}_n$ and Proposition 2.4.1(vii) we have $Y_n \cap A \preceq Y_n \cap B$, and therefore $A \cap Y' \preceq B \cap Y'$.

(viii) This item requires some amount of work. We begin by noting that finite additivity follows easily from Proposition 2.4.1(ix). Indeed, if A and B are disjoint, then

$$\frac{[A/F_n]+[B/F_n]}{[X/F_n]} \leq \frac{\big[(A\cup B)/F_n\big]}{[X/F_n]} \leq \frac{[A/F_n]+[B/F_n]+2}{[X/F_n]}.$$

and both the lower and the upper bounds converge to $\mathfrak{m}(A,x)+\mathfrak{m}(B,x)$ as $n\to\infty$. Together with Proposition 2.4.1(iv), this guarantees that $\mathfrak{m}(\bigcup_i A_i,x) \geq \sum_{i=0}^{\infty}\mathfrak{m}(A_i,x)$ mod $\mathscr{H}$. Indeed, since obviously $\bigcup_{i=0}^{m} A_i \preceq \bigcup_{i=0}^{\infty} A_i$ for all $m \in \mathbb{N}$, we have

$$\Big[\Big(\bigcup_{i=0}^{m} A_n\Big) \;/\; F_n\Big] \leq \Big[\Big(\bigcup_{i=0}^{\infty} A_n\Big) \;/\; F_n\Big] \quad \operatorname{mod} \mathscr{H}$$

and so $\mathfrak{m}(\bigcup_{i=0}^{\infty} A_i,x) \geq \mathfrak{m}(\bigcup_{i=0}^{m} A_i,x) = \sum_{i=0}^{m}\mathfrak{m}(A_i,x)$ mod $\mathscr{H}$ for all $m\in\mathbb{N}$. Since the left hand side does not depend on m, we get $\mathfrak{m}(\bigcup_{i=0}^{\infty} A_i,x) \geq \sum_{i=0}^{\infty}\mathfrak{m}(A_i,x)$ mod $\mathscr{H}$ (and, in particular, the right hand side converges modulo the Hopf's ideal). It remains to show the inequality in the other direction.

First we show the following claim: If $\mathfrak{m}(A,x) > \sum_{i=0}^{\infty}\mathfrak{m}(A_i,x)$ mod $\mathscr{H}$, where A_i are pairwise disjoint, then $\bigcup_{i=0}^{\infty} A_i \preceq A$ mod $\mathscr{H}$. If $\mathfrak{m}(A,x) > \sum_{i=0}^{\infty}\mathfrak{m}(A_i,x)$ mod $\mathscr{H}$, we have, in particular, that $\mathfrak{m}(A,x) > \mathfrak{m}(A_0,x)$ mod $\mathscr{H}$, so by item (vii) we have $A_0 \preceq A$ mod $\mathscr{H}$, i.e., there exists $B_0 \subseteq A$ such that $A_0 \sim B_0$ mod $\mathscr{H}$. We thus have

$$\mathfrak{m}(A,x) = \mathfrak{m}(B_0,x) + \mathfrak{m}(A\setminus B_0,x) > \sum_{i=0}^{\infty}\mathfrak{m}(A_i,x) \quad \operatorname{mod} \mathscr{H}.$$

Since $\mathfrak{m}(A_0,x) = \mathfrak{m}(B_0,x)$ mod $\mathscr{H}$ by (iii), we conclude that $\mathfrak{m}(A\setminus B_0,x) > \sum_{i=1}^{\infty}\mathfrak{m}(A_i,x)$ mod $\mathscr{H}$, so one may find $B_1 \subseteq A\setminus B_0$ such that $B_1 \sim A_1$ mod $\mathscr{H}$. Continuing the argument, we construct pairwise disjoint $B_i \subseteq A$ such that $B_i \sim A_i$ mod $\mathscr{H}$, hence $\bigcup_i A_i \preceq A$ mod $\mathscr{H}$ as claimed.

To finish the proof of item (viii), assume towards a contradiction that we have $\mathfrak{m}(\bigcup_i A_i,x) > \sum_i \mathfrak{m}(A_i,x)$ mod $\mathscr{H}$ for a certain family of pairwise disjoint sets A_i. We may pick $k\in\mathbb{N}$ so large that the set

$$S = \Big\{x \in X : \mathfrak{m}\Big(\bigcup_i A_i,x\Big) > \sum_i \mathfrak{m}(A_i,x) + 2^{-k}\Big\} \quad \text{is in compressible.}$$

By adding the set $X\setminus\bigcup_i A_i$ to family A_i, we may assume without loss of generality that $\bigcup_i A_i = X$. We have mod $\mathscr{H}$ the following inequalities

$$\mathfrak{m}(X\setminus F_k,x) = \mathfrak{m}(X,x) - 2^{-k} = \mathfrak{m}\Big(\bigcup_i A_i,x\Big) - 2^{-k} > \sum_i \mathfrak{m}(A_i,x),$$

which by the claim above implies that $\bigcup_i A_i \preceq X \setminus F_k$ mod $\mathscr{H}$. Since F_k is an E-complete section, we have, in fact, that $X = \bigcup A_i \prec X$ mod $\mathscr{H}$, i.e., $X \in \mathscr{H}$, and so (viii) is trivially valid mod $\mathscr{H}$. □

2.7 Uniqueness of local measures

We have built the local measure function $\mathfrak{m}(A, x)$ via an explicit construction. The goal of this section is to show that properties of $\mathfrak{m}$ listed in Proposition 2.6.2 identify $\mathfrak{m}$ uniquely. For the purpose of this section we define a *local measure function* on X to be any map $\mathfrak{n} : \mathcal{B} \times X \to \mathbb{R}^{\geq 0}$ such that for all Borel $A, B \in \mathcal{B}$

1. $\mathfrak{n}(A, \cdot) : X \to \mathbb{R}^{\geq 0}$ is Borel.
2. $\mathfrak{n}(X, x) = 1$ mod $\mathscr{H}$ and $\mathfrak{n}(\varnothing, x) = 0$ mod $\mathscr{H}$.
3. $\mathfrak{n}\big(\bigcup_n A_n, x\big) = \sum_n \mathfrak{n}(A_n, x)$ mod $\mathscr{H}$ for any pairwise disjoint family of Borel sets.
4. $\mathfrak{n}(A, x) = \mathfrak{n}(B, x)$ mod $\mathscr{H}$ for all $A \sim B$.
5. $\mathfrak{n}(A, x) = \mathfrak{n}(A, y)$ for all $x, y \in X$ such that $x \mathsf{E} y$.
6. If $S \subseteq X$ is E-invariant, then $\mathfrak{n}(A, x)\big|_S = \mathfrak{n}(A \cap S, x)\big|_S$ mod $\mathscr{H}$.

We note that item (6) implies that $\mathfrak{n}(A, x) = 0$ mod $\mathscr{H}$ for $x \in X \setminus [A]_{\mathsf{E}}$. Since local measures must attain non-negative values, additivity implies monotonicity: if $A \subseteq B$, then $\mathfrak{n}(A, x) \leq \mathfrak{n}(B, x)$ mod $\mathscr{H}$, because

$$\mathfrak{n}(B, x) = \mathfrak{n}(A, x) + \mathfrak{n}\big(B \setminus A, x\big) \mod \mathscr{H}.$$

Lemma 2.7.1. *Let $\mathfrak{n}$ be a local measure function on X and let $F \subseteq X$ be such that X can be partitioned into Borel sets $X = \bigcup_{i=1}^n \tilde{F}_i$ such that $F \sim \tilde{F}_i$ for all i. In this case $\mathfrak{n}(F, x) = 1/n$ mod $\mathscr{H}$.*

Proof. By item (4) $\mathfrak{n}(F, x) = \mathfrak{n}(\tilde{F}_i, x)$ mod $\mathscr{H}$ for all i, and also by (3) $\mathfrak{n}(X, x) = \sum_{i=1}^n \mathfrak{n}(\tilde{F}_i, x)$ mod $\mathscr{H}$. Since by (2) $\mathfrak{n}(X, x) = 1$ mod $\mathscr{H}$, we have modulo the Hopf's ideal

$$1 = \mathfrak{n}(X, x) = \sum_{i=1}^n \mathfrak{n}(\tilde{F}_i, x) = n\mathfrak{n}(F, x),$$

whence $\mathfrak{n}(F, x) = 1/n$ mod $\mathscr{H}$. □

Lemma 2.7.2. *If $A \approx nB$, then $n\mathfrak{n}(B, x) \leq \mathfrak{n}(A, x) \leq (n+1)\mathfrak{n}(B, x)$ mod $\mathscr{H}$.*

Proof. Recall that $A \approx nB$ means that we can partition $A = \bigsqcup_{i=1}^n A_i \sqcup R$ in such a way that $A_i \sim B$ and $R \prec B$. Modulo the Hopf's ideal we have

$$n\mathfrak{n}(B, x) = \sum_{i=1}^n \mathfrak{n}(B_i, x) \leq \mathfrak{n}(A, x) = \sum_{i=1}^n \mathfrak{n}(B_i, x) + \mathfrak{n}(R, x) \leq (n+1)\mathfrak{n}(B, x),$$

where the last inequality uses monotonicity and item (4). □

Proposition 2.7.3. *If $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are local measures on X, then $\mathfrak{n}_1(A, x) = \mathfrak{n}_2(A, x)$ mod $\mathscr{H}$ for all Borel $A \subseteq X$.*

Proof. We are going to show that for any $\epsilon > 0$ and any Borel $A \subseteq X$ one has

$$\left|\mathfrak{n}_1(A,x) - \mathfrak{n}_2(A,x)\right| < \epsilon \mod \mathscr{H}.$$

Pick n_0 so large that $2^{-n_0} < \epsilon$, and note that Lemma 2.7.1 implies

$$\mathfrak{n}_j(F_{n_0}, x) = 2^{-n_0} \mod \mathscr{H}, j = 1, 2,$$

where F_{n_0} is an element in the fundamental sequence. Take the decomposition $[A]_{\mathsf{E}} = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} Q_n$ associated with $[A/F_{n_0}]$. One has

$$\mathfrak{n}_1(A,x) = 0 = \mathfrak{n}_2(A,x) \mod \mathscr{H} \text{ for } x \notin [A]_{\mathsf{E}},$$

so it is enough to show that for each $m \in \mathbb{N}$

$$\left|\mathfrak{n}_1(A,x) - \mathfrak{n}_2(A,x)\right| < \epsilon \mod \mathscr{H} \text{ for } x \in Q_m.$$

Pick some $m_0 \in \mathbb{N}$. Item (6) ensures that $\mathfrak{n}_j(A,x) = \mathfrak{n}_j(A \cap Q_{m_0}, x) \bmod \mathscr{H}$ for $x \in Q_{m_0}$ and $j = 1, 2$. But $A \cap Q_{m_0} \approx m_0(F_{n_0} \cap Q_{m_0})$, which by Lemma 2.7.2 means that modulo $\mathscr{H}$ for $x \in Q_{m_0}$ we have for both $j = 1$ and $j = 2$

$$\mathfrak{n}_j(A,x) = \mathfrak{n}_j(A \cap Q_{m_0}, x) \in \left[m_0\mathfrak{n}_j(F_{n_0}, x), (m_0+1)\mathfrak{n}_j(F_{n_0}, x)\right] = [m_0 2^{-n_0}, (m_0+1)2^{-n_0}].$$

The latter yields that $\left|\mathfrak{n}_1(A,x) - \mathfrak{n}_2(A,x)\right| \leq 2^{-n_0} < \epsilon \bmod \mathscr{H}$ on Q_{m_0}, as claimed. □

2.8 From local to global

In this section we show how to concoct from the local measure function a genuine measure on X whenever X is in compressible, thus finishing the proof of Nadkarni's Theorem.

Let E be a cber on X, pick an action of a countable group $H \curvearrowright X$ which realizes E. By the standard "change of topology" technique we may endow X with a zero-dimensional Polish topology such that the action $H \curvearrowright X$ becomes continuous. Let $\mathcal{C}$ be a countable family of cl open subsets of X which

- forms a basis for the topology on X;
- is an algebra of sets, i.e., it is closed under finite unions, finite intersections, and complements;
- is H-invariant in the sense that $hC \in \mathcal{C}$ for all $h \in H$ and all $C \in \mathcal{C}$.

Pick a compatible metric d on X, and let $\mathcal{C}_k = \{C \in \mathcal{C} : \operatorname{diam} C \leq 1/k\}$. Note that each $\mathcal{C}_k$ is a sequential covering class (see Appendix B). For each $C \in \mathcal{C}$ and $k \geq 1$ we pick a (finite or infinite) partition $C = \bigsqcup_n C_n^k$ such that $C_n^k \in \mathcal{C}_k$ for all $n \in \mathbb{N}$. Since the family $\mathcal{C}$ is countable, we may select an E-invariant subset $Z \subseteq X$ such that $X \setminus Z$ is compressible and for all $x \in Z$ and all $C, D \in \mathcal{C}$ we have

(i) $\mathfrak{m}(\varnothing, x) = 0$;

(ii) $\mathfrak{m}(C,x) = \sum_{n=0}^{\infty} \mathfrak{m}(C_n^k, x)$;

(iii) $\mathfrak{m}(C,x) + \mathfrak{m}(D,x) = \mathfrak{m}(C \cup D, x)$ whenever $C \cap D = \varnothing$;

(iv) $C \underset{\mathsf{E}}{\sim} D \implies \mathfrak{m}(C,x) = \mathfrak{m}(D,x)$.

Items above are instances of corresponding items in Proposition 2.6.2 except that we may assume that they are true for each $x \in Z$, instead of holding mod $\mathscr{H}$.

Theorem 2.8.1. *Let $\tau_x : \mathcal{C} \to [0,1]$ be given by $\tau_x(C) = \mathfrak{m}(C,x)$ for each $x \in Z$, and let $\mu_x : \mathcal{B} \to [0,\infty]$ be the Carathéodory measure on Borel subsets of X associated with the outer measure constructed over τ (see Appendix B for the construction of the outer measure over τ). For each $x \in Z$, μ_x is an E-invariant Borel probability measure on X. Moreover, $\mu_x(C) = \tau_x(C)$ for all $C \in \mathcal{C}$.*

Proof. Each μ_x is a Borel measure on X by the Carathéodory's Theorem. Since $X \in \mathcal{C}$, to show that μ_x is a probability measure, it is enough to check the moreover part, i.e., that $\mu_x(C) = \tau_x(C)$ for all $C \in \mathcal{C}$. Pick $C \in \mathcal{C}$ and $\epsilon > 0$, we show that $\mu_x(C) \geq \tau_x(C) - \epsilon$. For each $k \geq 1$ we have a family $C_n^k \in \mathcal{C}_k$ such that $C = \bigsqcup_n C_n^k$ and $\tau_x(C) = \sum_n \tau_x(C_n^k)$, so we may pick p_n so large that $\tau_x(C) - \epsilon/2^{-k} \geq \sum_{n=0}^{p_k} \tau_x(C_n^k)$. Put $Y_N = \bigcap_{k=1}^N \bigcup_{n=0}^{p_k} C_n^k$, and note that $Y_N \supseteq Y_{N+1}$ for all $N \in \mathbb{N}$ and that

$$\tau_x(Y_N) \geq \tau_x(C) - \sum_{k=1}^N \epsilon/2^{-k} > \tau_x(C) - \epsilon.$$

Since each Y_N is covered by finitely many balls of diameter $\leq 1/N$ and since each C_n^k is closed, the intersection $Y := \bigcap_N Y_N$ is a compact subset of C.

We claim that $\mu_x(Y) > \tau_x(C) - \epsilon$, thus showing that also $\mu_x(C) > \tau_x(C) - \epsilon$. Pick $\delta > 0$ and let $D_j \in \mathcal{C}$, $j \in \mathbb{N}$, be a cover of Y such that $\mu_x(Y) + \delta > \sum_j \tau_x(D_j)$. Since Y is compact and each D_j is cl open, there is a finite subcover $\mathcal{D}_0, \dots, \mathcal{D}_M$ of Y. We therefore also have

$$\mu_x(Y) + \delta > \sum_{j=0}^M \tau_x(D_j).$$

Since Y is compact and $D_0 \cup \dots \cup D_M$ is open, there exists $\delta' > 0$ so small that $d(y_1, y_2) > \delta'$ for all $y_1 \in Y$ and all $y_2 \notin D_0 \cup \dots \cup D_M$. Since $\operatorname{dist}(Y, Y_N) \to 0$ as $N \to \infty$, one can find N_0 so large that $Y_{N_0} \subseteq D_0 \cup \dots \cup D_M$. But $Y_{N_0} \in \mathcal{C}$, so

$$\mu_x(Y) > \sum_{j=0}^M \tau_x(C_j) - \delta = \mathfrak{m}\Big(\bigcup_{j=0}^M D_j, x\Big) - \delta \geq \mathfrak{m}(Y_{N_0}, x) - \delta > \tau_x(C) - \epsilon - \delta.$$

Since δ is arbitrary, $\mu_x(Y) > \tau_x(C) - \epsilon$, and so $\mu_x(C) \geq \tau_x(C)$.

The opposite inequality is evident from the definition of μ_x, and so we have $\mu_x(C) = \tau_x(C)$ for all $C \in \mathcal{C}$. In particular, $\mu_x(X) = \tau_x(X) = \mathfrak{m}(X,x) = 1$, so μ_x is a probability measure on X.

Finally, we show that it is E-invariant. By Proposition 1.6.2, this is equivalent to showing that μ_x is H-invariant. Let

$$\mathcal{D} = \big\{A \in \mathcal{B} : \mu_x\big(h(A)\big) = \mu_x(A) \text{ for all } h \in H\big\}$$

be the family of H-invariant Borel subsets of X. The set $\mathcal{D}$ is a λ-system. By item (iv) in the choice of Z and H-invariance of $\mathcal{C}$, we have $\mathcal{C} \subseteq \mathcal{D}$. Dynkin's π-λ Theorem ensures that the σ-algebra generated by $\mathcal{C}$ is a subset of $\mathcal{D}$, thus $\mathcal{D} = \mathcal{B}$ and μ_x is E-invariant. □

For the record we now have a complete proof of the Nadkarni's Theorem.

Corollary 2.8.2. *If E is an incompressible cber, then E admits a probability invariant measure.*

Proof. Since E is incompressible, the set Z above cannot be empty, so there is some $x \in Z$, and μ_x is then a probability E-invariant measure. □

2.9 Ergodic decomposition

In this section we derive existence of an ergodic decomposition for any (aperiodic) cber E. This result is originally due to Veeravalli S. Varadarajan [Var63].

For an incompressible cber E, we have selected an E-invariant Borel subset $Z \subseteq X$ such that $X \setminus Z$ is compressible, and to each $x \in Z$ there corresponds an E-invariant Borel probability measure μ_x on X. We have also picked an H-invariant countable algebra of sets $\mathcal{C}$ which generates the σ-algebra of Borel sets. By construction $\mu_x(C) = \mathfrak{m}(C, x)$ for all $x \in Z$ and all $C \in \mathcal{C}$. Two comments are in order.

One observation is that while we have $\mu_x(C) = \mathfrak{m}(C, x)$ for all $x \in Z$ and all $C \in \mathcal{C}$, we also have $\mu_x(A) = \mathfrak{m}(A, x) \bmod \mathscr{H}$ for any Borel $A \subseteq X$. Here is one way to see it. By refining the topology on X, we may pick a countable algebra $\mathcal{C}'$ which contains $\mathcal{C}$, $A \in \mathcal{C}'$, and $\mathcal{C}'$ is an H-invariant clopen basis for a zero-dimensional topology on X. We can run the construction from Section 2.8 with respect to $\mathcal{C}'$ and get a subset $Z' \subseteq X$, such that $X \setminus Z'$ is compressible, and an assignment $Z' \ni x \mapsto \mu'_x \in \mathtt{INV}(\mathsf{E})$. Since we we have $\mu_x(C) = \mu'_x(C)$ for all $C \in \mathcal{C}$ and all $x \in Z \cap Z'$, Carathédory's Uniqueness Theorem (or just CUT for short) implies that $\mu_x = \mu'_x$ for all $x \in Z \cap Z'$, but $\mu'_x(A) = m(A, x)$ for all $x \in Z'$, thus $\mu_x(A) = \mathfrak{m}(A, x)$ mod $\mathscr{H}$.

Another convenient fact is that there is no loss in assuming that $Z = X$ (provided Z is non-empty, of course). For we may pick $x_0 \in Z$ and set $\mu_x = \mu_{x_0}$ for all $x \in X \setminus Z$.

Lemma 2.9.1. *The assignment $X \ni x \mapsto \mu_x \in \mathtt{INV}(\mathsf{E})$ satisfies the following properties.*

1. *The map $x \mapsto \mu_x(A)$ is Borel for any Borel $A \subseteq X$.*
2. *$\mu_x = \mu_y$ whenever $x \mathsf{E} y$.*
3. *For any $x \in X$ the set $S_x = \{y \in X : \mu_x = \mu_y\}$ is E-invariant and Borel.*
4. *If $\tilde{Z} \subseteq X$ is given by $\tilde{Z} = \big\{x \in X : \mu_x(S_x) = 1\big\}$, then $X \setminus \tilde{Z}$ is compressible.*
5. *Each measure μ_x, $x \in \tilde{Z}$, is ergodic.*
6. *The map $\tilde{Z} \to \mathtt{EINV}(\mathsf{E})$ is surjective, i.e., any ergodic measure appears as μ_x for some $x \in \tilde{Z}$.*
7. *For all $x \in \tilde{Z}$ the measure μ_x is the unique ergodic invariant probability measure for the restriction of E onto S_x.*

Proof. (1) Let $\mathcal{D}$ denote the set of Borel subsets $A \subseteq X$ for which that map $x \mapsto \mu_x(A)$ is Borel. Countable additivity of measures implies that $\mathcal{D}$ is necessarily a λ-system. By Theorem 2.8.1 $\mu_x(C) = \mathfrak{m}(C, x)$ for all $C \in \mathcal{C}$, and therefore item (1) of Proposition 2.6.2 implies that $\mathcal{C} \subseteq \mathcal{D}$. By Dynkin's π-λ Theorem, $\mathcal{B} \subseteq \mathcal{D}$, as claimed.

(2) Since $\mu_x(C) = \mu_y(C)$ for all $C \in \mathcal{C}$ whenever $x \mathsf{E} y$, one has $\mu_x = \mu_y$ by CUT.

(3) The set S_x is E-invariant by (2). Let for $C \in \mathcal{C}$ the set $S_{x,C}$ be given by

$$S_{x,C} = \big\{y \in X : \mu_x(C) = \mu_y(C)\big\}.$$

By CUT $S_x = \bigcap_{C \in \mathcal{C}} S_{x,C}$. Each $S_{x,C}$ is Borel, for

$$S_{x,C} = \big\{y \in X : \mathfrak{m}(x, C) = \mathfrak{m}(y, C)\big\},$$

and Proposition 2.6.2(1).

(4) In the notation above it is enough to prove that $\mu_x(S_{x,C}) = 1 \bmod \mathscr{H}$. Define for $n \in \mathbb{N}$

$$S_{x,C,n} = \big\{y \in X : [C/F_n](y) = [C/F_n](y)\big\}.$$

If $[C]_{\mathsf{E}} = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} Q_n$ is the decomposition associated with $[C/F_n]$, then

$$S_{x,C,n} = \begin{cases} \big(X \setminus [C]_{\mathsf{E}}\big) \cup Q_0 & \text{if } n = 0 \\ Q_n & \text{otherwise.} \end{cases}$$

In particular, $S_{x,C,n}$ is Borel. Note also that $\bigcap_n S_{x,C,n} \subseteq S_{x,C}$. For each C and n there are only countably many sets of the form $S_{x,C,n}$ and $S_{x,C,n} = S_{y,C,n}$ whenver $y \in S_{x,C,n}$. So it is enough to show that for each $\tilde{S}$ of the form $S_{x,C,n}$ one has $\mu_x(\tilde{S}) = 1 \bmod \mathscr{H}$ for $x \in \tilde{S}$. This follows from the fact that $\tilde{S}$ is E-invariant, so $\mathfrak{m}(\tilde{S}, x) = 1$ for all $x \in \tilde{S}$, and also $\mu_x(\tilde{S}) = \mathfrak{m}(\tilde{S}, x) \bmod \mathscr{H}$.

(5) Let $Y \subseteq X$ be invariant. We need to show that for any $x_0 \in \tilde{Z}$ either $\mu_{x_0}(Y) = 0$ or $\mu_{x_0}(Y) = 1$. We know that $\mathfrak{m}(Y, x) \in \{0, 1\}$ for all $x \in X$, and also $\mu_x(Y) = \mathfrak{m}(Y, x) \bmod \mathscr{H}$. Since S_{x_0} is incompressible (because μ_{x_0} is an invariant measure on S_{x_0}), there must exist some $y_0 \in S_{x_0}$ such that $\mu_{y_0}(Y) = \mathfrak{m}(Y, y_0)$, whence $\mu_{x_0}(Y) = \mu_{y_0}(Y) \in \{0, 1\}$.

(6) Let ν be an invariant ergodic probability measure on X. First we claim that for any $C \in \mathcal{C}$ the set

$$S_{\nu,C} = \big\{x \in X : \mu_x(C) = \nu(C)\big\}$$

is ν-full. Recall that for any invariant measure and any invariant set $Y \subseteq X$ we have $\nu(F_n \cap Y) = 2^{-n}\nu(Y)$. Pick $\epsilon > 0$ and m_0 so large that $2^{-m_0} < \epsilon$. Let Q_n, $n \in \mathbb{N} \cup \{\infty\}$, be the decomposition associated with $[C/F_n]$. By ergodicity of ν either $\nu(X \setminus [C]_{\mathsf{E}}) = 1$ or $\nu(Q_n) = 1$ for exactly one $n \in \mathbb{N}$ ($n = \infty$ is excluded, since Q_∞ is compressible). Suppose $\nu(Q_{n_0}) = 1$ for some $n_0 \in \mathbb{N}$. In this case $C \approx n_0 F_{m_0}$, and so

$$\nu(C) = \nu(C \cap Q_{n_0}) \in \big[2^{-m_0} n_0, 2^{-m_0}(n_0 + 1)\big].$$

Also

$$\mu_x(C \cap Q_{n_0}) \in \big[2^{-m_0} n_0 \mu_x(Q_{n_0}), 2^{-m_0}(n_0 + 1)\mu_x(Q_{n_0})\big] \text{ for all } x \in X.$$

Finally, since $\mu_x(Q_{n_0}) = 1 \bmod \mathscr{H}$ for $x \in Q_{n_0}$, we get that for ν-almost all $x \in X$

$$\big|\nu(C) - \mu_x(C)\big| < 2^{-m_0} < \epsilon.$$

The above analysis was done under the assumption that $\nu(Q_{n_0}) = 1$ for some $n_0 \in \mathbb{N}$. If $\nu(X \setminus [C]_{\mathsf{E}}) = 1$, then $\nu(C) = 0$, and also $\mu_x(X \setminus [C]_{\mathsf{E}}) = 1 \bmod \mathscr{H}$ for $x \in X \setminus [C]_{\mathsf{E}}$, so $\mu_x(C) = 0$ for ν-almost all $x \in X$. This shows that $\nu(S_{\nu,C}) = 1$, and therefore also

$$\nu\Big(\bigcap_{C \in \mathcal{C}} S_{\nu,C}\Big).$$

Since this intersection is incompressible, we may pick $z_0 \in \tilde{Z} \cap \bigcap_{C \in \mathcal{C}} S_{\nu,C}$. Since $\mu_{z_0}(C) = \nu(C)$ for all $C \in \mathcal{C}$, Carathéodory's Uniqueness Theorem ensures that $\mu_{z_0} = \nu$.

(7) Pick some $x \in \tilde{Z}$ and let ν be an ergodic invariant probability measure on S_x. By item (6) there is some $z \in \tilde{Z}$ such that $\mu_z = \nu$. Since $\mu_z(S_z) = 1$, the intersection $S_x \cap S_z$ is non-empty; whence $\mu_x = \mu_z = \nu$. $\square$

Following an earlier remark, we may redefine the assignment $x \mapsto \mu_x$ on $X \setminus \tilde{Z}$ by picking $z_0 \in \tilde{Z}$ and setting $\mu_x = \mu_{z_0}$ for all $z_x \in X \setminus \tilde{Z}$. With this twist Lemma 2.9.1 can be summarized into the following very important Ergodic Decomposition Theorem

Theorem 2.9.2. *Let* E *be an aperiodic incompressible cber on* X*. There exists an ergodic decomposition: a Borel surjection* $X \ni x \mapsto \mu_x \in \mathtt{EINV}(X)$ *such that*

(i) $\mu_x = \mu_y$ *whenever* $x \,\mathsf{E}\, y$*.*

(ii) For all $x \in X$ *the set* $\{y \in X : \mu_x = \mu_y\}$ *is Borel,* $\mu_x\big(\{y \in X : \mu_x = \mu_y\}\big) = 1$*, and* μ_x *is the unique ergodic invariant probability measure on this set.*

Moreover, such a decomposition is unique up to a compressible set: if $x \mapsto \mu'_x$ *is another ergodic decomposition, then* $\{x \in X : \mu_x \neq \mu'_x\}$ *is compressible.*

Proof. Existence of ergodic decomposition follows from Lemma 2.9.1 and the remark after it. To check uniqueness let $Z = \{x \in X : \mu_x \neq \mu'_x\}$, and note that Z is Borel and E-invariant. Suppose it is incompressible. By Nadkarni's Theorem there must be an invariant ergodic measure ν on Z. Since $x \to \mu_x$ and $x \to \mu'_x$ are surjections, there must be some $x_1, x_2 \in X$ such that $\mu_{x_1} = \nu = \mu'_{x_2}$. Since $\mu_{x_1}(S_{x_1}) = 1 = \mu'_{x_2}(S'_{x_2})$, where

$$S_{x_1} = \{x \in X : \mu_x = \mu_{x_1}\},$$
$$S'_{x_2} = \{x \in X : \mu'_x = \mu'_{x_2}\},$$

there is some $y \in S_{x_1} \cap Z \cap S'_{x_2}$. But this means that $\mu_y = \mu_{x_1} = \nu = \mu'_{x_2} = \mu'_y$, contradicting the definition of Z. □

Let ν be any (not necessarily invariant) probability measure on X. For a Borel set $A \subseteq X$ define $\hat{\nu}(A)$ by the formula

$$\hat{\nu}(A) = \int_X \mu_x(A)\, d\nu(x).$$

It is easy to check that $\hat{\nu}$ is an E-invariant probability measure on X. Additivity for $\hat{\nu}$ follows from Tonelli's Theorem.

Proposition 2.9.3. *Let* ν *be a probability measure on* X*. The measure* ν *is* E*-invariant if and only if* $\nu = \hat{\nu}$*.*

Proof. Sufficiency comes from the fact that $\hat{\nu}$ is E-invariant. So, suppose ν is E-invariant, we show that

$$\nu = \int_X \mu_x\, d\nu.$$

By Theorem 2.5.5, it is enough to check that $\nu(Y) = \hat{\nu}(Y)$ holds for all invariant $Y \subseteq X$. Since all μ_x are ergodic,

$$\hat{\nu}(Y) = \nu\Big(\big\{x \in X : \mu_x(Y) = 1\big\}\Big).$$

But $\{x : \mu_x(Y) = 1\} = Y \bmod \mathscr{H}$, so $\hat{\nu}(Y) = \nu(Y)$. □

Exercises

Exercise 2.1. Let E be a cber on X, let $A \subseteq X$ be an E-compressible set, and let $B \subseteq A$. Show that if $B \cap [x]_{\mathsf{E}} = A \cap [x]_{\mathsf{E}}$ for all $x \in B$, then B is also compressible.

Exercise 2.2. Show that $\mathscr{H}_{\mathsf{E}}$ is a Borel ideal for any cber E. In other words show that

- if $A \in \mathscr{H}$ and $B \subseteq A$ is Borel, then $B \in \mathscr{H}$;
- if $A_n \in \mathscr{H}$, $n \in \mathbb{N}$, then $\bigcup_n A_n \in \mathscr{H}$.

Exercise 2.3. Show that if $[A]_{\mathsf{E}} = [B]_{\mathsf{E}}$ and both A and B are compressible, then $A \sim B$.

Exercise 2.4. Show that $A \prec A$ if and only if the set A is compressible.

▸ *Exercise* 2.5. Let E be a cber on X. Let us say that a subset A of X is *syndetic* if there are $n \in \mathbb{N}$ and elements $f_i \in [\![\mathsf{E}]\!]$, $1 \leq i \leq n$, such that $A \subseteq \operatorname{dom}(f_i)$ and $X = \bigcup_{i=1}^n f_i(A)$. Show that E is compressible if and only if $A \sim B$ for all syndetic subsets $A, B \subseteq X$.

▸ *Exercise* 2.6. Let E be a cber on X. Show that E is compressible if and only if there is an aperiodic smooth equivalence relation E' on X such that $\mathsf{E}' \subseteq \mathsf{E}$, i.e., $x \,\mathsf{E}'\, y \implies x \,\mathsf{E}\, y$ for all $x, y \in X$.

Exercise 2.7. Give a proof of item (ix) from Proposition 2.4.1.

Exercise 2.8. Complete the proof of Lemma 2.6.1 by showing that $\lim_{n\to\infty} \frac{[A/F_n](x)}{[B/F_n](x)}$ for $x \in [A]_{\mathsf{E}} \cap [B]_{\mathsf{E}}$ is non-zero mod $\mathscr{H}$.

Exercise 2.9. Let E be a cber on X, and let $A, B \subseteq X$ be such that $\vartheta(A) \leq \vartheta(B)$ for all $\vartheta \in \mathtt{EINV}(\mathsf{E})$. Show that there exists $f \in [\mathsf{E}]$ such that $f(A) \subseteq f(B)$ mod $\mathscr{H}$, i.e., there is a co-compressible invariant set $Y \subseteq X$ such that $f(A \cap Y) \subseteq f(B \cap Y)$.

Show that if $\vartheta(A) = \vartheta(B)$ for all $\vartheta \in \mathtt{EINV}(\mathsf{E})$, then there is $f \in [\mathsf{E}]$ such that $f(A) = f(B)$ mod $\mathscr{H}$.

Chapter 3

Hyperfinite equivalence relations

3.1 Hyperfinite relations arise from $\mathbb{Z}$ actions

This chapter is devoted to hyperfinite relations. Most of the results are from [DJK94]. Recall that a Borel equivalence relation is finite, or just fber for short, if every equivalence class if finite.

Definition 3.1.1. A cber E on X is said to be *hyperfinite* if it can be written as an increasing union of finite equivalence relations, i.e., if there exist finite Borel equivalence relations F_n on X such that $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}$ and $\mathsf{E} = \bigcup_n \mathsf{F}_n$.

Note that the condition for the union in the definition of hyperfiniteness to be increasing is crucial, the notion would trivialize if this condition is dropped. Indeed, by the Feldman–Moore's Theorem 1.2.3 we can find a Borel action of a countable group $H \curvearrowright X$ such that $\mathsf{E} = \mathsf{E}_X^H$, and moreover, we have a countable family of elements $h_n \in H$ each having order at most 2 such that $x \,\mathsf{E}\, y \iff x = y$ or $h_x x = y$ for some n. Set $\mathsf{F}_n = \{(x, y) : h_n x = y\} \cup \Delta$. Since each h_n has order two, F is a Borel equivalence relation with each class having size at most two; and also $\mathsf{E} = \bigcup_n \mathsf{F}_n$. So, any cber is a union of finite relations, but as we shall see later, only very special cbers are *increasing* unions of fbers.

Example 3.1.2. Equivalence relation E_0, which served us well so far, is hyperfinite. Indeed, if we set F_n on $2^{\mathbb{N}}$ by declaring that $x \,\mathsf{F}_n\, y$ whenever $x(k) = y(k)$ for all $k \geq n$, then F_n is a finite equivalence relation and $\mathsf{E}_0 = \bigcup_n \mathsf{F}_n$.

The tail equivalence relation E_{t} is also hyperfinite, though this is less obvious. Details of the argument are postponed to Corollary 3.3.5.

An example of a non-hyperfinite cber is given by the Bernoulli shift of a non-abelian free group $F_k \curvearrowright 2^{F_k}$. The reason why such an action is not hyperfinite is best explained involving the notion of amenability, and is postponed to the next chapter.

Proposition 3.1.3. *Let E be a cber on X. The following are equivalent.*

(i) E *is hyperfinite.*

(ii) $\mathsf{E} = \bigcup_n \mathsf{F}_n$, *where F_n are finite Borel equivalence relations on X, $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}$, and each F_n-class has size at most n.*

(iii) $\mathsf{E} = \bigcup_n \mathsf{E}^n$, *where each E^n is a smooth cber on X and the union is increasing:* $\mathsf{E}^n \subseteq \mathsf{E}^{n+1}$.

(iv) There is a Borel action of $\mathbb{Z}$ on X such that $\mathsf{E} = \mathsf{E}_X^{\mathbb{Z}}$.

Proof. (i) ⇒ (ii) Suppose $\mathsf{E} = \bigcup_{n=1}^{\infty} \mathsf{F}_n$ is hyperfinite, and the union is increasing. We may assume that $\mathsf{F}_1 = \Delta$. For each n and $n \geq k \geq 1$ we set

$$X_k^n = \Big\{ x \in X \setminus \bigcup_{i=k+1}^{n} X_i : \big|[x]_{\mathsf{F}_k}\big| \leq n \Big\}.$$

In words, X_n^n consists of the points $x \in X$ whose F_n-equivalence class has size at most n; X_{n-1}^n collects those points whose F_n-class is bigger than n, but whose F_{n-1}-class has size at most n, etc. We may now set

$$\mathsf{F}_n' = \mathsf{F}_n\big|_{X_n^n} \cup \mathsf{F}_{n-1}\big|_{X_{n-1}^n} \cup \cdots \cup \mathsf{F}_1\big|_{X_1^n}.$$

It is easy to check that $\mathsf{E} = \bigcup_{n=1}^{\infty} \mathsf{F}_n'$, the union is increasing, and each F_n'-class has size at most n.

(ii) ⇒ (iii) Is immediate from Proposition 1.4.4.

(iii) ⇒ (i) Let us first suppose that E itself is smooth. Pick a countable group $H \curvearrowright X$ acting on X such that $\mathsf{E} = \mathsf{E}_X^H$, $H = \{h_n : n \in \mathbb{N}\}$, and let $s : X \to X$ be a Borel selector for E. We may show that E is hyperfinite by defining

$$x \,\mathsf{F}_n\, y \quad \text{whenever} \quad (x = y) \text{ or } \big(s(x) = s(y) \text{ and } h_k s(x) = x, h_m s(y) = y \text{ for some } k, m \leq n\big).$$

In a more verbose fashion, all F_n-classes consist of a single point except for the classes "around" the transversal points $s(X)$, which consist of elements $\{h_k s(x) : k \leq n\}$. As n growth, classes around the transversal grow and eventually exhaust all of E, thus showing that E is hyperfinite.

Now back to the general situation. Suppose $\mathsf{E} = \bigcup_n \mathsf{E}^n$ is represented as an increasing union of smooth equivalence relations, let $H^n \curvearrowright X$ be countable group actions such that $\mathsf{E}^n = \mathsf{E}_X^{H^n}$, $H^n = \{h_k^n : k \in \mathbb{N}\}$, and let $s_n : X \to X$ be a Borel selector for E^n. There is no loss in generality to assume that $\mathsf{E}^0 = \Delta$ is the trivial equivalence relation. We define F_n on X by setting

$$\begin{aligned} x \,\mathsf{F}_n\, y \iff & \exists m \leq n \text{ such that } x \;\mathsf{E}^m\; y \text{ and } \big(\exists k_0, \ldots, k_m \leq n \text{ such that } h_{k_0}^0 s_0 h_{k_1}^1 s_1 \cdots h_{k_m}^m s_m(x) = x\big) \text{ and} \\ & \big(\exists l_0, \ldots, l_m \leq n \text{ such that } h_{l_0}^0 s_0 h_{l_1}^1 s_1 \cdots h_{l_m}^m s_m(y) = y\big). \end{aligned}$$

In a more verbose fashion, equivalence relation F_n can be explained as follows. Since E^0 is assumed to be the trivial equivalence relation, $m = 0$ in the definition of F_n corresponds to $x = y$, so $\Delta \subseteq \mathsf{F}_n$. Let us take $m = 1$ next. This corresponds to the structure of equivalence classes described at the beginning of this argument under the assumption that E is smooth. More precisely, consult Figure 3.1, where each line corresponds to an E^1-class, and black dots represent the transversal given by s_1, i.e., points $x \in X$ such that $s_1(x) = x$. For $m = 1$ points x and y are F_n-equivalent if they lie in the same E^1-class, so $s_1(x) = s_1(y)$, and they are "n-around $s(x)$" in the sense that there are $h_{k_1}^1, h_{l_1}^1 \in H^1$ such that $x = h_{k_1}^1 s(x)$ and $y = h_{l_1}^1 s(x)$. In Figure 3.1 this corresponds to a thin rectangle around each dot. If $n = 1$, this completes the description of F_1.

When $n \geq 2$, points within each rectangle are F_1-equivalent, but taking $m = 2$ we see that some rectangles fall into a single F_1-class. Each block of lines in Figure 3.1 represents an E^2-class, and a hollow circle in each block corresponds to the transversal picked by s_2, i.e., a point $x \in X$ such that $s_2(x) = x$. We now look at points n-around each such x, i.e., points of the form $h_k^2 s(x)$ for $k \leq n$; this is depicted by dashed rectangles around hollow discs. We want to put points in a dashed rectangle into a single F_n-class, but this could violate the condition $\mathsf{F}_{n-1} \subseteq \mathsf{F}_n$, we need to ensure that each F_n-class is a union of F_{n-1}-classes. So instead of taking points in dashed rectangles, we take the points in the thin rectangles within orbits "n-around" discs. In other words, to each z in a dashed rectangle we apply s_1, which brings us to a black dot within the

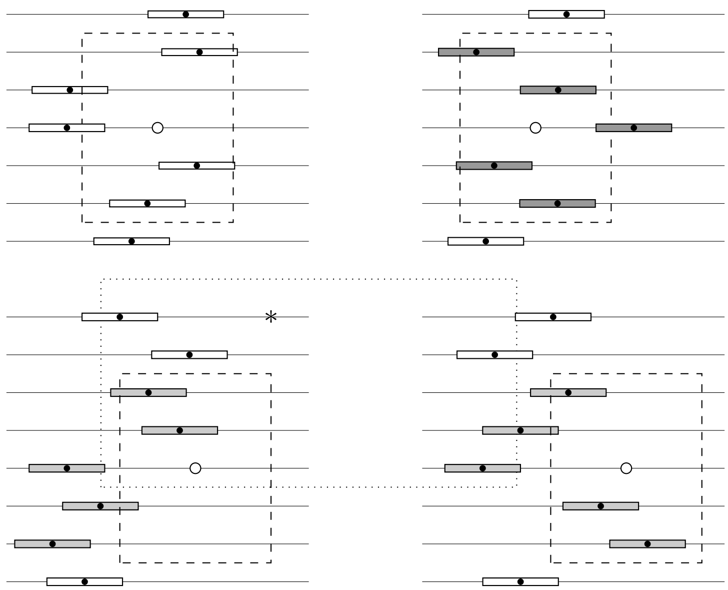

Figure 3.1: Increasing union of smooth cbers is hyperfinite.

same orbit. The dot may be inside or outside the dashed rectangle. We take all the thin rectangles around such dots and glue them into a single F_1-class. In the top-right E^2-class in Figure 3.1 thin rectangles that are glued together are depicted in darker gray.

If $n = 2$, this concludes the description of F_2. If $n \geq 3$, then we continue gluing some of the equivalence classes as prescribed by s_m for $3 \leq m \leq n$. For example, Figure 3.1 shows a single E^3-class, and the asterisk in the bottom-left E^2-class corresponds to the fixed point of s_3. We first look at the points that are "n-around" the asterisk in E^3, i.e., we consider points of the form $h^3_k s_3(x)$, $k \leq n$, depicted by the dotted rectangle. Again, we can't make this points into a single F_3-class as this may violate the condition for the union to be increasing, so instead we look which E^2-classes are spanned by the dotted rectangle, from each such class pick the F_2-class constructed up to this point and glue them into a single F_3-class. The union of lighter gray rectangles in the two bottom E^2-classes in Figure 3.1 constitutes a single F_3-class.

It is evident from the explanation above that each F_n is a finite Borel equivalence relation, they form an increasing sequence $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}$, and cover all of E, $\mathsf{E} = \bigcup_n \mathsf{F}_n$, thus witnessing its hyperfiniteness.

(iv) ⇒ (i) Let $\mathbb{Z}$ act in a Borel way on X, put $\mathsf{E} = \mathsf{E}^{\mathbb{Z}}_X$, and let $T : X \to X$ be the generator of the action, i.e., $x \mathsf{E} y$ if and only if $T^n x = y$ for some $n \in \mathbb{Z}$. By Proposition 1.4.2 we may decompose X in a periodic part and an aperiodic part, and since any finite cber is evidently hyperfinite, we may assume without loss of generality that E is aperiodic, i.e., the action $\mathbb{Z} \curvearrowright X$ is free. By Proposition 1.8.3, there exists a vanishing marker sequence $S_n \subseteq X$, $n \in \mathbb{N}$.

In geometric terms, S_n selects a subset of points from each orbit of T. On some orbits there can be a left-most point or a right-most point. In other words, let

$$D^n_l = \big\{x \in S_n : T^n x \not\in S_n \text{ for all } n < 0\big\} \text{ and } D^n_r = \big\{x \in S_n : T^n x \not\in S_n \text{ for all } n > 0\big\}.$$

The sets D^n_l, D^n_r pick at most one point from each orbit, so the restriction of E onto $\Big[\bigcup_n (D^n_l \cup D^n_r)\Big]_{\mathsf{E}}$ is smooth. Since we already know what to do with smooth pieces, one may assume that for each $n \in \mathbb{N}$ and each $x \in S_n$ there are $l < 0 < r$ such that $T^l x \in S_n$ and $T^r x \in S_n$, which means that each S_n partitions every orbit of T into finite intervals. We may therefore define functions

$$l_n : X \to \mathbb{N}$$

by setting

$$l_n(x) = \min\{l \in \mathbb{N} : T^{-l}x \in S_n\}.$$

These functions are Borel and we may define F_n by

$$x \,\mathsf{F}_n\, y \iff x \,\mathsf{E}\, y \text{ and } l_n(x) = l_n(y).$$

In words, $x \,\mathsf{F}_n\, y$ whenever x and y belong to the same intervals or the partition of $[x]_{\mathsf{E}}$ as determined by S_n. Since $S_n \supseteq S_{n+1}$, we have $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}$, and $\bigcap S_n = \varnothing$ ensures that $\mathsf{E} = \bigcup_n \mathsf{F}_n$.

(i) $\Rightarrow$ (iv) It remains to show that any hyperfinite equivalence relation is given by an action of $\mathbb{Z}$. We are going to build the action by constructing the graph for its generator. Let $\mathsf{E} = \bigcup_n \mathsf{F}_n$ be represented as an increasing union of finite equivalence relations. We construct a subset $G \subseteq X \times X$ as follows. The space X can be endowed with a Borel linear ordering, i.e., one may assume that $X = [0, 1]$. Let $m_n : X \to X$ and $M_n : X \to X$ be the functions that select the minimal point and the maximal point from the F_n-equivalence class of its argument:

$$m_n(x) = \min\{y \in [x]_{\mathsf{F}_n}\} \quad \text{ and } M_n(x) = \max\{y \in [x]_{\mathsf{F}_n}\}.$$

These functions are Borel. Let $\preceq_n$ be a quasi-order given by $x \preceq_n y$ whenever $m_{n-1}(x) \le m_{n-1}(y)$. Set

$$G_0 = \{(x, y) \in \mathsf{F}_0 : y \text{ is the successor of } x \text{ within } [x]_{\mathsf{F}_0}\}.$$

Every $y \in X$ which is not the minimal element of its F_0-class occurs in a unique pair $(x, y) \in G_0$ for some x; also every $x \in X$ which is not a maximal element of its F_0-class occurs in a unique pair of the form (x, y) for some y.

We now enlarge G_0 to G_1 by setting

$$G_1 = G_0 \sqcup \{(x, y) \in \mathsf{F}_1 : x = M_0(x),\ y = m_0(x), \text{ and } y \text{ is a } \preceq_1\text{-successor of } x \text{ in } [x]_{\mathsf{F}_1}\}.$$

Note that now every $y \in X$ which is not the $\preceq_1$-minimal element of its F_1-class occurs in a unique pair $(x, y) \in G_0$ for some x; similarly for not $\preceq_1$-maximal elements. The construction is continued in a similar fashion — we define

$$G_2 = G_1 \sqcup \{(x, y) \in \mathsf{F}_2 : x = M_1(x),\ y = m_1(x), \text{ and } y \text{ is a } \preceq_2\text{-successor of } x \text{ in } [x]_{\mathsf{F}_2}\}.$$

Set $G = \bigcup_n G_n$, and let

$$\begin{aligned} Z_m &= \{x \in X : (y, x) \notin G \text{ for any } y \in X \text{ such that } x \,\mathsf{E}\, y\}, \\ Z_M &= \{y \in X : (y, x) \notin G \text{ for any } x \in X \text{ such that } x \,\mathsf{E}\, y\}. \end{aligned}$$

Sets Z_m and Z_M intersect every E-class in at most one point and so $\mathsf{E}\big|_{[Z_m \cup Z_M]_{\mathsf{E}}}$ is smooth. The restriction of G onto $X \setminus [Z_m \cup Z_M]_{\mathsf{E}}$ is a graph of a Borel bijection, say

$$T : X \setminus [Z_m \cup Z_M]_{\mathsf{E}} \to X \setminus [Z_m \cup Z_M]_{\mathsf{E}}.$$

Since $[Z_m \cup Z_M]_{\mathsf{E}}$ is smooth, it is easy to extend T to an automorphism $T : X \to X$ such that the action $\mathbb{Z} \curvearrowright X$ given by T generates E. $\square$

Example 3.1.4. Consider the equivalence relation E_{V} on $\mathbb{R}$, called the *Vitali* equivalence relation, given by $x \,\mathsf{E}_{\mathrm{V}}\, y$ whenever $x - y \in \mathbb{Q}$. Clearly E_{V} is a cber. Using item (iii) of Proposition 3.1.3 it is easy to show that

E_V is hyperfinite (Exercise 3.3). Note that E_V is just the orbit equivalence relation of the (free) action of $\mathbb{Q}$ by translations on $\mathbb{R}$.

Another interesting example arises if we consider a multiplicative action of $\mathbb{Q}$. More precisely, let $\mathbb{Q}^\times = \{q \in \mathbb{Q} : q > 0\}$ be the multiplicative group of positive rationals and let it act on $\mathbb{R}^{>0}$ by multiplication. We may define the *Pythagorean* equivalence relation E_P to be the orbit equivalence relation of this action: $x \,\mathsf{E}_\mathrm{P}\, y$ if and only if $x/y \in \mathbb{Q}$. Pythagorean relation is also hyperfinite, but despite looking superficially similar to the Vitali equivalence relation showing its hyperfiniteness is much harder. This result is due to Su Gao and Steve Jackson [GJ15], we shall prove it in the following chapter.

3.2 Generators

Definition 3.2.1. Let $H \curvearrowright X$ be an action of a countable group on a standard Borel space X. A countable Borel partition $\mathcal{P} = \{P_i : i \in \mathbb{N}\}$ is said to be a *countable generator* for $H \curvearrowright X$ if for all distinct $x, y \in X$ there exists $h \in H$ and $i \in \mathbb{N}$ such that $hx \in P_i$ and $hy \notin P_i$.

Remark 3.2.2. Given any countable Borel partition, we can define a map $\zeta : X \to \mathbb{N}^H$ by setting $\zeta(x)(h)$ to be the unique $i \in \mathbb{N}$ such that $h^{-1}x \in P_i$. This map is an equivariant homomorphism into the shift action on $\mathbb{N}^H$, i.e., $\zeta(hx) = h\zeta(h)$ for all $x \in X$ and $h \in H$. A partition $\mathcal{P}$ is a countable generator if and only if the corresponding map $\zeta : X \to \mathbb{N}^H$ is an embedding.

For the rest of this section we work with free Borel actions of $\mathbb{Z}$ on a standard Borel space X. The automorphism of X which corresponds to $1 \in \mathbb{Z}$ under this action will be denoted by T. We also let $\mathsf{E} = \mathsf{E}_X^\mathbb{Z}$ to denote the orbit equivalence of $\mathbb{Z} \curvearrowright X$.

Definition 3.2.3. A Borel set $A \subseteq X$ is *recurrent* if for all $x \in A$ there are $m < 0 < n$ such that $T^m x \in A$ and $T^n x \in A$. In other words, A is recurrent if its intersection with any orbit of T is either empty or bi-infinite.

Our first observation is that for any Borel $A \subseteq X$ there is a subset $A' \subseteq A$ such that A' is recurrent and $A \setminus A'$ is smooth. Indeed, if the intersection $A \cap [x]_\mathsf{E}$ fails to be bi-infinite, then it either has the largest or the smallest element in the ordering inherited from $\mathbb{Z}$ (recall that the action $\mathbb{Z} \curvearrowright X$ is free). Therefore we may pick these endpoints in a Borel fashion by setting

$$\tilde{A} = \{x \in A : T^n x \notin A \text{ for all } n \geq 1\} \cup \{x \in A : T^n x \notin A \text{ for all } n \leq -1\}.$$

The set $\tilde{A}$ intersects every orbit of T in at most two points, and therefore is smooth; whence so is its saturation $[\tilde{A}]_\mathsf{E}$. One may set $A' = A \setminus [\tilde{A}]_\mathsf{E}$ for the required recurrent subset. We have used the same idea earlier in the proof of the implication (iv) ⇒ (i) in Proposition 3.1.3.

Importance of recurrent sets lies in the idea of the induced transformation. If $A \subseteq X$ is recurrent we define the *first return time* map $t_A : X \to \mathbb{N}$ by

$$t_A(x) = \min\{k \geq 1 : T^k x \in A\}.$$

The *induced automorphism* $T_A : A \to A$ is the map $T_A(x) = T^{t_A(x)}x$. Recurrency of A ensures that $t_A(x)$ is defined for any x, and is also responsible for surjectivity of T_A. Checking that T_A is a Borel bijection is easy and is left for Exercise 3.4.

With any recurrent set we also associate the canonical *return time partition* $\mathcal{R}_t = \{R_n : n \geq 1\}$ of A, $A = \bigsqcup_{n=1}^\infty R_n$, given by $R_n = t_A^{-1}(n) \cap A$. This partition of A gives rise to the partition of $[A]_\mathsf{E}$ once we add sets of the form $T^j R_k$ for all $k \geq 1$ and all $0 \leq j < k$. This partition is called the *Kakutani–Rokhlin partition* and it gives the following graphical representation of the automorphism $T : [A]_\mathsf{E} \to [A]_\mathsf{E}$ depicted in Figure 3.2.

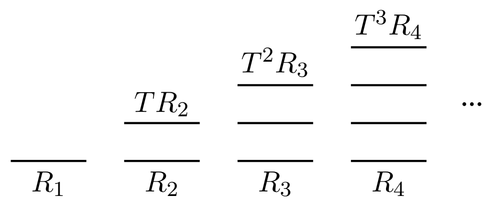

Figure 3.2: Kakutani–Rokhlin partition of X.

The base of the partition consists of the set $A = \bigsqcup_{i=1}^{\infty} R_i$. Whithin a tower on top of some R_k the automorphism acts by lifting a point by one level. The top of each tower is mapped to the base, i.e., if $x \in T^{k-1}R_k$, then $T_x \in R_n$ for some $n \geq 1$; note also that the value of n is typically different for different $x \in T^{k-1}R_k$.

Lemma 3.2.4. *Let $\mathcal{P} = \{P_1, \dots, P_m\}$ be a finite partition of X and let $A \subseteq X$ be an E-complete recurrent subset of X. There exists a countable Borel partition $A = \bigsqcup_n A_n$ such that each atom of $\mathcal{P}$ is a disjoint union of translates of A_n: $P_k = \bigsqcup_{m=1}^{\infty} T^{i_m} A_{j_m}$ for some $i_m \in \mathbb{N}$, $j_m \in \mathbb{N}$.*

Proof. We start with the Kakutani–Rokhlin partition associated with A. Since A is assumed to be E-complete, i.e., $[A]_{\mathsf{E}} = X$, one has a partition of the whole phase space.

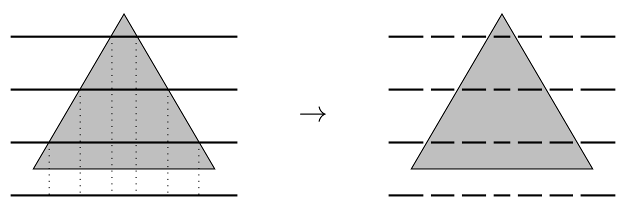

Figure 3.3: Refining a tower of the Kakutani–Rokhlin partition.

Take the common refinement of the Kakurani–Rokhlin partition and $\mathcal{P}$, transfer the atoms of this partition to the base, and take the partition of the base they generate. Figure 3.3 shows a refinement of one Kakutani–Rokhlin tower. It is clear that the resulting partition of the base A satisfies the conclusion of the lemma. $\square$

Theorem 3.2.5. *Any aperiodic automorphism of a standard Borel space admits a countable generator.*

Proof. By the proof of Lemma 2.5.1 we can find a Borel partition of $X = \bigsqcup_n F'_n$ into E-complete sets. By perturbing these sets on a smooth piece we may furthermore assume that all F'_n are recurrent. We may now select a countable family of Borel subset $B_n \subseteq X$ which separate points: for each $x, y \in X$ there is $n \in \mathbb{N}$ such that $x \in B_n$ and $y \notin B_n$. Let $\mathcal{P}_n$ be the partition of X into B_n and $X \setminus B_n$. To a set F'_n and partition $\mathcal{P}_n$ we apply Lemma 3.2.4 to find a partition $F'_n = \bigsqcup_{i \in \mathbb{N}} \tilde{A}^n_i$. This results in a partition of X

$$X = \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{i \in \mathbb{N}} A^n_i,$$

which we may reenumerate as $X = \bigsqcup_{i \in \mathbb{N}} A^n_i$. By construction for each B_n there are sequences of natural numbers $(i_k), (j_k)$ such that $B_n = \bigsqcup_k T^{i_k} A_{j_k}$. This partition is therefore a countable generator for the action, because the family $\{B_n : n \in \mathbb{N}\}$ separates points. $\square$

In view of Remark 3.2.2, Theorem 3.2.5 implies that any aperiodic action $\mathbb{Z} \curvearrowright X$ is isomorphic to a restriction of the shift $\mathbb{Z} \curvearrowright \mathbb{N}^{\mathbb{Z}}$ onto an invariant subset. In particular, any aperiodic hyperfinite cber is isomorphic to the restriction of $\mathsf{E}^{\mathbb{Z}}_{\mathbb{N}^{\mathbb{Z}}}$ onto an invariant subset, but it is worth stressing that the latter is a much weaker statement.

3.3 Bi-embeddability

We have shown in Intermezzo I that E_0 can be embedded into any non-smooth cber. The goal of this section is to prove for hyperfinite relations a converse to this. We are going to show that any hyperfinite relation can be embedded into E_0. This result is originally due to Randall Dougherty, Steve Jackson, and Alexander Kechris [DJK94].

It is helpful to take a slightly different perspective on E_0. Recall that $x \,\mathsf{E}_0\, y$ holds whenever $x(k) = y(k)$ for all sufficiently large $k \in \mathbb{N}$. The fact that $\mathbb{N}$ has a natural linear ordering on it is irrelevant[1] for the definition of E_0, as it can be equivalently described by saying that $x \,\mathsf{E}_0\, y$ whenever the set $\{i : x(i) \neq y(i)\}$ is finite. A binary sequence $x \in 2^{\mathbb{N}}$ can be identified with the subset $\{i \in \mathbb{N} : x(i) = 1\}$. With this in mind, E_0 is a cber on the family of all subsets of $\mathbb{N}$ where two subsets $A, B \subseteq \mathbb{N}$ are E_0 equivalent if and only if the symmetric difference $A \triangle B$ is finite.

For any countable set A we let $\mathsf{E}_0(A)$ to denote a cber on $A^{\mathbb{N}}$ given by $x \ \mathsf{E}_0(A)\ y$ if and only if the set $\big\{i \in \mathbb{N} : x(i) \neq y(i)\big\}$ is finite. The discussion above shows that E_0 is the same as $\mathsf{E}_0(2)$ in this notation. Our first lemma shows that $\mathsf{E}_0(A)$ can be embedded into $\mathsf{E}_0(2)$ for any countable A.

Lemma 3.3.1. *For any countable set A one has* $\mathsf{E}_0(A) \sqsubseteq \mathsf{E}_0(2)$.

Proof. Any $x \in A^{\mathbb{N}}$ is a function $\mathbb{N} \to A$. Note that $x \ \mathsf{E}_0(A)\ y$ if and only if $\mathrm{graph}(x) \triangle \mathrm{graph}(y)$ is finite. The map $x \mapsto \mathrm{graph}(x)$ is therefore an embedding $\mathsf{E}_0(A)$ into E on $2^{\mathbb{N}\times A}$ given by $z_1 \,\mathsf{E}\, z_2$ whenever the set of $i \in \mathbb{N} \times A$ such that $z_1(i) \neq z_2(i)$ is finite (we identify sequences in $2^{\mathbb{N}\times A}$ with subsets of $\mathbb{N} \times A$). Since $\mathbb{N} \times A$ is countable, E is clearly isomorphic to $\mathsf{E}_0(2)$. □

Lemma 3.3.2. *Let* E *be the orbit equivalence relation induced by the shift action on* $\mathbb{N}^{\mathbb{Z}}$*,i.e.,* $\mathsf{E} := \mathsf{E}^{\mathbb{Z}}_{\mathbb{N}^{\mathbb{Z}}}$. *There exists an* E*-invariant Borel subset* $Y \subseteq \mathbb{N}^{\mathbb{Z}}$ *such that* $\mathbb{N}^{\mathbb{Z}} \setminus Y$ *is smooth and* $\mathsf{E}|_Y \sqsubseteq \mathsf{E}_0$.

Proof. Let $\mathbb{N}^{<\mathbb{N}}$ denote the set of all finite sequences of natural numbers. We endow this set with the lexicographical ordering. More formally, given $x, y \in \mathbb{N}^{<\mathbb{N}}$ we set

$$x < y \iff \big(x(i) = y(i) \text{ for all } i < \min\{|x|, |y|\} \text{ and } |x| < |y|\big) \text{ or } \\ \big(x(j) < y(j) \text{ where } j = \min\{i : x(i) \neq y(i)\}\big).$$

Note that this ordering induces a well-ordering on $\mathbb{N}^n$ for each $n \in \mathbb{N}$. Given $x \in \mathbb{N}^{\mathbb{Z}}$ and $u \in \mathbb{N}^{<\mathbb{N}}$ we say that u occurs in x at $k \in \mathbb{Z}$ if $x(k+i) = u(i)$ for all $0 \leq i < |u|$; we say that u occurs in x if it occurs in x at some k. One says that u occurs bi-infinitely often in x if the set of $k \in \mathbb{Z}$ such that u occurs in x at k is unbounded both from below and from above.

Our first obseration is than for any $u \in \mathbb{N}^{<\mathbb{N}}$ the set of $x \in \mathbb{N}^{\mathbb{Z}}$ in which u occurs at some $k \in \mathbb{Z}$ but does not occur bi-infinitely often is smooth. This is because a transversal for the restricion of E onto such set can be obtained by picking those x where the smallest/largest occurance takes place at $k = 0$. Since the conclusion of the lemma is claimed to hold only up to a smooth set, we may concentrate on the set Z_1 of those x where each $u \in \mathbb{N}^{<\mathbb{Z}}$ either does not occur at all or occurs bi-infinitely often.

Let $f_n : Z_1 \to \mathbb{N}^n \subseteq \mathbb{N}^{<\mathbb{N}}$ be the function that assigns to $x \in Z_1$ the smallest $u \in \mathbb{N}^n$ which occurs in x. A direct inspection of the definition shows that f_n is Borel. Observe that $f_n(x) = f_n(y)$ whenever $x \,\mathsf{E}\, y$ and note that $f_{n+1}(x)|_n = f_n(x)$ for all $x \in Z_1$. We may therefore define a function $f : Z_1 \to \mathbb{N}^{\mathbb{N}}$ to be the

[1] Note that the linear order on $\mathbb{N}$ is important for the definition of E_t.

limit of $f_n(x)$, i.e., $f(x)|_n = f_n(x)$ for all $n \in \mathbb{N}$. Employing the same idea as before, we note that the set of $x \in Z_1$, where

$$\{k \in \mathbb{Z} : x(k+i) = f(x)(i) \text{ for all } i \in \mathbb{N}\}$$

is non-empty and bounded from below, is smooth, as a transversal is given by

$$\{x \in Z : x(i) = f(x)(i) \text{ for all } i \in \mathbb{N} \text{ and for all } k < 0 \text{ there is } i \in \mathbb{N} \text{ such that } x(k+i) \neq f(x)(i)\}.$$

We may therefore neglect it, and set Z_2 to consist of those $x \in Z_1$ for which either $f(x)$ does not occur in x, or the set of points where it occurs in x is unbounded from below.

One last reduction comes from the observation that if the set of $k \in \mathbb{Z}$ such that $f(x)$ occurs in x at k is unbounded from below, then x is periodic. By Proposition 1.4.4 the restriction of E onto finite orbits is smooth, so we may finally put Y to be the set of all $x \in Z_2$ such that $f(x)$ does not occur in x. The set $\mathbb{N}^{\mathbb{Z}} \setminus Y$ is smooth, and Y is E-invariant. We are going to construct an embedding $\mathsf{E}|_Y \sqsubseteq \mathsf{E}_0(\mathbb{N}^{<\mathbb{N}})$. By Lemma 3.3.1 this is enough to imply $\mathsf{E}|_Y \sqsubseteq \mathsf{E}_0$.

Given a sequence $x \in Y$ we construct sequences $r_n(x) \in \mathbb{N}^{<\mathbb{N}}$, $n \in \mathbb{N}$, as follows. Start with $k_0^x = 0$ and set

$$k_{n+1}^x = \begin{cases} \text{smallest } k > 0 \text{ such that } f_{n+1}(x) \text{ occurs at } k & \text{if } n \text{ is even,} \\ \text{largest } k < 0 \text{ such that } f_{n+1}(x) \text{ occurs at } k & \text{if } n \text{ is odd.} \end{cases}$$

Note that

$$\cdots k_{2n}^x \leq k_{2n-2}^x \leq \cdots \leq k_4^x \leq k_2^x < 0 < k_1^x \leq k_3^x \leq \cdots \leq k_{2n-1}^x \leq k_{2n+1}^x \leq \cdots,$$

because $f_{n+1}(x)$ extends $f_n(x)$, and $k_{2n}^x \to -\infty$, $k_{2n+1}^x \to \infty$ as $n \to \infty$ for each $x \in Y$, because $f(x)$ does not occur in x. Define

$$r_n(x) = \begin{cases} x|_{[k_{n+1}^x, k_n^x]} & \text{if } n \text{ is odd,} \\ x|_{[k_n^x, k_{n+1}^x]} & \text{if } n \text{ is even.} \end{cases}$$

Direct inspection shows that the map $x \mapsto r_n(x)$ is Borel, and we may therefore define $\xi : Y \to (\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}}$ by setting $\xi(x)(n) = r_n(x)$. Note that $r_{n+1}(x)$ is of the form $u^\frown r_n(x)$ for some $u \in \mathbb{N}^{<\mathbb{N}}$ when n is even, and it is of the form $r_n(x)^\frown u$, when n is odd. Note also that $r_0(x) = [0, k_1^x]$. This implies that ξ is injective.

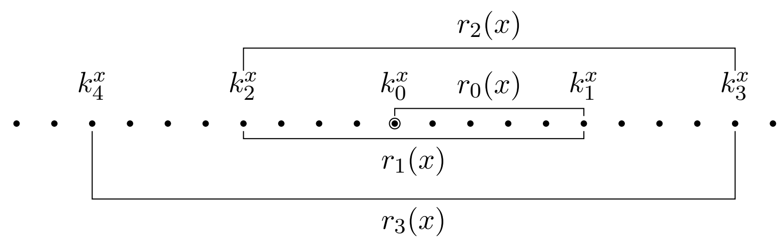

Figure 3.4: Spiral structure of segments of x cut by r_n^x.

We claim that $x \mathsf{E} y$ if and only if $r_n(x) = r_n(y)$ for all sufficiently large n. In other words, we claim that ξ witnesses $\mathsf{E}|_Y \sqsubseteq \mathsf{E}_0(\mathbb{N}^{<\mathbb{N}})$. Pick $x, y \in Y$ such that $x \mathsf{E} y$. Let $m \in \mathbb{Z}$ be such that $x(m+i) = y(i)$ for all $i \in \mathbb{Z}$. By changing the roles of x and y we may assume that $m \in \mathbb{N}$. Pick N so large that $k_{2n}^x, k_{2n}^y < -m$ and $k_{2n+1}^x, k_{2n+1}^y > m$ for all $n \geq N$. One has to have $k_p^x = k_p^y + m$ for all $p \geq 2N$. Indeed, suppose for instance $p = 2n$ for some $n \geq N$. By the definition of k_{2n}^y we know that $f_{2n}(y)$ occurs in y at $k_{2n}^y < -m$. Since $f_{2n}(y) = f_{2n}(x)$, and since $x(m+i) = y(i)$, we see that $f_{2n}(x)$ occurs in x at $k_{2n}^y + m$ which is still below 0. As according to the definition k_{2n}^x is supposed to be the largest negative index at which $f_{2n}(x)$

occurs in x, we get $k^x_{2n} \geq k^y_{2n} + m$. Similarly, $f_{2n}(y)$ occurs in y at $k^x_{2n} - m < 0$, and therefore by the definition of k^y_{2n} we obtain $k^y_{2n} \geq k^x_{2n} - m$. These two inequalities imply $k^y_{2n} + m = k^x_{2n}$. The argument for showing $k^x_{2n+1} = k^y_{2n+1} + m$ for $n \geq N$ is completely analogous. We have shown that $x \mathsf{E} y$ implies $r_n(x) = r_n(y)$ for all large enough n.

For the other direction suppose that $r_n(x) = r_n(y)$ for all $n \geq N$. We may assume N is even, and let $m \in \mathbb{N}$ be such that $k^x_N + m = k^y_N$. Let also $u_n(x) \in \mathbb{N}^{<\mathbb{N}}$ be such that

$$r_{n+1}(x) = \begin{cases} u_n(x)^\frown r_n(x) & \text{if } n \text{ is even,} \\ r_n(x)^\frown u_n(x) & \text{if } n \text{ is odd.} \end{cases}$$

Sequences $u_n(y)$ are defined similarly. Since $r_n(x) = r_n(y)$ for all $n \geq N$, one has $u_n(x) = u_n(y)$ for all $n \geq N$. By the choice of m we have

$$x(i+m) = y(i) \quad \text{for all } i \in [k^x_N, k^x_{N+1}].$$

Since $u_n(x) = u_n(y)$ for all $n \geq N$, it is easy to check that $x(i+m) = y(i)$ is true for all $i \in \mathbb{Z}$, thus $x \mathsf{E} y$. □

Theorem 3.3.3. *Any hyperfinite cber* E *embeds into* E_0.

Proof. We have shown that every aperiodic hyperfinite E can be (invariantly) embedded into $\mathsf{E}^{\mathbb{Z}}_{\mathbb{N}^\mathbb{Z}}$ on $\mathbb{N}^{\mathbb{Z}}$. Recall that by Proposition 1.4.4 the periodic part of any cber is smooth. In Lemma 3.3.2 we have shown that $\mathsf{E}^{\mathbb{Z}}_{\mathbb{N}^\mathbb{Z}}|_Y \sqsubseteq \mathsf{E}_0$ for some invariant subset $Y \subseteq \mathbb{N}^{\mathbb{Z}}$ such that $\mathbb{N}^{\mathbb{Z}} \setminus Y$ is smooth.

Let $\mathsf{E}_0 \times 2$ be a cber on $2^{\mathbb{N}} \times \{0,1\}$ which makes (x,α) equivalent to (y,β) if and only if $x \mathsf{E}_0 y$ and $\alpha = \beta$. We first observe that $\mathsf{E}_0 \times 2 \sqsubseteq \mathsf{E}_0$ as witnessed by the map $2^{\mathbb{N}} \times 2 \ni (x,\alpha) \mapsto \zeta(x,\alpha) \in 2^{\mathbb{N}}$,

$$\zeta(x,\alpha)(n) = \begin{cases} x(n/2) & \text{if } n \text{ is even,} \\ \alpha & \text{otherwise.} \end{cases}$$

To prove the argument it is therefore enough to show that any smooth cber can be embedded into E_0. This is requested in Exercise 3.2. □

Let $\mathsf{E}_{\mathrm{t}}(\mathbb{N})$ be the "tail equivalence relation on $\mathbb{N}$", i.e., for $x, y \in \mathbb{N}^{\mathbb{N}}$ one has $x \, \mathsf{E}_{\mathrm{t}}(\mathbb{N}) \, y$ whenever there are $k_1, k_2 \in \mathbb{N}$ such that $x(k_1 + m) = y(k_2 + m)$ for all $m \in \mathbb{N}$.

Theorem 3.3.4. *The cber* $\mathsf{E}_{\mathrm{t}}(\mathbb{N})$ *is hyperfinite.*

Proof. We first show that $\mathsf{E}_{\mathrm{t}}(\mathbb{N}) \sqsubseteq \mathsf{E}_0(\mathbb{N})$. In the spirit of the proof of Theorem 3.3.3, we pick a linear ordering on $\bigcup_n \mathbb{N}^n$ that extends the lexicographical ordering on $\mathbb{N}^n$ and satisfies $s \leq t$ for any t that extends s. For $x \in \mathbb{N}^{\mathbb{N}}$ let $u^x_n \in \mathbb{N}^n$ be the minimal word that occurs in x infinitely often, and let $k^x_n \in \mathbb{N}$ be the place of the first occurrence of u^x_n in x. Set $k^x_0 = 0$. Similarly to the proof of Theorem 3.3.3, one shows that the set of x where $k^x_n \not\to \infty$ as $n \to \infty$ is smooth. So we may restrict our attention to the subset $Z \subseteq \mathbb{N}^{\mathbb{N}}$ of those x for which $k^x_n \to \infty$ as $n \to \infty$.

Pick a bijection $\langle \cdot \rangle : \bigcup_n \mathbb{N}^n \to \mathbb{N}$. For $n \geq 1$ and $x \in Z$ let

$$r^x_n = \big\langle x|_{[k^x_{n-1}, k^x_n - 1]} \big\rangle.$$

Consider now the map $g : Z \to \mathbb{N}^{\mathbb{N}}$ given by $g(x) = (r^x_1, r^x_2, \ldots)$. It is easy to check that g is injective Borel reduction witnessing $\mathsf{E}_{\mathrm{t}}(\mathbb{N}) \sqsubseteq \mathsf{E}_0(\mathbb{N})$. By Lemma 3.3.1, this implies $\mathsf{E}_{\mathrm{t}}(\mathbb{N}) \sqsubseteq \mathsf{E}_0$. □

Corollary 3.3.5. *The tail equivalence relation* E_{t} *is hyperfinite.*

Proof. This is immediate from Theorem 3.3.4, since $\mathsf{E}_t \sqsubseteq \mathsf{E}_t(\mathbb{N})$. □

Theorem 3.3.6. *Up to an isomorphism tail equivalence relation E_t is the unique non-smooth compressible hyperfinite Borel equivalence relation.*

Proof. Let E be a non-smooth compressible hyperfinite Borel equivalence relation on a standard Borel space X. By Theorem 3.3.3 E can be embedded into E_t, i.e., there is a Borel $A \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_t|_A$ is isomorphic to E. But by Proposition 2.2.6 $\mathsf{E}_t|_A$ is isomorphic to $\mathsf{E}_t|_{[A]_\mathsf{E}}$. The conclusion is that E is isomorphic to a restriction of E_t onto an invariant subsets, and similarly E_t is isomorhic to a restriction of E onto an invariant subset of X. The Schröder–Bernstein construction presents an isomorphism between E_t and E. □

3.4 Rokhlin's Lemma

Lemma 3.4.1. *Let T be an aperiodic Borel automorphism of a standard Borel space X. There exists a recurrent complete Borel subset $A \subseteq X$ such that $TA \cap A = \varnothing$.*

Proof. We apply Proposition 1.8.5 and pick a subset $F \subseteq X$ such that $F \sim X \setminus F$. By changing F on a smooth set if necessary, we may assume that both F and $X \setminus F$ are recurrent. This means that for any $x \in F$ there are$k_1 < 0 < k_2$ and $m_1 < 0 < m_2$ such that $T^{k_i}x \notin F$ and $T^{m_i}x \in F$. Set

$$A = \{x \in F : T^{-1}x \notin F\}.$$

It is easy to see that A is a recurrent complete set and $TA \cap A = \varnothing$ by construction. □

Lemma 3.4.2. *Let T be an aperiodic Borel automorphism of a standard Borel space X. For any $n \geq 1$ there exists a Borel complete recurrent subset $A \subseteq X$ such that $T^iA \cap A = \varnothing$ for all $1 \leq i < n$.*

Proof. Let $A_1 \subseteq X$ by obtained by applying Lemma 3.4.1. Since A_1 is recurrent, we may consider the induced map T_{A_1} and apply the same lemma to T_{A_1} producing a complete recurrent Borel $A_2 \subseteq A_1$ such that $T_{A_1}A_2 \cap A_2 = \varnothing$. It is straightforward to check that A_2 is also a complete recurrent subset with respect to T and $T^iA \cap A = \varnothing$ for $1 \leq i \leq 3$. Repeating the same construction, we get a nested $A_{n+1} \subseteq A_n$ sequence of complete recurrent Borel sets such that $T^iA_n \cap A_n = \varnothing$ for all $1 \leq i < 2^n$. The lemma follows. □

Theorem 3.4.3 (Rokhlin's Lemma)**.** *Let $T : X \to X$ be an aperiodic automorphism. For any $\epsilon > 0$ and any $n \geq 1$ there is a Borel subset $B \subseteq X$ such that $B \cap T^iB = \varnothing$ for all $1 \leq i < n$ and*

$$\vartheta\Big(X \setminus \bigcup_{i=0}^{n-1} T^iB\Big) < \epsilon \quad \textit{for all invariant probability measures } \vartheta \textit{ on } X.$$

Proof. Given n and $\epsilon > 0$ pick N so large that $1/N < \epsilon$. Lemma 3.4.2 guarantees existence of a complete set $A \subseteq X$ such that $T^iA \cap A = \varnothing$ for all $1 \leq i < 2Nn$. Set

$$B = \big\{T^{nj}x : x \in A, 0 \leq j \leq \lfloor t_A(x)/n \rfloor - 1\big\}.$$

Note that $t_B(x) \in [n, 2n)$ for all $x \in B$, and also

$$X \setminus \bigcup_{i=0}^{n-1} T^iB \subseteq \big\{T^jx : x \in A \text{ and } t_A(x) - 2n \leq j < t_A(x)\big\} =: Y.$$

Since $T^{2nj}Y \cap Y = \varnothing$ for all $1 \leq j < 2Nn/2n = N$, we conclude that $\vartheta(Y) \leq 1/N < \epsilon$ for all invariant probability measures ϑ on X. □

Lemma 3.4.4. *Let $T : X \to X$ be an aperiodic automorphism, $\epsilon \in (0,1]$, $n \geq 1$, and let $B \subseteq X$ be such that $T^i B \cap B = \varnothing$ for all $1 \leq i < n$ and $\vartheta\big(X \setminus \bigcup_{i=0}^{n-1} T^i B\big) < \epsilon$ for all pie measures ϑ on X. For any $\delta \in (0, \epsilon]$ there exists a subset $B' \subseteq B$ such that for all pie measures ϑ on X one has*

$$\epsilon > \vartheta\Big(X \setminus \bigcup_{i=0}^{n-1} T^i B'\Big) \geq \epsilon - \delta.$$

Proof. Set $\alpha = \frac{1-\epsilon}{n}$, $\beta = 1/n$, and note that $\vartheta(B) \in (\alpha, \beta]$ for all pie measures ϑ on X. Pick positive $\delta' < \delta/n$ and observe that for any $b \geq \alpha$ if r is such that $1 - nrb = \epsilon$, then $r \in (0,1]$ and

$$1 - nrc \in (\epsilon - \delta, \epsilon] \quad \text{for all } c \in [b, b + \delta'].$$

Pick an increasing sequence α_m, $m = 0, \ldots, M$, such that $\alpha_0 = \alpha$, $\alpha_M = \beta$, and $\alpha_{m+1} - \alpha_m \leq \delta'$. Select an ergodic decomposition $x \mapsto \mu_x$, and set for $0 \leq m < M$

$$Q_m = \big\{x \in X : \mu_x(B) \in (\delta_m, \delta_{m+1}]\big\}.$$

Note that these sets partition X into invariant Borel pieces. For each m let r_m be such that

$$1 - r_m n \delta_m = \epsilon, \quad \text{i.e.,} \quad r_m = \frac{1-\epsilon}{n\delta_m} \in [0,1].$$

We apply Corollary 2.5.4 and find a subset $B'_m \subseteq B \cap Q_m$ such that for any pie measure ϑ on Q_m one has $\vartheta(B'_m) = r_m \vartheta(B \cap Q_m)$. Since $\mu_x(B \cap Q_m) \in (\delta_m, \delta_{m+1}]$, we have for all $x \in Q_m$

$$\mu_x\Big(Q_m \setminus \bigcup_{i=0}^{n-1} T^i B'_m\Big) = 1 - n r_m \mu_x(B \cap Q_m) \in (\epsilon - \delta, \epsilon).$$

Set $B' = \bigcup_m B'_m$. The construction ensures that

$$\mu_x\Big(X \setminus \bigcup_{i=0}^{n-1} T^i B'\Big) \in (\epsilon - \delta, \epsilon) \quad \text{for all } x \in X.$$

Since μ_x exhausts all pie measures on X, the lemma follows. □

Theorem 3.4.5 (Strong Rokhlin's Lemma)**.** *Let $T : X \to X$ be an aperiodic automorphism. For any $\epsilon \in (0,1)$ and any $n \geq 1$ there is a recurrent Borel subset $B \subseteq X$ such that $B \cap T^i B = \varnothing$ for all $1 \leq i < n$ and*

$$\vartheta\Big(X \setminus \bigcup_{i=0}^{n-1} T^i B\Big) = \epsilon \quad \textit{for all pie measures } \vartheta \textit{ on } X.$$

Proof. We begin with an application of Theorem 3.4.3 an select a subset $A_1 \subseteq X$ such that $T^i A_1 \cap A_0 = \varnothing$ for $1 \leq i < n$ and $\vartheta\big(X \setminus \bigcup_{i=0}^{n-1} T^i A_1\big) < \epsilon$ for all pie measures ϑ on X. Lemma 3.4.4 lets us find a subset $A_2 \subseteq A_1$ such that $\vartheta\big(X \setminus \bigcup_{i=0}^{n-1} T^i A_1\big) \in (\epsilon - 1/2, \epsilon)$ for all pie measures ϑ on X. Applying Lemma 3.4.4 again we find $A_2 \subseteq A_1$ such that $\vartheta\big(X \setminus \bigcup_{i=0}^{n-1} T^i A_2\big) \in (\epsilon - 1/4, \epsilon)$, and construct inductively a nested sequence $A_{m+1} \subseteq A_m$ such that $\vartheta\big(X \setminus \bigcup_{i=1}^{n-1} T^i A_m\big) \in (\epsilon - 2^{-m}, \epsilon)$ for all ϑ. The set $B = \bigcap_m A_m$ clearly works, except that B may not be recurrent; by altering B on a smooth set, we can make B recurrent. □

3.5 Von Neumann automorphisms

Definition 3.5.1. An *ordered partition* of a set X is a tuple $\mathcal{P} = (D_1, \ldots, D_n)$ such that $X = \bigsqcup_i D_i$. In plain words, it is a partition with a specified order on its pieces. The first element D_1 will be called the *base* of $\mathcal{P}$, and D_n will be referred to as the *top* of $\mathcal{P}$. An ordered partition is said to be *dyadic* if the number of its pieces is a power of 2.

Let E be a cber on X. A *partial von Neumann automorphism* on X is a pair $(\mathcal{P}, \xi)$, where $\mathcal{P}$ is a dyadic ordered partition, $\mathcal{P} = (D_1, \ldots, D_{2^n})$, and $\xi \in [\![\mathsf{E}]\!]$ is such that

- $\mathrm{dom}(\xi) = \bigcup_{i=1}^{2^n-1} D_i$;
- $\xi(D_i) = D_{i+1}$ for all $1 \leq i < 2^n$.

We say that a partial von Neumann automorphism $(\mathcal{P}_2, \xi_2)$ extends $(\mathcal{P}_1, \xi_1)$ if

- $\mathcal{P}_2$ refines $\mathcal{P}_1$;
- the base of $\mathcal{P}_2$ is a subset of the base of $\mathcal{P}_1$;
- ξ_2 extends ξ_1.

An automorphism $S : X \to X$ is said to be a *weak von Neumann automorphism* if there exists a sequence of partial von Neumann automorphisms $(\mathcal{P}_n, \xi_n)$, $n \in \mathbb{N}$, such that for all $n \in \mathbb{N}$

1. $\mathcal{P}_n$ has 2^n-many elements;
2. $(\mathcal{P}_{n+1}, \xi_{n+1})$ extends $(\mathcal{P}_n, \xi_n)$;
3. S extends all of ξ_n.

The sequence of partial von Neumann automorphisms $(\mathcal{P}_n, \xi_n)$ as above will be called an *approximating sequence* for S. Since partial automorphisms ξ_n in an approximating sequence are readily reconstructed from S, we shall sometimes abuse the terminology and refer to the sequence of partitions $\mathcal{P}_n$ alone as an approximating sequence.

A weak von Neumann automorphism $S : X \to X$ is said to be a *strong von Neumann automorphism* if there exists an approximating sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ such that partitions $\mathcal{P}_n$ separate points in X: for all $x, y \in X$ there are $n \in \mathbb{N}$ and D_i^n — an element of $\mathcal{P}_n$ — such that $x \in D_i^n$ and $y \notin D_i^n$.

Odometer would be the canonical example of a strong von Neumann automorphism. But before discussing this important example, we would like to make a few simple observations about partial von Neumann automorphisms. Suppose $(\mathcal{P}_2, \xi_2)$ extends $(\mathcal{P}_1, \xi_1)$, and let us assume that $|\mathcal{P}_1| = 2^n$, $|\mathcal{P}_2| = 2^{n+1}$, i.e., $\mathcal{P}_2$ has twice as many elements as $\mathcal{P}_1$ does. Since $\mathcal{P}_2$ has to refine $\mathcal{P}_1$, the base D_1^1 of $\mathcal{P}_1$ is a union of some elements of $\mathcal{P}_2$, say

$$D_1^1 = D_{i_1}^2 \cup \cdots \cup D_{i_k}^2.$$

According to the definition of extension for partial von Neumann automorphisms, the base D_1^2 of $\mathcal{P}_2$ has to be a subset of D_1^1, so we may assume $i_1 = 1$. Also, as ξ_2 has to extend ξ_1, one sees that for each $1 \leq j \leq 2^n$ sets D_j^1 are partitioned as

$$D_j^1 = \xi_1^{j-1}(D_1^2) \sqcup \xi_1^{j-1}(D_{i_2}^2) \sqcup \xi_1^{j-1}(D_{i_3}^2) \sqcup \cdots \sqcup \xi_1^{j-1}(D_{i_k}^2).$$

Since all these sets $\xi_1^{j-1}(D_{i_l}^2) = \xi_2^{j-1}(D_{i_l}^2)$ must be elements of $\mathcal{P}_2$, an since we assume that $|\mathcal{P}_2| = 2|\mathcal{P}_1|$, we may conclude that $k = 2$, i.e., $\mathcal{P}_2$ partitions D_1^1 into two pieces, $D_1^1 = D_1^2 \sqcup D_{i_2}^2$, and moreover, $i_2 = 2^n + 1$, as ξ_2 must extend ξ_1.

Here is a picture that explains the discussion above. If $|\mathcal{P}_2| = |\mathcal{P}_1|$, then $\mathcal{P}_2$ is obtained as follows. The base of $\mathcal{P}_1$ is partitioned into two pieces, $D^1_1 = D^2_1 \sqcup D^2_{2^n+1}$, this partitions generates partitions of all levels D^i_i via the map ξ_1. This results in the tower $\mathcal{P}_1$ being split into two sub-towers. The partitions $\mathcal{P}_2$ is obtained by stacking the right sub-tower of $\mathcal{P}_1$ on top of its left sub-tower as show in Figure 3.5.

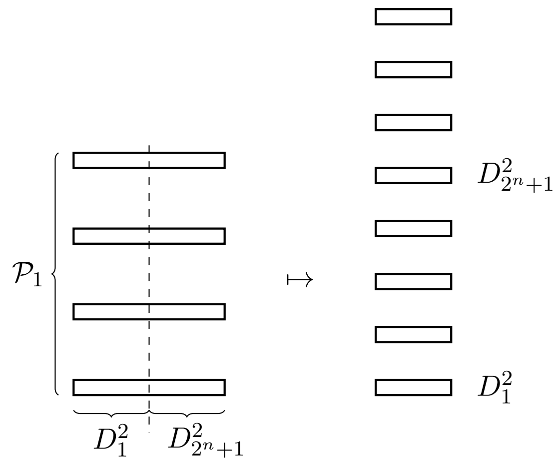

Figure 3.5: Extension of a partial von Neumann automorphism

To summarize, extension $(\mathcal{P}_2, \xi_2)$ is uniquely defined by specifying two things: a partition of the base of $\mathcal{P}_1$ into two pieces $D^1_1 = D^2_1 \sqcup D^2_{2^n+1}$, and a map $\zeta : \xi_1^{2^n-1}(D^1) \to D^2_{2^n+1}$, which specifies how the top of the left sub-tower is mapped onto the base of the right sub-tower. The converse is also true: any partition of the base $D^1_1 = D^2_1 \sqcup D^2_{2^n+1}$ into two equidecomposable pieces, and any map $\zeta : \xi_1^{2^n-1}(D^1) \to D^2_{2^n+1}$, $\zeta \in [\![\mathsf{E}]\!]$, give rise to a unique extension $(\mathcal{P}_2, \xi_2)$ of $(\mathcal{P}_1, \xi_1)$.

A similar picture is valid in general, when $\mathcal{P}_2$ is not necessarily twice the size of $\mathcal{P}_2$. Since $\mathcal{P}_2$ must refine $\mathcal{P}_1$, and since ξ_2 has to extend ξ_1, it is easy to deduce that $|\mathcal{P}_2| = n|\mathcal{P}_1|$ for some $n \in \mathbb{N}$, and in this case $\mathcal{P}_2$ induces a partition of the base of $\mathcal{P}_1$ into n pieces. This partition, when transferred by ξ_1 to each level of $\mathcal{P}_1$, defines a partition of $\mathcal{P}_1$ into n towers, and $\mathcal{P}_2$ is obtained by stacking these towers on top of each other.

Note that if $\mathcal{P}_2$ partitions the base of $\mathcal{P}_1$ into four pieces, then we can first consider a coarser partition of the base of $\mathcal{P}_1$ into two pieces, and define an extension $(\mathcal{P}', \xi')$ of $(\mathcal{P}_1, \xi_1)$; the pair $(\mathcal{P}_2, \xi_2)$ will then be an extension of $(\mathcal{P}', \xi')$. So, in this case we can find an intermediate extension between $\mathcal{P}_1$ and $\mathcal{P}_2$. A similar argument proves the following lemma.

Lemma 3.5.2. *Let $(\mathcal{P}_k, \xi_k)$ and $(\mathcal{P}_l, \xi_l)$ be partial von Neumann automorphisms such that $|\mathcal{P}_k| = 2^k$, $|\mathcal{P}_l| = 2^l$, $k \leq l$, and $(\mathcal{P}_l, \xi_l)$ extends $(\mathcal{P}_k, \xi_k)$. There exist partial von Neumann automorphisms $(\mathcal{P}_i, \xi_i)$, $k < i < l$, such that for all $k \leq i < l$*

1. *$(\mathcal{P}_{i+1}, \xi_{i+1})$ extends $(\mathcal{P}_i, \xi_i)$;*

2. *$|\mathcal{P}_{i+1}| = 2|\mathcal{P}_i|$.*

Proof. Exercise 3.5. □

Example 3.5.3. As promised, we now show that odometer $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an example of a strong von Neumann automorphism. Periodic partitions $\mathcal{P}_n = \big(D^n_1, \ldots, D^n_{2^n}\big)$ are given by cylindrical sets

$$D^n_i := C[s_{i,n}] = \big\{x \in 2^{\mathbb{N}} : x(j) = s_{i,n}(j) \text{ for } 0 \leq j < |s_{i,n}|\big\},$$

where $s_{i,n}$ is the reverse of the binary expansion of i with enough zeroes added to ensure that $|s_{i,n}| = n$. For example, if $i = 7$ and $n = 4$, then $s_{7,4} = 1110$. Set $\xi_n = \sigma|_{\bigcup_{i=1}^{2^n-1} D_i^n}$. A direct inspection show that $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is indeed an approximating sequence for σ, and it is clear that partitions $\mathcal{P}_n$ separate points of $2^{\mathbb{N}}$.

The following proposition shows that odometer is indeed *the* example of a von Neumman automorphism, as any strong von Neumann automorphism is isomorphic to a restriction of the odometer onto an invariant subset.

Proposition 3.5.4. *Let $T : X \to X$ be a strong von Neumann automorphism. There exists a σ-invariant subset $Y \subseteq 2^{\mathbb{N}}$ and a Borel bijection $\phi : X \to Y$ such that $\phi \circ T(x) = \sigma \circ \phi(x)$ for all $x \in X$.*

Proof. Let $(\mathcal{P}_n, \xi_n)$ be an approximating sequence for T such that partitions $\mathcal{P}_n = (D_i^n)_{i=1}^{2^n}$ separate points of X. For any $n \in \mathbb{N}$ and any $x \in X$ we may find the unique $k_n(x)$ such that $x \in D_{k_n(x)}^n$. Define the map $\phi : X \to 2^{\mathbb{N}}$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if } 1 \leq k_{n+1}(x) \leq 2^n, \\ 1 & \text{if } 2^n < k_{n+1}(x) \leq 2^{n+1}. \end{cases}$$

Observe that knowing the segment $\phi(x)|_n$, one may reconstruct $k_{n+1}(x)$ uniquely. Since sets $D_{k_n(x)}^n$ separate points, the map ϕ is injective, Borel, and, as one readily checks, it is also equivariant. □

For any partition $\mathcal{P} = \{D_i : 1 \leq i \leq N\}$ and a set $Q \subseteq X$ we let $\mathcal{P} \cap Q$ to denote the partition induced on Q,

$$\mathcal{P} \cap Q = \{D_i \cap Q : 1 \leq i \leq N\}.$$

Definition 3.5.5. Let $\mathcal{P} = \{D_i : 1 \leq i \leq N\}$ be a family of subsets of X. For a set $A \subseteq X$ we define the *inner* and *outer covers* of A by elements of $\mathcal{P}$:

$$\mathcal{A}^\circ(\mathcal{P}, A) = \bigcup_{\substack{D_i \in \mathcal{P} \\ D_i \subseteq A}} D_i \quad \text{— inner cover,}$$

$$\mathcal{A}^\bullet(\mathcal{P}, A) = \bigcup_{\substack{D_i \in \mathcal{P} \\ D_i \cap A \neq \varnothing}} D_i \quad \text{— outer cover.}$$

The definition does not require elements of $\mathcal{P}$ to be disjoint, but typically $\mathcal{P}$ will be a partition of X, or a restriction of a partition onto an invariant set.

We close this section with the following useful sufficient condition for the an approximating sequence to separate points.

Lemma 3.5.6. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be a sequence of partitions of X such that $\mathcal{P}_{n+1}$ extends $\mathcal{P}_n$; let also $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X such that*

- *sets A_n separate points;*
- *each element A_n occurs in the sequence $(A_n)_{n\in\mathbb{N}}$ infinitely often.*

If $\limsup_{n\to\infty}\big(\mathcal{A}^\bullet(\mathcal{P}_n, A_n) \setminus \mathcal{A}^\circ(\mathcal{P}_n, A_n)\big) = \varnothing$, then the sequence $(\mathcal{P}_n)$ separates points: for all $x, y \in X$ there are $n \in \mathbb{N}$ and $D \in \mathcal{P}_n$ such that $x \in D$ and $y \notin D$.

Proof. Pick $x, y \in X$, and let $k \in \mathbb{N}$ be such that $x \in A_k$ and $y \notin A_k$. Since $\limsup_n \big(\mathcal{A}^\bullet(\mathcal{P}_n, A_n) \setminus \mathcal{A}^\circ(\mathcal{P}_n, A_n)\big) = \varnothing$, we may find N so large that

$$x, y \notin \mathcal{A}^\bullet(\mathcal{P}_n, A_n) \setminus \mathcal{A}^\circ(\mathcal{P}_n, A_n) \quad \text{for all } n \geq N.$$

By assumption, the set A_k occurs infinitely often in the sequence (A_n), so we find $n_0 \geq N$ such that $A_{n_0} = A_k$. Let $D_i^{n_0}, D_j^{n_0} \in \mathcal{P}_{n_0}$ be such that $x \in D_i^{n_0}$ and $y \in D_j^{n_0}$. We claim that $i \neq j$, which will witness that partitions $\mathcal{P}_n$ separate points. Indeed, if $i = j$, then

$$D_i^{n_0} \cap A_{n_0} =\neq \varnothing \quad \text{because} \quad x \in D_i^{n_0} \cap A_{n_0},$$

on the other hand $y \in D_i^{n_0} \setminus A_{n_0}$, whence

$$D_i^{n_0} \subseteq \mathcal{A}^\bullet(\mathcal{P}_{n_0}, A_{n_0}) \setminus \mathcal{A}^\circ(\mathcal{P}_{n_0}, A_{n_0}),$$

contradicting the choice of N. □

3.6 Weak von Neumann automorphisms

Lemma 3.6.1. *Let $T : X \to X$ be an aperiodic Borel automorphism. There exists a co-compressible invariant subset $Y \subseteq X$ and weak von Neumann automorphism $S : Y \to Y$ such that $[T|_Y] = [S]$.*

Proof. We are going to construct sequences of subsets $A_n, B_n, C_n \subseteq X$ and induced automorphisms $T'_n = T_{A_n}$ and $T_n = T_{B_n}$ with the following properties for all $n \geq 1$ and all pi measures ϑ on X:

(a) $A_{n+1} \subseteq A_n$ and $B_{n+1} \subseteq B_n$;

(b) $C_{n+1} = B_n \setminus B_{n+1}$ and $C_1 = X \setminus B_1$;

(c) sets A_n and B_n are T-recurrent;

(d) $T'_n A_{n+1} \cap A_{n+1} = \varnothing$ and $TA_1 \cap A_1 = \varnothing$;

(e) $\vartheta(B_n) = \dfrac{2^n + 1}{2^{n+1}}$;

(f) $\vartheta\big(A_n \setminus (A_{n+1} \cup T'_{n+1} A_{n+1})\big) = 2^{-2(n+1)}$ and $\vartheta\big(X \setminus (A_1 \sqcup TA_1)\big) = 2^{-2} = 1/4$;

(g) $B_{n+1} = \displaystyle\bigsqcup_{i=0}^{2^{n+1}-1} T_n^i A_{n+1}$ and $B_1 = A_1 \sqcup TA_1$.

The base of the construction is provided by Theorem 3.4.5, which allows us to pick a recurrent A_1 such that $TA_1 \cap A_1 = \varnothing$ and $\vartheta\big(X \setminus (A_1 \cup TA_1)\big) = 1/4$ for all pi measures ϑ. We set $B_1 = A_1 \cup TA_1$ and $C_1 = X \setminus B_1$. In Figure 3.6 the set A_1 is depicted in light gray. We set $T'_1 = T_{A_1}$ and note that B_1 must be recurrent since so is $A_1 \subseteq B_1$. Note also that

$$\vartheta(B_1) = 3/4 = \frac{2^1 + 1}{2^2}$$

in compliance with item (e).

At the second step of the construction we apply Theorem 3.4.5 to the automorphism $T'_1 : A_1 \to A_1$ and find a recurrent Borel subset $A_2 \subseteq A_1$ such that $T'_1 A_2 \cap A_2 = \varnothing$ and $\vartheta\big(A_1 \setminus (A_2 \sqcup T'_1 A_2)\big) = 2^{-4} = 1/6.$

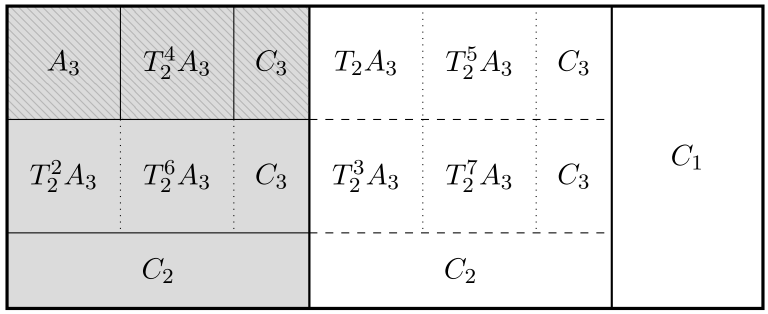

Figure 3.6: Construction of sets A_n, B_n, and C_n.

We set $T_2' = T_{A_2}$ and note that T_2' is equal to the automorphism induced by T_1' onto A_2, let $B_2 = \bigsqcup_{i=0}^{3} T_1^i A_2$ and $C_2 = B_1 \setminus B_2$. The set A_2 is dashed in Figure 3.6. The construction continues in the same fashion.

Let $B = \bigcap_n B_n$ and note that $\vartheta(B) = \lim \vartheta(B_n) = 1/2$ for all pi measure ϑ on X. Set $\mathcal{P}_n = \{T_B^i(A_n \cap B) : 0 \leq i < 2^n\}$, and notice that $\mathcal{P}_n$ witness that $T_B : B \to B$ is a weak von Neumann automorphism. Since $\vartheta(B) = \vartheta(X \setminus B)$, we may apply Exercise 2.9 and find a co-compressible invariant set $Y \subseteq X$ and an automorphism $f \in [T]$ such that $f(B \cap Y) = f(Y \setminus B)$. Finally, we are ready to define $S : Y \to Y$ by setting

$$S(x) = \begin{cases} f(x) & \text{if } x \in B, \\ T_B \circ f^{-1}(x) & \text{if } x \in Y \setminus B. \end{cases}$$

We leave the details of checking that S satisfies the conclusions of the lemma for the reader. □

Lemma 3.6.2 (mod $\mathscr{H}$)**.** *Let $(\mathcal{P}, \xi)$ be a partial von Neumann automorphism, let $A \subseteq X$ be a Borel set, let $\epsilon > 0$, and let $x \mapsto \mu_x$ be an ergodic decomposition for* E. *There exist an extension $(\mathcal{P}', \xi')$ of $(\mathcal{P}, \xi)$ and an invariant Borel partition $X = \bigsqcup_n Q_n$ (which is coarser than the ergodic partition) such that for all $x \in X$ and all $n \in \mathbb{N}$ one has*

$$\mu_x\Big(\mathcal{A}^\bullet\big(\mathcal{P}' \cap Q_n, A \cap Q_n\big) \setminus \mathcal{A}^\circ\big(\mathcal{P}' \cap Q_n, A \cap Q_n\big)\Big) < \epsilon.$$

Proof. Let $\mathcal{P} = (R_1, \ldots, R_{2^k})$. Define $\mathcal{C}_i$ to be the partition of R_1 generated by $\xi^{-i+1}(A \cap R_i)$:

$$R_1 = \xi^{-i+1}(A \cap R_i) \sqcup \xi^{-i+1}(R_i \setminus A),$$

see Figure 3.7. Let $R_1 = \bigsqcup_{i=1}^{l} B_i$ be the partition of R_1 generated by all of $\mathcal{C}_i$, $1 \leq i \leq 2^k$. Note that $l \leq 2^{2^k}$. By refining sets B_i if necessary, we may assume that $l = 2^{2^k}$. The partition of X will be defined by breaking X into pieces, where $\mu_x(B_i)$ is almost constant for all $i \leq l$.

Let $\delta = 2^{-L}$ for L so large that $2^k \cdot 2^{2^k} \cdot \delta < \epsilon$. The partition is indexed by vectors $\vec{p} \in \mathbb{N}^l$ and is given by

$$Q_{\vec{p}} = \Big\{x \in X : \mu_x(B_i) \in \big[\delta\vec{p}(i), \delta\vec{p}(i) + \delta\big) \text{ for all } i \leq l\Big\}.$$

Fix a vector $\vec{p} \in \mathbb{N}^l$. Using Exercise 2.9, we may find Borel subsets $B_{i,j} \subseteq B_i \cap Q_{\vec{p}}$, $1 \leq j \leq \vec{p}(i)$, such that $\mu_x(B_{i,j}) = \delta$ for all i, j and all $x \in Q_{\vec{p}}$. Let $C_i = (B_i \cap Q_{\vec{p}}) \setminus \bigsqcup_{j=1}^{\vec{p}(i)} B_{i,j}$ be the part of B_i within $Q_{\vec{p}}$ that is not covered by any $B_{i,j}$; note that $\mu_x(C_j) < \delta$ for all $x \in Q_{\vec{p}}$. Recall that $\mu_x(R_1) = 2^{-k}$ for all $x \in X$. Since $\mu_x(B_{i,j}) = \delta$, we get

$$\mu_x\Big(\bigcup_{i=1}^{l} C_i\Big) = 2^{-k} - \delta\Big(\sum_{i=1}^{l} \vec{p}(i)\Big).$$

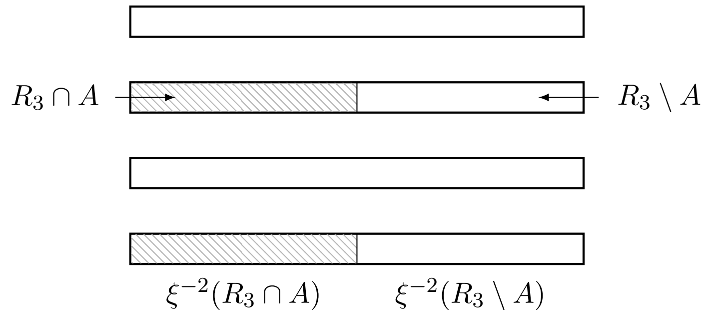

Figure 3.7: Illustration of partition $\mathcal{C}_3$.

Since 2^{-k} is an integer multiple of δ, we get that $\mu_x\big(\bigcup_{i=1}^l C_i\big) = N\delta$ for some $N \in \mathbb{N}$, $N \leq l$, and all $x \in Q_{\vec{p}}$. The latter implies (via Exercise 2.9) that $\bigcup_i C_i$ can be partitioned into sets $B_{0,j}$, $1 \leq j \leq N$, such that $\mu_x(B_{0,j}) = \delta$ for all $x \in Q_{\vec{p}}$ and all j. Sets $B_{i,j}$ form a partition of $R_1 \cap Q_{\vec{p}}$. We will no longer need indices i, j, so let us re-enumerate sets $B_{i,j}$ into a partition $R_1 \cap Q_{\vec{p}} = \bigsqcup_{i=1}^q G_i$, which satisfy the following properties for all $x \in Q_{\vec{p}}$:

(a) $\mu_x(G_i) = \delta$ for all i;

(b) $\mu_x\Big((B_i \cap Q_{\vec{p}}) \setminus \bigsqcup_{G_j \subseteq B_i} G_j\Big) = \mu_x\Bigg((B_i \cap Q_{\vec{p}}) \setminus \bigsqcup_{j=1}^{\vec{p(i)}} B_{i,j}\Bigg) < \delta$;

(c) $\mu_x\Big((B_i \cap Q_{\vec{p}}) \setminus \bigsqcup_{G_j \cap B_i \neq \varnothing} G_j\Big) \leq \mu_x\Bigg(\bigsqcup_{j=1}^{\vec{p}(i)} B_{i,j} \cup \bigsqcup_{j=1}^{N} B_{0,j} \setminus B_i\Bigg) \leq \mu_x\Big(\bigsqcup_{j=1}^{N} B_{0,j}\Big) = N\delta \leq l\delta$.

By Exercise 2.9 we may find (mod $\mathscr{H}$) automorphisms $f_j \in [\mathsf{E}]$ such that

$$f_j\big(\xi^{2^k-1} G_j\big) = G_{j+1} \cap Y \quad \text{for all } 1 \leq j < q.$$

We are now ready to define $(\mathcal{P}', \xi')$ on $Q_{\vec{p}}$ by setting

$$\mathcal{P}' \cap Q_{\vec{p}} = \big(G_1, \xi(G_1), \ldots, \xi^{2^k-1}(G_1), G_2, \xi(G_2), \ldots, \xi^{2^k-1}(G_2), \ldots, G_q, \xi(G_q), \ldots, \xi^{2^k-1}(G_q)\big),$$

and declaring for $x \in Q_{\vec{p}}$

$$\xi'(x) = \begin{cases} \xi(x) & \text{if } x \in \xi^i(G_j) \text{ for } 0 \leq i < 2^k - 1, \\ f_j(x) & \text{if } x \in \xi^{2^k-1}(G_j), j < q. \end{cases}$$

It is evident from the construction that $(\mathcal{P}', \xi')$ is an extension of $(\mathcal{P}, \xi)$. Also,

$$\mathcal{A}^\bullet\big(\mathcal{P}' \cap Q_{\vec{p}}, A \cap Q_{\vec{p}}\big) \setminus \mathcal{A}^\circ\big(\mathcal{P}' \cap Q_{\vec{p}}, A \cap Q_{\vec{p}}\big) \subseteq \bigcup_{i=0}^{2^k-1} \bigcup_{j=1}^{N} \xi^i\big(B_{0,j}\big).$$

Therefore for all $x \in Q_{\vec{p}}$

$$\mu_x\Big(\mathcal{A}^\bullet\big(\mathcal{P}' \cap Q_{\vec{p}}, A \cap Q_{\vec{p}}\big) \setminus \mathcal{A}^\circ\big(\mathcal{P}' \cap Q_{\vec{p}}, A \cap Q_{\vec{p}}\big)\Big) \leq 2^k l\delta < \epsilon.$$

$\square$

Lemma 3.6.3 (mod $\mathscr{H}$)**.** *Any partial von Neumann automorphism* $(\mathcal{P}, \xi)$ *can be extended to a weak von Neumann automorphism* $S : X \to X$ *such that* $\mathsf{E}_X^S = \mathsf{E}$.

Proof. Let $\mathcal{P} = (D_1, \ldots, D_{2^k})$. Use Lemma 3.6.1 to find a weak von Neumann automorphism $J : D_1 \to D_1$ such that $\mathsf{E}_{D_1}^J = \mathsf{E}|_{D_1}$. Define $S : X \to X$ by

$$Sx = \begin{cases} \xi(x) & \text{if } x \in D_i \text{ for } i < 2^k, \\ J \circ \xi^{-2^k+1}(x) & \text{if } x \in D_{2^k}. \end{cases}$$

It is straightforward to check that $S : X \to X$ is a weak von Neumann automorphism and $\mathsf{E}_X^S = \mathsf{E}$. □

Definition 3.6.4. Let $(\mathcal{P}, \xi)$ be a partial von Neumann automorphism, $\mathcal{P} = (D_1, \ldots, D_{2^k})$. A *fiber* over $x \in D_1$, $\mathcal{F}(x)$, is the set of points $\mathcal{F}(x) = \{\xi^i x : 0 \leq i < 2^k\}$. Given $y_1, y_2 \in X$, we say that y_1 and y_2 are the same $\mathcal{P}$-fiber if there is $x \in D_1$ such that $y_1, y_2 \in \mathcal{F}(x)$. Note that if $(\mathcal{P}', \xi')$ extends $(\mathcal{P}, \xi)$ and y_1, y_2 are in the same $\mathcal{P}$-fiber, then y_1 and y_2 are also in the same $\mathcal{P}'$-fiber.

Lemma 3.6.5. *Let* $T : X \to X$ *be an aperiodic Borel automorphism such that* $\mathsf{E}_X^T = \mathsf{E}$*, and let* $S : X \to X$ *be a weak von Neumann automorphism such that* $\mathsf{E}_X^S = \mathsf{E}$*. Let also* $\mathcal{P}_n$ *be an approximating sequence for* S*, we assume that* $|\mathcal{P}_n| = 2^n$*. For any ergodic decomposition* $x \to \mu_x$ *and any* $\epsilon > 0$ *there exists a countable invariant Borel partition* Q_n*,* $n \in \mathbb{N}$*, and naturals* $r_n \in \mathbb{N}$*, such that*

(i) $\{Q_n : n \in \mathbb{N}\}$ *is coarser than the partition associated with the ergodic decomposition.*

(ii) For any $x \in Q_n$

$$\mu_x\big(\{y \in Q_n : y \textit{ and } Ty \textit{ are in the same } \mathcal{P}_{r_n}\textit{-fiber}\}\big) \geq 1 - \epsilon.$$

Proof. Exercise. □

3.7 Classification of hyperfinite relations

Recall that for an ergodic decomposition $x \mapsto \mu_x$ we define

$$\Xi_x = \{y \in X : \mu_x = \mu_y\}.$$

Theorem 3.7.1 (mod $\mathscr{H}$)**.** *Let* E *be a non-compressible hyperfinite Borel equivalence relation on* X*, and let* $x \mapsto \mu_x$ *be an ergodic decomposition for* E*. There exists a weak von Neumann automorphism* $S : X \to X$ *such that* $\mathsf{E}_X^S = \mathsf{E}$ *and for all* $x \in X$ *the restriction* $S|_{\Xi_x}$ *is a strong von Neumann automorphism.*

Proof. Pick a sequence $\epsilon_n > 0$, $n \in \mathbb{N}$, such that $\sum_n \epsilon_n < 0$, $\epsilon_0 = 1$, and let $T : X \to X$ be an aperiodic Borel automorphism that generates E. Fix a sequence $(A_n)_{n \in \mathbb{N}}$ of Borel sets $A_n \subseteq X$ which separate points and is such that each A_k occurs in the sequence infinitely often. We are going to construct (mod $\mathscr{H}$) the following objects:

- Weak von Neumann automorphism $S_n : X \to X$ such that $\mathsf{E}_X^{S_n} = \mathsf{E}$.
- Approximating sequences $(\mathcal{P}_{n,m}, \xi_{n,m})_{m \in \mathbb{N}}$ for each S_n; $\mathcal{P}_{n,m} = (D_1^{n,m}, \ldots, D_{2^m}^{n,m})$.
- A tree of partitions, i.e., E-invariant Borel sets $(Q_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ indexed by finite sequences of natural numbers.
- Naturals $r_t \in \mathbb{N}$ for each $t \in \mathbb{N}^{<\mathbb{N}}$.

These objects will satisfy the following properties for all $n \in \mathbb{N}$ and all $t \in \mathbb{N}^n$:

(1) $Q_{t^\frown i} \subseteq Q_t$ for all $i \in \mathbb{N}$.

(2) $X = \bigsqcup_{t \in \mathbb{N}^n} Q_t$ and this partition is coarser than the one associated with the ergodic decomposition.

(3) $r_t \geq n$.

(4) $(\mathcal{P}_{n+1,k} \cap Q_t, \xi_{n+1,k}|_{Q_t}) = (\mathcal{P}_{n,k} \cap Q_t, \xi_{n,k}|_{Q_t})$ for all $0 \leq k \leq r_t$. In particular

$$\{x \in Q_t : S_{n+1}x \neq S_n x\} \subseteq D^{n,r_t}_{2^{r_t}}.$$

(5) For any $x \in Q_t$

$$\mu_x\big(\{y \in Q_t : y \text{ and } Ty \text{ are not in the same } \mathcal{P}_{r_t}\text{-fiber}\}\big) \leq \epsilon_n.$$

(6) For any $x \in Q_t$

$$\mu_x\Big(\mathcal{A}^\bullet\big(\mathcal{P}_{n,r_t} \cap Q_t, A_n \cap Q_t\big) \setminus \mathcal{A}^\circ\big(\mathcal{P}_{n,r_t} \cap Q_t, A_n \cap Q_t\big)\Big) \leq \epsilon_n.$$

For the base of this construction we may take $Q_\varnothing = X$, use Lemma 3.6.1 to find $S_0 : X \to X$ which generates E, set $r_\varnothing = 0$ and note that $\epsilon_0 = 1$ ensures that items (5) and (6) are trivially fulfilled.

For the induction step suppose sets Q_t have been constructed for all $t \in \mathbb{N}^n$ and S_k for $k \leq n$ have been defined. Pick some $t \in \mathbb{N}^n$ and restrict S_n onto Q_t. We may apply Lemma 3.6.2 to the partial von Neumann automorphism $(\mathcal{P}_{n,r_t} \cap Q_t, \xi_{n,r_t}|_{Q_t})$ and set $A_{n+1} \cap Q_t$. This results in a partial von Neumann automorphism $(\mathcal{P}', \xi')$ on Q_t which extends $(\mathcal{P}_{n,r_t}, \xi_{n,r_t})$ and a partition $Q_t = \bigsqcup_n \tilde{Q}_n$. This extension satisfies item (6) above on each $\tilde{Q}_n$. Let L be such that $|\mathcal{P}'| = 2^L$. An application of Lemma 3.6.3 allows us to find a weak von Neumann automorphism $S_{n+1}|_{Q_t}$ that extends $(\mathcal{P}', \xi')$ and generates E on Q_t. Finally, we may apply Lemma 3.6.5 to the restriction of T onto each of $\tilde{Q}_n$ and the automorphism $S_{n+1}|_{\tilde{Q}_n}$, which yields a partition $Q_{t^\frown i}$ of Q_t into invariant Borel sets and naturals $r_{t^\frown i} \in \mathbb{N}$ for which the analog of item (5) is fulfilled. Without loss of generality we may assume that $r_{t^\frown i} \geq \max\{L, n+1\}$.

Performing the same operation for each $t \in \mathbb{N}^n$, we obtain the weak von Neumann automorphism $S_{n+1} : X \to X$, approximations $(\mathcal{P}_{n+1,m}, \xi_{n+1,m})$ and the tree of partitions $(Q_t)_{t \in \mathbb{N}^{\leq n+1}}$. Routine inspection shows that all items above are satisfied.

We define sets Z_i, $i = 1, 2, 3$, to be the following limits:

$$Z_1 = \limsup_{n \to \infty} \bigcup_{t \in \mathbb{N}^n} D^{n,r_t}_{2^{r_t}},$$

$$Z_2 = \limsup_{n \to \infty} \bigcup_{t \in \mathbb{N}^n} \{y \in Q_y : y \text{ and } Ty \text{ are not in the same } \mathcal{P}_{n,r_t}\text{-fiber}\},$$

$$Z_3 = \limsup_{n \to \infty} \bigcup_{t \in \mathbb{N}^n} \mathcal{A}^\bullet\big(\mathcal{P}_{n,r_t} \cap Q_t, A_n \cap Q_t\big) \setminus \mathcal{A}^\circ\big(\mathcal{P}_{n,r_t} \cap Q_t, A_n \cap Q_t\big).$$

Items (3), (5), and (6) ensure that for all $x \in X$ one has $\mu_x(Z_i) = 0$ for $i = 1, 2,$ and 3 respectively. Saturations of sets Z_i are therefore compressible, and by throwing them away we may for notational convenience assume that $Z_i = \varnothing$, $i = 1, 2, 3$. Item (4) together with $Z_1 = \varnothing$ implies that for any $x \in X$ there is $N(x)$ so large that for all $n \geq N(x)$ one has $S_n x = S_{n+1} x$. We may therefore define $S : X \to X$ by setting $Sx = S_{N(x)} x$. The rest of the argument will show that S is the desired automorphism.

It is clear that S is a weak von Neumann automorphism, as partitions $\mathcal{P}_{n,n}$ form an approximating sequence. Pick some $\Xi = \Xi_{x_0}$. Since partitions $X = \bigsqcup_{t \in \mathbb{N}^n} Q_t$ are coarser than the partition associated with the ergodic decomposition, for each n there exists the unique $t_n \in \mathbb{N}^n$ such that $\Xi \subseteq Q_t$. For brevity let r_{t_n} be denoted simply by r_n. The assumption $Z_3 = \varnothing$ together with Lemma 3.5.6 ensures that $S|_\Xi$ is a strong von Neumann automorphism.

It remains to check that $\mathsf{E}_X^S = \mathsf{E}$. It is evident from the construction that $x \,\mathsf{E}_X^S\, y \implies x \,\mathsf{E}\, y$, we show the inverse implication by checking that for each $x \in \Xi$ one has $x \,\mathsf{E}_X^S\, Tx$. As $Z_2 = \varnothing$, there is N so large that x and Tx are the same $\mathcal{P}_{n,r_n}$-fiber for all $n \geq N$. But $S|_\Xi$ extends $\xi_{n,r_n}|_\Xi$, therefore x and Tx are in the same orbit of S, as claimed. □

Theorem 3.7.2 (mod $\mathscr{H}$). *Let* E *be an aperiodic hber on* X*. Suppose* E *is not compressible, and let* $Z = \mathtt{EINV}(\mathsf{E})$ *viewed as a standard Borel space. Let* Δ_Z *denote the trivial equivalence relation on* Z*:* $z_1 \Delta_Z z_2 \iff z_1 = z_2$*. The relation* E *is isomorphic to* $\mathsf{E}_0 \times \Delta_Z$*.*

Proof. Pick an ergodic decomposition $x \mapsto \mu_x$ for E and apply Theorem 3.7.1 to find a weak von Neumann automorphism $S : X \to X$ such that $\mathsf{E} = \mathsf{E}_X^S$ and $S|_{\Xi_x}$ is a strong von Neumann automorphism for all $x \in X$. Let $\mathcal{P}_n$ be an approximating sequence for S. For each $x \in X$ partitions $\mathcal{P}_n \cap \Xi_x$ separate points in Ξ_x. Following the proof of Proposition 3.5.4, we define the map $\phi : X \to 2^{\mathbb{N}}$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if } x \in D_i^{n+1} \text{ for some } i \leq 2^n, \\ 1 & \text{otherwise.} \end{cases}$$

As shown in the proof of Proposition 3.5.4, the map $\phi|_{\Xi_x} : \Xi_x \to 2^{\mathbb{N}}$ is an embedding of $\mathsf{E}|_{\Xi_x}$ into E_0. Since E_0 is uniquely ergodic, the image $\phi(\Xi_x)$ is co-compressible in $2^{\mathbb{N}}$ for every $x \in X$. We define the map $\zeta : X \to 2^{\mathbb{N}} \times Z$ by setting

$$\zeta(x) = \big(\xi(x), \mu_x\big).$$

It is straightforward to check that ζ is an isomorphism (mod $\mathscr{H}$) of cbers E and $\mathsf{E}_0 \times \Delta_Z$. □

Theorem 3.7.3. *Let* E_i*,* $i = 1, 2$*, be non-smooth aperiodic hbers on* X_i*,* $i = 1, 2$*. If* $\big|\mathtt{EINV}(\mathsf{E}_1)\big| = \big|\mathtt{EINV}(\mathsf{E}_2)\big|$*, then* E_1 *is isomorphic to* E_2*.*

Proof. If $\mathtt{EINV}(\mathsf{E}_i)$ is empty, then the theorem follows from Theorem 3.3.6, so we may assume that E_i admit finite invariant measures. Pick invariant Borel subsets $Y_i \subseteq X_i$ such that $\mathsf{E}_i|_{Y_i}$ is isomorphic to E_{t}. Note that $\mathsf{E}_i|_{X_i \setminus Y_i}$ has the same number of pie measures as E_i does. Using Theorem 3.7.2, we find subsets $W_i \subseteq X_i \setminus Y_i$ such that $\mathsf{E}_1|_{W_1}$ is isomorphic to $\mathsf{E}_2|_{W_2}$. Since both $X_1 \setminus W_1$ and $X_2 \setminus W_2$ are non-smooth by the choice of Y_i, we extend this isomorphism to witness $\mathsf{E}_1 \cong \mathsf{E}_2$. □

Here is a complete list, up to an isomorphism, of non-smooth aperiodic hyperfinite equivalence relations: E_{t}, $\mathsf{E}_0 \times \Delta_{\{0,1,2,\ldots,n-1\}}$ for some $n \in \mathbb{N}$, $\mathsf{E}_0 \times \Delta_{\mathbb{N}}$, $\mathsf{E}_0 \times \Delta_{2^{\mathbb{N}}}$.

Exercises

Exercise 3.1. Show that any weak von Neumann automorphism is aperiodic.

Exercise 3.2. Let E be a smooth cber. Show that $\mathsf{E} \sqsubseteq \mathsf{E}_0$.

Exercise 3.3. Using item (iii) of Proposition 3.1.3 show that the Vitali equivalence relation on $\mathbb{R}$ given by $x \,\mathsf{E}_{\mathrm{V}}\, y \iff x - y \in \mathbb{Q}$ is hyperfinite.

Exercise 3.4. Check that the induced automorphism $T_A : A \to A$ defined for a recurrent Borel set $A \subseteq X$ is indeed a Borel automorphism of A.

Exercise 3.5. Prove Lemma 3.5.2.

Chapter 4

Hyperfinite actions

4.1 Amenable equivalence relations

We begin by introducing a notion of an amenable equivalence relation, using an analog of the Reiter's condition. Appendix C reviews the notion of amenability for countable groups.

Definition 4.1.1. A cber E on X is said to be *amenable* if there are Borel functions $\phi_n : \mathsf{E} \to \mathbb{R}^{\geq 0}$ such that

- $\sum_{y \in [x]_{\mathsf{E}}} \phi_n(x,y) = 1$ for all $x \in X$;
- $\sum_{y \in [x]_{\mathsf{E}}} \left|\phi_n(x,y) - \phi_n(x',y)\right| \to 0$ as $n \to \infty$ for all $(x,x') \in \mathsf{E}$.

Proposition 4.1.2. *Let G be a countable group, and let $G \curvearrowright X$ be a Borel action on a standard Borel space. If G is amenable, then E_X^G is amenable.*

Proof. Let E denote the orbit equivalence relation E_G^X. According to Reiter's condition, there are functions $f_n \in \ell^1_+(G)$, $||f_n||_1 = 1$, such that $\left|\left|f_n - gf_n\right|\right|_1 \to 0$ for all $g \in G$. We define $\phi_n : \mathsf{E} \to \mathbb{R}^{\geq 0}$ by setting

$$\phi_n(x,y) = \sum_{\substack{g \in G \\ gy = x}} f_n(g).$$

For all $x \in X$ one has

$$\sum_{y \in [x]_{\mathsf{E}}} \phi_n(x,y) = \sum_{y \in [x]_{\mathsf{E}}} \sum_{\substack{g \in G \\ gy = x}} f_n(g) = \sum_{g \in G} f_n(g) = 1.$$

Also, for the second item from the definition of an amenable relation, take $(x,x') \in \mathsf{E}$, and pick $h \in G$ such that $hx = x'$.

$$\begin{aligned}\sum_{y \in [x]_{\mathsf{E}}} \left|\phi_n(x,y) - \phi_n(x',y)\right| = &\sum_{y \in [x]_{\mathsf{E}}} \left|\sum_{\substack{g \in G \\ gy = x}} f_n(g) - \sum_{\substack{g \in G \\ gy = x'}} f_n(g)\right| = \\ &\sum_{y \in [x]_{\mathsf{E}}} \left|\sum_{\substack{g \in G \\ gy = x}} f_n(g) - \sum_{\substack{g \in G \\ gy = x}} f_n(hg)\right| \leq \\ &\sum_{y \in [x]_{\mathsf{E}}} \sum_{\substack{g \in G \\ gy = x}} \left|f_n(g) - f_n(hg)\right| = \\ &||f_n - h^{-1}f_n||_1 \to 0 \text{ as } n \to \infty.\end{aligned}$$

The relation E is therefore amenable. □

Since any hyperfinite relation is generated by an action of $\mathbb{Z}$, and since the group of integers is amenable, the following is an immediate corollary of Proposition 4.1.2.

Corollary 4.1.3. *Any hyperfinite cber is amenable.*

The next proposition is a partial converse to Proposition 4.1.2.

Proposition 4.1.4. *Let G be a countable group acting in a Borel way on a standard Borel space X. Suppose the action is free, and assume that it admits a pie measure, call it μ. If the relation $\mathsf{E} = \mathsf{E}_G^X$ is amenable, then so is the group G.*

Proof. Let $\phi_n : \mathsf{E} \to \mathbb{R}^{\geq 0}$ be the functions from the definition of amenability for E. We verify amenability of G via the Reiter's condition by defining $f_n : G \to \mathbb{R}^{\geq 0}$ via

$$f_n(g) = \int_X \phi_n(x, g^{-1}x)\, d\mu(x).$$

Maps f_n are seen to be in $\ell^1_+(G)$ and satisfy $||f_n||_1 = 1$, as

$$\sum_{g\in G}\int_X \phi_n(x, g^{-1}x)\, d\mu(x) = \int_X \sum_{g\in G}\phi_n(x, g^{-1}x)\, d\mu(x) = \int_X \sum_{y\in[x]_{\mathsf{E}}} \phi_n(x, y)\, d\mu(x) = \int_X \mathbb{1}\, d\mu(x) = 1.$$

Finally, for any $h \in G$ one has

$$\begin{aligned}||f_n - hf_n||_1 = &\sum_{g\in G}\big|f_n(g) - f_n(h^{-1}g)\big| = \sum_{g\in G}\Big|\int_X \phi_n(x, g^{-1}x)\, d\mu(x) - \int_X \phi_n(x, g^{-1}hx)\, d\mu(x)\Big| = \\ &\sum_{g\in G}\Big|\int_X \phi_n(x, g^{-1}x)\, d\mu(x) - \int_X \phi_n\big(h^{-1}x, g^{-1}x\big)\, d\mu(x)\Big| \leq \\ &\int_X \Big(\sum_{g\in G}\Big|\phi_n(x, g^{-1}x) - \phi_n\big(h^{-1}x, g^{-1}x\big)\Big|\Big) d\mu(x) = \\ &\int_X \Big(\sum_{y\in[x]_{\mathsf{E}}}\Big|\phi_n(x, y) - \phi_n\big(h^{-1}x, y\big)\Big|\Big) d\mu(x).\end{aligned}$$

The last expression converges to 0 as $n \to \infty$. Indeed, set

$$\xi_n(x) = \sum_{y\in[x]_{\mathsf{E}}}\big|\phi_n(x, y) - \big(h^{-1}x, y\big)\big|.$$

By assumption on functions ϕ_n, one has $\xi_n(x) \to 0$ pointwise. Evidently $0 \leq \xi_n(x) \leq 2$ for all $x \in X$. Therefore, by Dominated Converges Theorem, one has $\int \xi_n\, d\mu \to 0$, as required. □

To give an example of a non-hyperfinite equivalence relation, it is therefore enough to construct a free measure preserving action of a non-amenable group, e.g., $F_2 = \langle a, b\rangle$. The natural candidate would be a Bernoulli shift, $F_2 \curvearrowright 2^{F_2}$, but this action is not free. Fortunately, this obstacle is easy to overcome.

Proposition 4.1.5. *Any infinite countable group G admits a free Borel probability measure preserving action on a standard Borel space. In fact, if*

$$\mathrm{Free}(2^G) = \big\{x \in 2^G : hx \neq x \textit{ for all } h \in G\big\},$$

then $\mu\big(\mathrm{Free}\big(2^G\big)\big) = 1$ for the Bernoulli measure on 2^G.

Proof. We aim at showing that $\mu\big(\mathrm{Free}(2^G)\big) = 1$. Since G is countable and

$$\mathrm{Free}(2^G) = \bigcap_{h \in G} \{x \in 2^G : hx \neq x\},$$

it is enough to check that for any fixed $h \in G$ one has

$$\mu\big(\{x \in 2^G : hx \neq x\}\big) = 1.$$

Note that

$$\{x \in 2^G : hx = x\} = \{x \in 2^G : x(g) = x(h^{-1}g) \text{ for all } g \in G\}.$$

We split the verification into two cases. If h is of infinite order, we may take $g = h^{2n+1}$ in the above to get

$$\{x \in 2^G : hx = x\} \subseteq \{x \in 2^G : x(h^{2n}) = x(h^{2n+1}) \text{ for all } n \in \mathbb{N}\},$$

where the right-hand side clearly has measure 0 with respect to μ.

If h has finite order, then we may choose $h_n \in G$ from different cosets of $\langle h \rangle$, which ensures that conditions $x(h^{-1}h_n) = x(h_n)$ are pairwise independent for distinct n. This allows us to conclude that for all h of finite order

$$\mu\big(\{x \in 2^G : hx = x\}\big) = 0.$$

Thus $\mu\big(\mathrm{Free}(2^G)\big) = 1$, as claimed. □

Propositions 4.1.4 and 4.1.5 together with Corollary 4.1.3 and the fact that the free group $F_2 = \langle a, b \rangle$ is not amenable (see Appendix C), imply that $F_2 \curvearrowright \mathrm{Free}(2^{F_2})$ generates a non-hyperfinite cber.

Corollary 4.1.6. *The cber* E *given by the action* $F_2 \curvearrowright \mathrm{Free}(2^{F_2})$ *of the free group on the free part of its Bernoulli shift is not hyperfinite.*

4.2 Borel graphs

In the next section we show that all orbit equivalence relations arising from groups of polynomial growth are hyperfinite. The result is due to Steve Jackson, Alexander Kechris, and Alain Louveau [JKL02]. Our presentation in this section and the next one follows closely pp. 15–17 of [JKL02]. We begin by reviewing some notions from Borel combinatorics.

Definition 4.2.1. A *Borel graph* on a standard Borel space X is a Borel set $\mathcal{G} \subseteq X \times X$ such that $\Delta_X \subseteq \mathcal{G}$ and $(x, y) \in \mathcal{G} \implies (y, x) \in \mathcal{G}$ for all $x, y \in X$. In other words, a Borel graph is a symmetric and reflexive Borel relation. Given a graph $\mathcal{G}$ and a point $x \in X$, the *neighborhood* of x in $\mathcal{G}$ is denoted by $[x]_{\mathcal{G}}$ and is given by

$$[x]_{\mathcal{G}} = \{y \in X : (x, y) \in \mathcal{G}\}.$$

A subset $A \subseteq X$ is $\mathcal{G}$*-independent* if $(x, y) \notin \mathcal{G}$ for all distinct $x, y \in A$. An independent set A is said to be a *maximal* independent set if moreover $A \cup \{z\}$ is not independent for any $z \in X \setminus A$, which is equivalent to $[z]_{\mathcal{G}} \cap A \neq \varnothing$ for all $z \in X$.

A graph $\mathcal{G}$ is said to be *locally finite* if $[x]_{\mathcal{G}}$ is finite for all $x \in X$.

Lemma 4.2.2. *For any locally finite Borel graph on a standard Borel space there exists a Borel maximal independent set.*

Proof. Let $\mathcal{G}$ be a graph on X, and let $(B_n)_{n=0}^{\infty}$ be a sequence of subsets of X such that the family $\{B_n : n \in \mathbb{N}\}$ is closed under finite intersection and separates points in X. Let $\xi : X \to \mathbb{N}$ be given by

$$\xi(x) = \min\big\{n : [x]_{\mathcal{G}} \cap B_n = \{x\}\big\}.$$

Using Luzin–Novikov's Theorem, one checks that the map ξ is Borel. Note that $\xi^{-1}(n)$ is an independent subset of X for every $n \in \mathbb{N}$. Define $Y_n \subseteq X$ inductively by setting $Y_0 = \xi^{-1}(0)$ and

$$Y_{n+1} = Y_n \sqcup \big\{y \in \xi^{-1}(n+1) : [y]_{\mathcal{G}} \cap Y_n = \varnothing\big\}.$$

Sets Y_n are Borel, and $Y = \bigsqcup_n Y_n$ is seen to be a maximal independent subset of X. $\square$

Given a graph $\mathcal{G}$ on X, we denote by $\mathcal{G}^2$ a graph on the same space X given by

$$\mathcal{G}^2 = \big\{(x, y) : (x, z) \in \mathcal{G} \text{ and } (z, y) \in \mathcal{G} \text{ for some } z \in X\big\}.$$

Note that if $\mathcal{G}$ is a locally finite Borel graph on X, then so is $\mathcal{G}^2$.

Definition 4.2.3. Let $\mathcal{G}_n$ be a sequence of locally finite Borel graphs on X. We say that $(\mathcal{G}_n)_{n\in\mathbb{N}}$ satisfies *Weiss' condition* if $\mathcal{G}_n^2 \subseteq \mathcal{G}_{n+1}$, $n \in \mathbb{N}$, and there exists $K \in \mathbb{N}$ such that for any $x \in X$ there are infinitely many $n \in \mathbb{N}$ for which any $\mathcal{G}_n$-independents subsets of $[x]_{\mathcal{G}_{n+2}}$ has size at most K.

Lemma 4.2.4. *Let E be a cber on X, and let $(\mathcal{G}_n)$ be a sequence of Borel graphs satisfying Weiss' condition such that $\mathsf{E} = \bigcup_n \mathcal{G}_n$. The relation E is hyperfinite.*

Proof. By Lemma 4.2.2, we may select $\mathcal{G}_n$-independent subsets $Z_n \subseteq X$. Luzin–Novikov's Theorem lets us find Borel maps $\pi_n : X \to X$ such that $\pi_n(x) \in [x]_{\mathcal{G}_n} \cap Z_n$ for all $x \in X$. Maps π_n are finite-to-one. Define fbers F_n by

$$x \,\mathsf{F}_n\, y \iff \pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_0(x) = \pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_0(y).$$

Relations F_n are nested, so $\mathsf{E}' = \bigcup_n \mathsf{F}_n$ is hyperfinite. Clearly $\mathsf{E}' \subseteq \mathsf{E}$. While E' is not necessarily equal to E, we shall show that any E-class contains finitely many E'-classes, which by Jackson's Theorem implies that E is hyperfinite.

Let $K \in \mathbb{N}$ be the constant in the definition of Weiss' condition. Suppose towards a contradiction that there is an E-class that contains at least $K + 1$ many E'-classes. Pick $x_0, \ldots, x_K$ which are pairwise E-equivalent and E'-inequivalent. Let n be so large that $x_i \in [x_0]_{\mathcal{G}_n}$ for all i. By increasing n if necessary, we may assume that any $\mathcal{G}_n$-independent subset of $[x_0]_{\mathcal{G}_{n+2}}$ has size at most K. Set

$$y_i = \pi_n \circ \cdots \circ \pi_0(x_i).$$

By assumption on points x_i, all elements y_i are distinct elements of Z_n, therefore $\{y_i : 0 \le i \le K\}$ is a $\mathcal{G}_n$-independent set of size $K + 1$. But $y_i \in [x_i]_{\mathcal{G}_{n+1}}$, and therefore $y_i \in [x_0]_{\mathcal{G}_{n+2}}$, contradicting the choice of the constant K. $\square$

4.3 Groups of polynomial growth

Definition 4.3.1. Let G be a finitely generated group, and let $S \subseteq G$ be a finite symmetric generating set for G containing 1. The group G has *polynomial growth* d if $b_n = O\big(n^d\big)$, where

$$b_n = \Big|\big\{g \in G : g = s_1 \cdots s_n \text{ for some } s_i \in S\big\}\Big|.$$

The property of having polynomial growth d is independent of the choice of generating set.

We shall also need the following technical condition.

Definition 4.3.2. We say that a countable group G has *mild growth* K, $K \in \mathbb{N}$, if there is a sequence of finite subsets $C_n \subseteq G$ such that for all $n \in \mathbb{N}$

(i) C_n is symmetric: $C_n^{-1} = C_n$;

(ii) $1 \in C_n$;

(iii) $C_n^2 \subseteq C_{n+1}$;

(iv) $G = \bigcup_n C_n$;

(v) there are infinitely many n such that $|C_{n+4}| \leq K|C_n|$.

Usefulness of this definition for our purposes is illustrated by the following proposition.

Proposition 4.3.3. *Let G be a countable group of mild growth K. Any orbit equivalence relation arising from an action of G is hyperfinite.*

Proof. Let $\mathsf{E} = \mathsf{E}_X^G$ be an orbit equivalence relation of a Borel action $G \curvearrowright X$. Pick a sequence $C_n \subseteq G$ witnessing that G has mild growth K. Set

$$\mathcal{G}_n = \big\{(x, gx) : x \in X \text{ and } g \in C_n\big\}.$$

Since C_n are symmetric and contain the unit of G, each $\mathcal{G}_n$ is a Borel graph; it is locally finite, because C_n is finite. Also, $G = \bigcup_n C_n$ implies $\mathsf{E} = \bigcup_n \mathcal{G}_n$. In view of Lemma 4.2.4, to show that E is hyperfinite, it is enough to check that $\mathcal{G}_n$ satisfies Weiss' condition. By assumption $C_n^2 \subseteq C_{n+1}$, therefore $\mathcal{G}_n^2 \subseteq \mathcal{G}_{n+1}$. We claim that for any n such that $|C_{n+4}| \geq K|C_n|$ one has that any $\mathcal{G}_{n+1}$-independent subset of $[z]_{\mathcal{G}_{n+3}}$ has size at most K for all $z \in X$. Indeed, suppose towards a contradiction, there is a $\mathcal{G}_{n+1}$-independent set $\{x_0, \ldots, x_K\} \subseteq [z]_{\mathcal{G}_{n+3}}$. Let $g_i \in C_{n+3}$ be such that $g_i z = x_i$. Note that $C_n g_i \cap C_n g_j = \varnothing$ for $i \neq j$, as if $h_1, h_2 \in C_n$ are such that $h_1 g_i = h_2 g_j$, then $h_2^{-1} h_1 \in C_{n+1}$ satisfies $h_2^{-1} h_1 x_i = x_j$, contradicting $\mathcal{G}_{n+1}$-independence of x_i and x_j. Since $C_n g_i \subseteq C_{n+4}$ are pairwise disjoint, we get $|C_{n+4}| \geq (K+1)|C_n|$, which is impossible. Thus $\mathcal{G}_n$ satisfies Weiss' condition and E is hyperfinite. $\square$

As the following lemma shows, the class of groups that have mild growth K is closed under inductive limits.

Lemma 4.3.4. *Let G be a countable group, and suppose that $G = \bigcup_n G_n$ is written as an increasing union of groups each having mild growth K (note that K is assumed to be independent of n). The group G also has mild growth K.*

Proof. Let $C_{k,n}$ be a sequence of subsets of G_k witnessing that G_k has mild growth K. We construct inductively a mild growth witness D_n, $n \in \mathbb{N}$, for G. The step of induction will construct 5 sets at a time: $D_{5m}, D_{5m+1}, \ldots, D_{5m+4}$. At each step we ensure that $|D_{5m+4}| \leq K|D_{5m}|$. In other words, item (v) in the definition of mild growth will be satisfied for all n such that $n = 0 \mod 5$.

We need to take into account all sets $C_{k,n}$, so we start by enumerating all $C_{k,n}$ in a sequence, i.e., we pick a bijection $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $\alpha(n) = \big(\alpha_1(n), \alpha_2(n)\big)$. The base of inductive construction of sets D_n is no different from the step of induction, so we show the latter. Suppose we have constructed D_i for $i < 5m$, and we aim at defining $D_{5m}, \ldots, D_{5m+4}$. Let $C = C_{\alpha_1(5m), \alpha_2(5m)}$. Pick N_1 so large that $D_{5m-1} \subseteq G_{N_1}$ and $\alpha_1(5m) \leq N_1$. Since the sequence $C_{N_1,n}$ witnesses the mild growth of G_{N_1}, one can find N_2 so large that

$$\big(D_{5m-1} \sqcup C\big)^2 \subseteq C_{N_1, N_2}$$

and $|C_{N_1,N_2+4}| \leq K|C_{N_1,N_2}|$. We set $D_{5m+i} = C_{N_1,N_2+i}$ for $i = 0, \dots, 4$. Evidently, sets D_i witness the mild growth of G. □

The primary example of groups with mild growth are the groups of polynomial growth.

Proposition 4.3.5. *If G is a finitely generated group of polynomial growth d, then G has mild growth $16^d + 1$.*

Proof. Let $S \subseteq G$ be a symmetric generating set for G and let $a \in \mathbb{R}^{\geq 0}$ be such tat $|S^n| \leq an^d$. Set $C_n = S^{2^n}$. We claim that the sequence C_n witnesses the mild growth of G. Only item (v) requires checking. Set $K = 16^d + 1$. Suppose towards a contradiction that $|C_{n+4}| > K|C_n|$ for all $n \geq n_0$. Therefore also

$$|C_{n+8}| > K|C_{n+4}| > K^2|C_n|,$$

and more generally $|C_{n+4m}| > K^m|C_n|$ for all $n \geq n_0$. One thus has for all $m \in \mathbb{N}$

$$K^m|C_{n_0}| < |C_{n_0+4m}| = \left|S^{2^{n_0+4m}}\right| \leq a2^{n_0 d+4md} = a2^{n_0 d} \cdot (16^d)^m.$$

The latter is possible only when $K \leq 16^d$, contradicting $K = 16^d + 1$. □

Corollary 4.3.6. *All actions of finitely generated nilpotent groups are hyperfinite.*

Proof. By a well-known theorem of Joseph Wolf [Wol68], all finitely generated nilpotent groups have polynomial growth. Therefore Propositions 4.3.5 and 4.3.3 imply that such groups have hyperfinite actions only. □

Corollary 4.3.7. *All Borel actions of $\mathbb{Q}^d$ are hyperfinite.*

Proof. While the group $\mathbb{Q}^d$ is not finitely generated, it can be written as an increasing union of subgroups

$$\mathbb{Q}^d = \bigcup_n \big(\mathbb{Z}[1/n!]\big)^d,$$

each having polynomial growth d. Proposition 4.3.5, Lemma 4.3.4, and Proposition 4.3.3 altogether imply that all Borel action of $\mathbb{Q}^d$ are hyperfinite. □

Appendix A

Spaces of Measures

Definition A.1. Let $(X, \mathcal{B})$ be a standard Borel space. Recall that a *signed measure* or a *charge* on X is a function $\mu : \mathcal{B} \to \mathbb{R}$ such that $\mu(\varnothing) = 0$ and μ is countably additive, i.e., $\mu\big(\bigcup_n A_n\big) = \sum_n \mu(A_n)$ for all pairwise disjoint families $A_n \in \mathcal{B}$.

A charge μ is said it to be a *measure* if $\mu(A) \geq 0$ for all $A \in \mathcal{B}$.

Theorem A.2 (Hahn)**.** *Let μ be a charge on $(X, \mathcal{B})$. There exists a Borel partition $X = P \sqcup N$ such that $\mu(A \cap P) \geq 0$ and $\mu(A \cap N) \leq 0$ for all $A \in \mathcal{B}$. Moreover, such a partition is essentially unique in the sense that if $X = P' \sqcup N'$ is another such partition, then $\mu(A \cap P \cap Q') = 0 = \mu(A \cap P' \cap Q)$ for all $A \in \mathcal{B}$.*

For a charge μ let $X = P \sqcup Q$ be the decomposition as in Hahn's Theorem. Set $\mu^+ : \mathcal{B} \to \mathbb{R}^{\geq 0}$ to be $\mu^+(A) = \mu(A \cap P)$ and define $\mu^- : \mathcal{B} \to \mathbb{R}^{\geq 0}$ by $\mu^- = -\mu(A \cap N)$. The functions μ^+ are μ^- are, in fact, measures, $\mu = \mu^+ - \mu^-$, and a decomposition of this form (called the Jordan decomposition) is unique, i.e., if $\mu = \nu^+ - \nu^-$, where ν^+ and ν^- are measures on $(X, \mathcal{B})$, then $\nu^+ = \mu^+$ and $\nu^- = \mu^-$. The *variation* of a charge μ is the measure $|\mu| = \mu^+ + \mu^-$, and the *total variation* of μ is the real $||\mu|| = |\mu|(X)$. The set $\mathcal{C}(X)$ of all charges on X is a Banach space when endowed with the norm $||\mu||$; the set $\mathcal{M}(X) \subseteq \mathcal{C}(X)$ of measures on X forms a closed cone in $\mathcal{C}(X)$ (we include the zero measure in $\mathcal{M}(X)$).

Let X be a compact Polish space, and let $C(X)$ denote the Banach space of continuous real-valued functions.

Theorem A.3 (Riesz, Markov, Kakutani)**.** *The dual $C(X)^*$ to the space $C(X)$ is isometric to the Banach space $\mathcal{C}(X)$ of charges on X.*

In particular, by Alaoglu's Theorem, the unit Ball $\mathcal{C}_1(X)$ in $\mathcal{C}(X)$ is a compact metrizable space in the weak$*$ topology. Since $\mathcal{M}(X)$ is closed in $\mathcal{C}(X)$ in the weak$*$ topology, the set $\mathcal{M}_1(X) = \mathcal{C}_1(X) \cap \mathcal{M}(X)$ is also weak$*$ compact.

Appendix B

Existence and uniqueness of measures

In this appendix we would like to recall some standard notions from measure theory, which are often used to construct Borel measures on metric spaces. Proofs of the following theorems can be found in any standard textbook in measure theory.

Definition B.1. An *outer measure* on a set X is a map $\mu^* : 2^X \to [0, \infty]$ such that

- $\mu^*(\varnothing) = 0$;
- $\mu^*(A) \le \mu^*(B)$ whenever $A \subseteq B$;
- $\mu^*\Big(\bigcup_n A_n\Big) \le \sum_n \mu^*(A_n)$ for any countable family $A_n \subseteq X$.

The classical Carathéodory's Theorem gives a way of constructing a measure out of an outer measure.

Theorem B.2 (Carathéodory's Theorem)**.** *Let μ^* be an outer measure on X, and let $\mathcal{B}$ be the set of all $Y \subseteq X$ such that $\mu^*(Y) = \mu^*(Y \cap A) + \mu^*\big(Y \cap (X \setminus A)\big)$ for all $A \subseteq X$. The set $\mathcal{B}$ is a σ-algebra and μ^* restricted onto $\mathcal{B}$ is a σ-additive measure on $\mathcal{B}$.*

We call the σ-algebra $\mathcal{B}$ the Carathéodory σ-algebra, and the restriction of μ^* onto $\mathcal{B}$ the Carathéodory measure associated with μ^*.

Definition B.3. Let (X, d) be a metric space. An outer measure μ^* on X is said to be a *metric outer measure* if $\mu^*(A \sqcup B) = \mu^*(A) + \mu^*(B)$ for all $A, B \subseteq X$ such that $d(A, B) := \inf\{d(a, b) : a \in A, b \in B\} > 0$.

Theorem B.4. *If μ^* is a metric outer measure on a metric space (X, d), then the Carathéodory σ-algebra contains all Borel sets, and so the restriction of the Carathéodory measure associated with μ^* onto the Borel σ-algebra gives a Borel measure on X.*

The following is an important method of constructing metric outer measures. We say that $\mathcal{C} \subseteq 2^X$ is a *sequential covering class* if there exists a countable family $C_k \in \mathcal{C}$ such that $X = \bigcup_k C_k$. Let (X, d) be a metric space, $\mathcal{C} \subseteq 2^X$ be such that for each $n \in \mathcal{C}$ the family

$$\mathcal{C}_n = \big\{C \in \mathcal{C} : \operatorname{diam}(C_n) < 1/n\big\} \text{ is a sequential covering class.}$$

Let also $\tau : \mathcal{C} \to [0, \infty]$ be any function such that $\tau(\varnothing) = 0$. Let $\mu_n^* : 2^X \to [0, \infty]$ be define by

$$\mu_n^*(A) = \inf\Big\{\sum_{k=0}^{\infty} \tau(C_k) : A \subseteq \bigcup_k C_k, C_k \in \mathcal{C}_n\Big\}.$$

Note that $\mu_n^*(A) \ge \mu_{n+1}^*(A)$ for all $n \in \mathbb{N}$ and all $A \subseteq X$, so we may set $\mu^*(A) = \lim_{n\to\infty} \mu_n^*(A)$.

Theorem B.5. *The function* $\mu^* : 2^X \to [0, \infty]$ *defined above is a metric outer measure on* X.

Definition B.6. Recall that a family $\mathcal{D} \subseteq 2^X$ of subsets of X is said to be a λ-*system* if

1. $X \in \mathcal{D}$;
2. if $A \in \mathcal{D}$, then $X \setminus A \in \mathcal{D}$;
3. if $A_n \in \mathcal{D}$ are pairwise disjoint, then $\bigcup_n A_n \in \mathcal{D}$.

A family $\mathcal{P} \subseteq 2^X$ is a π-*system* if is closed under finite intersections: $A \cap B \in \mathcal{P}$, whenever A and B belong to $\mathcal{P}$.

Here is a typical way how λ-systems arise in measure theory. Let μ and ν be two probability measures on X. The family of measurable sets

$$\mathcal{D} = \big\{ A \subseteq X : \mu(A) = \nu(A) \big\}$$

is easily seen to be a λ-system.

Theorem B.7 (Dynkin's π-λ theorem)**.** *Let* $\mathcal{P}$ *be a* π*-system on* X*,* $\mathcal{D}$ *be a* λ*-system on* X*, and suppose that* $\mathcal{P} \subseteq \mathcal{D}$*. If* $\sigma(\mathcal{P})$ *is the* σ*-algebra generated by* $\mathcal{P}$*, then* $\sigma(\mathcal{P}) \subseteq \mathcal{D}$*.*

Here is a useful immediate corollary of Dynkin's theorem.

Theorem B.8 (Carathéodory's Uniqueness Theorem)**.** *Let* μ *and* ν *be Borel probability measures on a standard Borel space* X*, let*

$$\mathcal{D} = \big\{ A \subseteq X : A \text{ is Borel and } \mu(A) = \nu(A) \big\}.$$

If there is a π*-system* $\mathcal{P} \subseteq \mathcal{D}$ *such that* $\mathcal{P}$ *generates the Borel* σ*-algebra on* X*, then* $\mu = \nu$*.*

In particular, two probability measures on $2^{\mathbb{N}}$ which agree on all clopen sets must be equal.

Appendix C

Amenable groups

Definition C.1. A *finitely additive measure* on a set X is a map $\mu : 2^X \to [0,1]$ such that

(i) $\mu(\varnothing) = 0$, $\mu(X) = 1$;

(ii) $\mu(A \sqcup B) = \mu(A) + \mu(B)$ for all disjoint subsets $A, B \subseteq X$.

If $H \curvearrowright X$ is an action of a countable group on X, we say that a finitely additive measure μ is H-invariant, if $\mu(hA) = \mu(A)$ for all $A \subseteq X$ and all $h \in H$.

Definition C.2. A *mean* on $\ell^\infty(X)$ is a functional $\tau : \ell^\infty(X) \to \mathbb{R}$ such that $\tau(f) \geq 0$ for all $f \in \ell^\infty_+(X)$ and $\tau(\mathbb{1}) = 1$. Suppose $H \curvearrowright \ell^\infty(X)$ by isometries. We say that a mean τ is H-invariant, if $\tau(hf) = \tau(f)$ for all $h \in H$ and $f \in \ell^\infty(X)$.

There is a duality between finitely additive measures on X and means on $\ell^\infty(X)$. Any mean τ on $\ell^\infty(X)$ gives rise to a measure μ_τ on X by the formula $\mu_\tau(A) = \tau(\chi_A)$, where χ_A is the characteristic function of A. Also, if μ is a finitely additive measure on X, then one can define a functional τ_μ on $\ell^\infty(X)$ by setting for $f \in \ell^\infty(X)$

$$\tau_\mu(f) = \int_X f(x)\, d\mu(x).$$

The functional τ_μ is easily seen to be a mean. Maps $\mu \mapsto \tau_\mu$ and $\tau \mapsto \mu_\tau$ are inverses of each other. Moreover, these maps preserve invariance of actions in the following sense. Suppose we have a group H acting on X. This action can be lifted to an action on $\ell^\infty(X)$ by $hf(x) = f(h^{-1}x)$. A finitely additive measure μ on X is H-invariant if and only if the mean τ_μ is H-invariant. This duality will let us speak of finitely additive measures or means depending on what is more convenient in the particular situation.

Definition C.3. A countable group G is *amenable* if there exists an invariant finitely additive measure for the action $G \curvearrowright G$ by left multiplication.

The notion of amenability is of fundamental importance and has a huge number of equivalent reformulations. The following lemma lists several of them. The proof is based on [Nam64], and our presentation follows 2.8 of [Tao10].

Lemma C.4. *Let G be a countable group. The following conditions are equivalent.*

(i) G is amenable;

(ii) for any finite $F \subseteq G$ and any $\epsilon > 0$ there exists a finitely supported $\nu \in \ell^1_+(G)$ such that $||\nu||_1 = 1$ and $||\nu - f\nu||_1 < \epsilon$ for all $f \in F$.

(iii) for any finite $F \subseteq G$ and any $\epsilon > 0$ there exists a finite set $K \subseteq G$ such that

$$\sup_{f\in F} \frac{\big|fK \bigtriangleup K\big|}{|K|} < \epsilon.$$

Proof. (i) ⇒ (ii) Suppose towards a contradiction that there is a finite set $F \subseteq G$ and $\epsilon > 0$ such that for every finitely supported $\nu \in \ell^1_+(G)$ of norm 1 one has $\sup_{f\in F} ||\nu - f\nu||_1 \geq \epsilon$. The same inequality is seen to be true for all, not necessarily finitely supported, $\nu \in \ell^1_+(G)$ of norm 1.

Consider the set

$$Z = \big\{(\nu - f\nu)_{f\in F} : \nu \in \ell^1_+(G), ||\nu||_1 = 1\big\} \subseteq \big(\ell^1(G)\big)^{|F|}.$$

This set is convex, and by assumption it is bounded away from 0. Hahn-Banach separation theorem guarantees existence of a linear functional $\lambda \in \big(\ell^\infty(G)\big)^{|F|}$ such that on Z one has

$$\lambda\big((\nu - f\nu)_{f\in F}\big) \geq 1.$$

Let $\lambda_f \in \ell^\infty(G)$ be such that $\lambda = (\lambda_f)_{f\in F}$. We therefore have for all $\nu \in \ell^1_+(G)$, $||\nu||_1 = 1$:

$$1 \leq \sum_{f\in F} \lambda_f(\nu - f\nu) = \sum_{f\in F} \big(\lambda_f(\nu) - \lambda_f(f\nu)\big) = \sum_{g\in G}\sum_{f\in F} \lambda_f(g)\nu(g) - \sum_{g\in G}\sum_{f\in F} \lambda_f(g)\nu(f^{-1}g) =$$
$$\sum_{g\in G}\sum_{f\in F} \lambda_f(g)\nu(g) - \sum_{g\in G}\sum_{f\in F} \lambda_f(fg)\nu(g) = \sum_{g\in G}\bigg(\sum_{f\in F} \lambda_f(g) - \lambda_f(fg)\bigg)\nu(g).$$

The above inequality is true for all $\nu \in \ell^1_+(G)$, $||\nu||_1 = 1$. Taking $\nu = \delta_g$, we deduce that for all $g \in G$ one has

$$\sum_{f\in F} \lambda_f(g) - \lambda_f(fg) \geq 1,$$

which implies that

$$\sum_{f\in F} (\lambda_f - f^{-1}\lambda_f) - \mathbb{1} \geq 0.$$

By assumption, there exists an invariant mean τ on $\ell^\infty(G)$. Thus

$$0 \leq \tau\Big(\sum_{f\in F}(\lambda_f - f^{-1}\lambda_f) - \mathbb{1}\Big) = \sum_{f\in F}\big(\tau(\lambda_f) - \tau(f^{-1}\lambda_f)\big) - 1 = -1.$$

This contradiction proves the implication.

(ii) ⇒ (iii) Fix a finite set $F \subseteq G$ and $\epsilon > 0$. By assumption there is a finitely supported $\nu \in \ell^1_+(G)$ such that

$$\sup_{f\in F} ||\nu - f\nu||_1 < \frac{\epsilon}{|F|}.$$

We may find nested sets $A_1 \supset A_2 \supset \cdots \supset A_k$, $A_1 = \operatorname{supp}\nu$, and $c_i > 0$ such that $\nu = \sum_{i=1}^k c_i\chi_{A_i}$. One has

$$\sum_{i=1}^k c_i|A_i| = 1.$$

Also, observe that

$$(\nu - f\nu)(g) = \sum_{i=1}^k c_i\big(\chi_{A_i\setminus fA_i}(g) - \chi_{fA_i\setminus A_i}(g)\big).$$

Note that all the summands above have the same sign, because sets A_i are nested. Using this, we have

$$||\nu - f\nu||_1 = \sum_{g\in G}\left|\sum_{i=1}^{k} c_i\big(\chi_{A_i\setminus fA_i}(g) - \chi_{fA_i\setminus A_i}(g)\big)\right| =$$
$$\sum_{g\in G}\sum_{i=1}^{k} c_i\left|\chi_{A_i\setminus fA_i}(g) - \chi_{fA_i\setminus A_i}(g)\right| =$$
$$\sum_{i=1}^{k} c_i\sum_{g\in G}\left|\chi_{A_i\setminus fA_i}(g) - \chi_{fA_i\setminus A_i}(g)\right| =$$
$$\sum_{i=1}^{k} c_i|A_i \vartriangle fA_i|.$$

Therefore, for all $f \in F$

$$\sum_{i=1}^{k} c_i|fA_i \vartriangle A_i| \le \frac{\epsilon}{|F|} = \frac{\epsilon}{|F|}\sum_{i=1}^{k} c_i|A_i|.$$

Summing over all $f \in F$, one has

$$\sum_{i=1}^{k} c_i\sum_{f\in F}|fA_i \vartriangle A_i| \le \epsilon\sum_{i=1}^{k} c_i|A_i|.$$

By pigeon-hole principle, there is i such that $\sum_{f\in F}|fA_i \vartriangle A_i| \le \epsilon|A_i|$, as claimed.

(iii) $\Rightarrow$ (i) By assumption, there is a sequence of finite subsets $F_n \subseteq G$ such that

$$\frac{|gF_n \vartriangle F_n|}{|F_n|} \to 0 \text{ as } n \to \infty \text{ for all } g \in G.$$

Pick a non-principal ultrafilter ω on $\mathbb{N}$, and define the mean τ by

$$\tau(\nu) = \lim_{n\to\omega}\nu\Big(\frac{\chi_{F_n}}{|F_n|}\Big).$$

It is straightforward to check that τ is invariant. □

Definition C.5. A countable group G satisfies *Reiter's condition* if for any finite $F \subseteq G$ and any $\epsilon > 0$ there exists $\nu \in \ell^1_+(G)$, $||\nu||_1 = 1$, such that

$$\sup_{f\in F}||\nu - f\nu||_1 < \epsilon.$$

A group satisfies *Følner's condition* if for any finite $F \subseteq G$ and $\epsilon > 0$ there exists a finite set $K \subseteq G$ such that

$$\sup_{f\in F}\frac{|fK \vartriangle K|}{|K|} < \epsilon.$$

Lemma C.4 establishes equivalence between amenability and the two conditions introduced in the definition above.

Example C.6. The group $\mathbb{Z}$ is amenable, as $\{1, \ldots, n\}$, $n \in \mathbb{N}$, forms a sequence of Følner sets. On the other hand, we claim that the free group $F_2 = \langle a, b\rangle$ is not amenable. Indeed, suppose towards a contradiction that μ is a finitely additive invariant measure on F_2. Let $S(a), S(a^{-1}), S(b), S(b^{-1})$ be sets consisting of elements of F_2, which start with the corresponding letter, i.e.,

$$F_2 = S(a) \sqcup S(a^{-1}) \sqcup S(b) \sqcup S(b^{-1}) \sqcup \{e\}.$$

First of all, note that $\mu(\{f\}) = 0$ for any $f \in \langle a, b\rangle$. Since

$$S(a) = aS(a) \sqcup aS(b) \sqcup aS(b^{-1}) \sqcup \{a\},$$

invariance of μ implies

$$\mu\big(S(a)\big) = \mu\big(S(a)\big) + \mu\big(S(b)\big) + \mu\big(S(b^{-1})\big),$$

hence $\mu\big(S(b)\big) = \mu\big(S(b^{-1})\big) = 0$. Similarly, $\mu\big(S(a)\big) = \mu\big(S(a^{-1})\big) = 0$. We conclude $\mu(F_2) = 0$, which is absurd.

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