Lecture notes on
Topological full groups of Cantor minimal systems

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LECTURE 1

Introduction to the topic

Throughout the text X denotes a Cantor space. When convenient we shall take a concrete realization of X, e.g., $2^{\mathbb{N}}$ or $2^{\mathbb{Z}}$. The group of homeomorphisms of X is denoted by $\mathrm{Homeo}(X)$. The natural numbers $\mathbb{N}$ start with 0.

1. Minimal homeomorphisms

Definition 1.1. A homeomorphism $\phi \in \mathrm{Homeo}(X)$ is called *periodic*, if every orbit of ϕ is finite. It is called *aperiodic*, if all its orbits are infinite. We say that ϕ *has period* n, if every orbit of ϕ has precisely n points; in this case $\phi^n = \mathrm{id}$. A homeomorphism $\phi \in \mathrm{Homeo}(X)$ is said to be *minimal* if every its orbit is dense: $\overline{\mathrm{Orb}_\phi(x)} = X$ for all $x \in X$. Note that minimal homeomorphisms are always aperiodic.

Proposition 1.2. *For a homeomorphism $\phi \in \mathrm{Homeo}(X)$ the following conditions are equivalent:*

(i) ϕ is minimal.

(ii) Every forward orbit of ϕ is dense: $\overline{\{\phi^n(x)\}}_{n\in\mathbb{N}} = X$ for all $x \in X$.

(iii) There are no nontrivial closed invariant subspaces of X: if $F \subseteq X$ is closed and $\phi(F) = F$, then either $F = \varnothing$ or $F = X$.

(iv) For any non-empty clopen $U \subseteq X$ there is $N \in \mathbb{N}$ such that $X = \bigcup_{i=0}^{N} \phi^i(U)$.

Proof. (i) ⇒ (iii) Let $F \subseteq X$ be a closed non-empty invariant subset with $x \in F$. By invariance $\mathrm{Orb}_\phi(x) \subseteq F$, hence $X = \overline{\mathrm{Orb}_\phi(x)} \subseteq F$.

(iii) ⇒ (ii) Pick $x \in X$ and let $R = \overline{\{\phi^n(x)\}}_{n\in\mathbb{N}}$; note that $\phi(R) \subseteq R$. If $F = \bigcap_{n\in\mathbb{N}} \phi^n(R)$, then

$$\phi(F) = \bigcap_{n\geq 1} \phi^n(R) = F$$

and therefore $F = X$, whence $R = X$.

(ii) ⇒ (iv) If U is open and non-empty, then $F = \sim \bigcup_{n\in\mathbb{Z}} \phi^n(U)$ is closed, invariant and $F \cap U = \varnothing$, hence $F = \varnothing$. Therefore $\bigcup_{n\in\mathbb{Z}} \phi^n(U) = X$, which by compactness implies $\bigcup_{|n|\leq M} \phi^n(U) = X$ for some M. Hence

$$X = \phi^M(X) = \bigcup_{n=0}^{2M} \phi^n(U)$$

(iv) ⇒ (i) For any $x \in X$ the set $\sim \overline{\mathrm{Orb}_\phi(x)}$ is open, invariant, and does not contain x, hence must be empty. □

Example 1.3. The *odometer* $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a homeomorphism defined as follows. For $x \in 2^{\mathbb{N}} \setminus \{\mathbf{1}\}$, where $\mathbf{1}$ is the constant sequence of ones, let n be the smallest integer such that $x(n) = 0$. The image $\sigma(x)$ is then defined by

$$\sigma(x)(i) = \begin{cases} 0 & \text{if } i < n, \\ 1 & \text{if } i = n, \\ x(i) & \text{if } i > n. \end{cases}$$

Set $\sigma(\mathbf{1}) = \mathbf{0}$. For examples if $x = 1110^\frown y$, then $\sigma(x) = 0001^\frown y$.

Exercise 1.4. Check that $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a homeomorphism. Show that it is minimal.

Example 1.5. Another important example is the shift $s : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ defined by $s(x)(i) = x(i+1)$. It is easy to see that s is indeed a homeomorphism.

Exercise 1.6. Show that s is *not* minimal, but s is transitive: there is $x \in 2^{\mathbb{Z}}$ such that the orbit $\mathrm{Orb}_\phi(x)$ is dense in $2^{\mathbb{Z}}$.

While the shift homeomorphism is not minimal, it has lots of minimal subshifts. We say that a sequence $x \in 2^{\mathbb{Z}}$ is *homogeneous* if for every finite sequence $\alpha \in 2^{<\omega}$ that occurs in x there is a number $N(\alpha)$ such that any interval of length $N(\alpha)$ in x contains α.

Theorem 1.7. *Let $x \in 2^{\mathbb{Z}}$ be a binary sequence, and let $Y = \overline{\mathrm{Orb}_s(x)}$. The subshift $(Y, s|_Y)$ is minimal if and only if x is homogeneous.*

PROOF. Suppose $x \in X$ is homogeneous and pick a $y \in Y$. Our goal is to show that $\mathrm{Orb}_s(y)$ is dense in Y. For this it is enough to show that $x \in \overline{\mathrm{Orb}_s(y)}$. Pick a segment α of x. By homogeneity there is an integer $N(\alpha)$ such that any segment of x of length $N(\alpha)$ contains a subsegment α. Pick any subsegment β of y of length $N(\alpha)$. Since $y \in Y$, this subsegment β must also occur in x, whereby using homogeneity we see that α occurs in y. Therefore $x \in \overline{\mathrm{Orb}_s(y)}$.

For the other direction we show the contrapositive. Suppose x is not homogeneous. It means that there is a segment α of x and infinitely many segments β_n of x such that the length of β_n growth and β_n does not contain the subsegment α. Assume for convenience that the length of β_n is $2n+1$. Let $y_n \in X$ be such that $y_n|_{[-n,n]} = \beta_n$ and α does not occur in y_n. By compactness of X there is a $y \in X$ and $(n_k)_{k\in\mathbb{N}}$ such that $y_{n_k} \to y$. It is now easy to see that $y \in \overline{\mathrm{Orb}_s(x)}$ and that $x \notin \overline{\mathrm{Orb}_s(y)}$, whence $s|_Y$ is not minimal. □

Proposition 1.8. *For any $\phi \in \mathrm{Homeo}(X)$ there is a closed non-empty $F_0 \subseteq X$ such that $\phi(F_0) = F_0$ and $(F_0, \phi|_{F_0})$ is minimal.*

PROOF. Let
$$\mathcal{F} = \{\, F \subseteq X \mid F \text{ is closed, non-empty, and } \phi(F) = F \,\}$$
be the family of closed invariant subsets ordered by inclusion. Note that if $(F_\lambda)_{\lambda\in\Lambda}$ is a chain in $\mathcal{F}$, then $\bigcap_\lambda F_\lambda$ also belongs to $\mathcal{F}$. Hence by Zorn's lemma we can find a minimal element $F_0 \in \mathcal{F}$. The system $(F_0, \phi|_{F_0})$ is minimal by item (iii) of Proposition 1.2. □

2. Full groups

Definition 1.9. Let $\phi \in \mathrm{Homeo}(X)$ be a homeomorphism of a Cantor space X. The *full group* of ϕ is denoted by $[\phi]$ and is defined to be
$$[\phi] = \{\, g \in \mathrm{Homeo}(X) \mid \forall x \in X\ \exists n(x) \in \mathbb{Z} \quad g(x) = \phi^{n(x)}(x) \,\}.$$
With an element $g \in [\phi]$ we associate the *cocycle* $n = n_g : X \to \mathbb{Z}$ given by $g(x) = \phi^{n(x)}(x)$. Note that if ϕ is aperiodic, then the cocycle is uniquely defined. The *topological full group* of ϕ is denoted by $[\![\phi]\!]$ and is the subgroup of those $g \in [\phi]$ for which the cocycle n_g is continuous (or, more formally, can be chosen to be continuous) with respect to the discrete topology on the integers:
$$[\![\phi]\!] = \{\, g \in [\phi] \mid n_g : X \to \mathbb{Z} \text{ is continuous} \,\}.$$

Proposition 1.10. *Let $\phi \in \mathrm{Homeo}(X)$ be any homeomorphism. An element $g \in \mathrm{Homeo}(X)$ is in the topological full group $g \in [\![\phi]\!]$ if and only if there are clopen sets $A_1, \ldots, A_m$ and integers $k_1, \ldots, k_m \in \mathbb{Z}$ such that $X = A_1 \sqcup \cdots \sqcup A_m$ and $g|_{A_i} = \phi^{k_i}|_{A_i}$.*

PROOF. If $g \in [\![\phi]\!]$, then the cocycle $n_g : X \to \mathbb{Z}$ can be chosen to be continuous, and therefore the image $n_g(X)$ is finite; let $k_1, \ldots, k_m \in \mathbb{Z}$ be the integers in the image of n_g. We set $A_i = n_g^{-1}(k_i)$ and the necessity is proved. For the sufficiency we note that the cocycle n_g can be constructed by setting $n_g|_{A_i} = k_i$. If the decomposition of X into the sets A_i is clopen, then the cocycle n_g is continuous. □

Definition 1.11. The *support* of a homeomorphism $\phi \in \mathrm{Homeo}(X)$ is defined to be the complement of the interior of the set of fixed points, or equivalently
$$\mathrm{supp}(\phi) = \overline{\{\, x \in X \mid \phi(x) \neq x \,\}}.$$
Note that support of an aperiodic homeomorphism is necessarily all of X.

In general support of a homeomorphism is not necessarily open. The following proposition shows that elements of the topological full group of a minimal homeomorphism are special in this sense.

Proposition 1.12. *Let $\phi \in \mathrm{Homeo}(X)$ be minimal. The support $\mathrm{supp}(g)$ of any $g \in [\![\phi]\!]$ is a clopen subset of X.*

PROOF. Pick a $g \in [\![\phi]\!]$ and find clopen subsets A_i for $i \in I$ such that $g|_{A_i} = \phi^i|_{A_i}$, where $I \subset \mathbb{N}$ is finite. The support of g is then given by

$$\mathrm{supp}(g) = \bigcup_{i \in I \setminus \{0\}} A_i,$$

and is therefore clopen. □

Proposition 1.13. *Let $\phi \in \mathrm{Homeo}(X)$ be minimal. For any $g \in [\![\phi]\!]$ and any $n \in \mathbb{N}$ the set*

$$X_n = \{ x \in X \mid \mathrm{Orb}_g(x) \text{ has cardinality } n \}$$

is clopen.

PROOF. Let $\mathcal{P} = (A_i)_{i \in I}$ be a clopen partition of X such that $g|_{A_i} = \phi^i|_{A_i}$, where $I \subset \mathbb{N}$ is finite. Let $(B_j)_{j=1}^N = \bigvee_{k=0}^n \phi^{-k}(\mathcal{P})$ be the refinement of the partitions $\phi^{-k}(\mathcal{P})$ for $0 \leq k \leq n$. For each B_j there is an integer m_j such that $g|_{B_j} = \phi^{m_j}|_{B_j}$. Let $x \in X_n$ and let $j_0, \dots, j_n$ be such that $\phi^k(x) \in B_{j_k}$ for all $0 \leq k \leq n$. By the definition of X_n we have $g^n(x) = x$ and therefore

$$\phi^{\sum_{k=0}^n m_{j_k}}(x) = x,$$

which is possible only if $\sum_{k=0}^n m_{j_k} = 0$, whence $B_{j_0} \subseteq X_n$. This shows that X_n is open.

Since

$$X_n = \{ x \in X \mid g^n(x) = x \} \setminus \bigcup_{m<n} \{ x \in X \mid g^m(x) = x \},$$

the set X_n is also closed. □

Proposition 1.14. *Let $f \in \mathrm{Homeo}(X)$ be a periodic homeomorphism of period n. There exists a clopen set $A \subseteq X$ such that $X = \bigsqcup_{i=0}^{n-1} f^i(A)$.*

PROOF. For any point $x \in X$ we can find a clopen neighbourhood $U_x \subseteq X$ such that $f^i(U_x) \cap U_x = \varnothing$ for all $1 \leq i < n$. By compactness of X there is a finite family $x_1, \dots, x_N \in X$ such that $X = \bigcup_{j \leq N} U_{x_j}$. We now construct sets A_j inductively. Put $A_1 = U_{x_1}$, and

$$A_{j+1} = A_j \cup \left(U_{x_{j+1}} \setminus \bigcup_{i=0}^{n-1} f^i(A_j) \right).$$

It is now straightforward to see that $A = A_N$ satisfies the conclusion of the proposition. □

3. Kakutani–Rokhlin partitions

We would like to describe an important space decomposition construction that is attributed to Kakutani and Rokhlin. Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism and let $D \subseteq X$ be a non-empty clopen subset. We define the *first return* function $t_{D,\phi} = t_D : D \to \mathbb{N}$ by

$$t_D(x) = \min\{ n \geq 1 \mid \phi^n(x) \in D \}.$$

By minimality of ϕ, the function t_D is well-defined and continuous. We can therefore find a number N, positive integers $k_1, \dots, k_N$, and a partition $D = D_1 \sqcup \dots \sqcup D_N$ into non-empty clopen subsets such that $t_D|_{D_i} = k_i$. The space X can then be written as a disjoint union of sets (see Figure 1)

$$X = D_1 \sqcup \phi(D_1) \sqcup \dots \sqcup \phi^{k_1 - 1}(D_1) \sqcup D_2 \sqcup \phi(D_2) \sqcup \dots \sqcup \phi^{k_2-1}(D_2) \sqcup \ldots \sqcup D_N \sqcup \phi(D_N) \sqcup \ldots \sqcup \phi^{k_N-1}(D_N).$$

One refers to the family $D_i, \phi(D_i), \dots, \phi^{k_i-1}(D_i)$ as to the *tower over* D_i. The number k_i is then the *height* of this tower. The set D_i is the *base* of the tower, and $\phi^{k_i-1}(D_i)$ is its *top*. Note that every point in the top level of some tower goes under the action of ϕ to a base of a (possibly different) tower.

Exercise 1.15. Draw the Kakutani–Rokhlin partition of the odometer σ over the cylindrical set $D = \{ x \in 2^{\mathbb{N}} \mid x(i) = 0,\ i \leq n \}$ for some fixed n.

When building a Kakutani–Rokhlin partition it is sometimes useful to assume that the obtained partition is finer than a given partition $\mathcal{P}$. The following proposition assures that this can always be done.

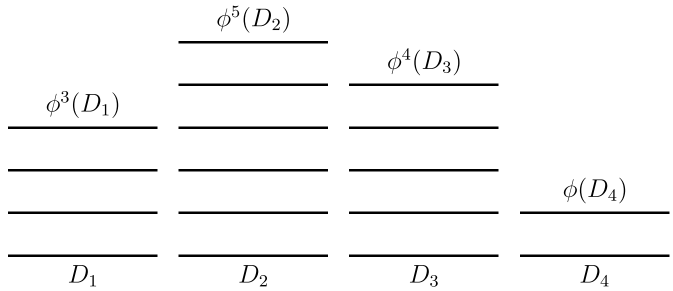

Figure 1. A Kakutani–Rokhlin partition of X with base D.

Proposition 1.16. *Let $\phi \in \mathrm{Homeo}(X)$ be minimal, let $D \subseteq X$ be a clopen subset, and let $\mathcal{P}$ be a partition of X. There are positive integers $K, J_1, \ldots, J_K$ and clopen subsets $D(i,j) \subseteq X$ indexed by pairs (i,j) satisfying $1 \le i \le K$ and $0 \le j < J_i$ such that*

(i) $X = \bigsqcup_{i,j} D(i,j)$ *and this partition is finer than* $\mathcal{P}$*;*
(ii) $D = \bigsqcup_i D(i,0)$*;*
(iii) $\phi\big(D(i,j)\big) = D(i,j+1)$ *for all* $1 \le i \le K$ *and* $0 \le j < J_i - 1$*;*
(iv) $\phi\big(D(i,J_i - 1)\big) \subseteq D$ *for all* $1 \le i \le K$*.*

Proof. The Kakutani–Rokhlin partition over the base D described above satisfies all the items except possibly for the first one: it may not refine the partition $\mathcal{P}$. We shall now explain how the Kakutani–Rokhlin partition can be refined.

Suppose we are given sets $\widetilde{D}(i,j)$ for $1 \le i \le \widetilde{K}$ and $0 \le j < \widetilde{J}_i$ that partition X and that satisfy all the items above with the exception that we do not require for this partition to be finer than $\mathcal{P}$. Take a base of one of the towers $\widetilde{D}(i,0)$. If we are given a partition of $\widetilde{D}(i,0)$ into non-empty clopen sets $\widetilde{D}(i,0) = \bigsqcup_p F_p$, where $1 \le p \le M$, then we can divide the ith tower into M towers (see Figure 2). This will naturally define

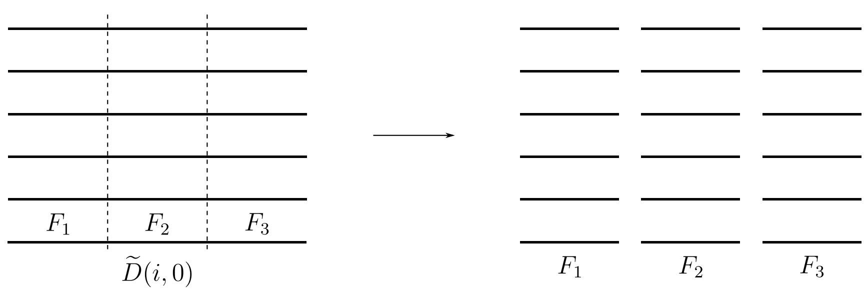

Figure 2. Refining a Kakutani–Rokhlin partition.

a refined Kakutani–Rokhlin partition with $K + M - 1$ many towers.

To obtain a partition that is finer than $\mathcal{P}$ we do as follows. For each level $\widetilde{D}(i,j)$ let $\mathcal{F}_{i,j}$ be the partition of $\widetilde{D}(i,j)$ induced by $\mathcal{P}$:

$$\mathcal{F}_{i,j} = \{\, \widetilde{D}(i,j) \cap P_k \mid P_k \in \mathcal{P} \text{ and } \widetilde{D}(i,j) \cap P_k \text{ is non-empty} \,\}.$$

Let $\mathcal{C}_{i,j}$ be the partition of $\widetilde{D}(i,0)$ obtained by transferring down the partition $\mathcal{F}_{i,j}$:

$$\mathcal{C}_{i,j} = \big\{\, \phi^{-j}\big(\widetilde{D}(i,j) \cap P_k\big) \;\big|\; \widetilde{D}(i,j) \cap P_k \in \mathcal{F}_{i,j} \,\big\}.$$

Let finally $\mathcal{C}$ be the partition of D generated by all the partitions $\mathcal{C}_{i,j}$. Note that by construction $\mathcal{C}$ is finer that the partition given by the sets $\widetilde{D}(i,0)$.

Suppose for example that the partition $\widetilde{D}(i,j)$ has three towers of height 4, 6 and 6 respectively (see Figure 3), and the partition $\mathcal{P}$ has four pieces P_k, $1 \le k \le 4$ which are shown in Figure 3. The little bars show how $\widetilde{D}(i,j)$ is partitioned into $\mathcal{F}_{i,j}$ and dashed lines show how the partitions $\mathcal{F}_{i,j}$ give rise to the partition $\mathcal{C}$ of the base.

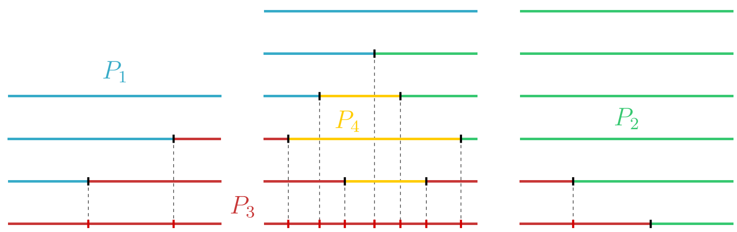

FIGURE 3. Refining the Kakutani–Rokhlin partition according to the partition $\mathcal{P}$ of four pieces.

We now refine the Kakutani–Rokhlin partition $\widetilde{D}(i,j)$ by splitting towers according to the partition $\mathcal{C}$ as explained in Figure 2, and obtain a new Kakutani–Rokhlin partition $D(i,j)$ for $1 \le i \le K$, $1 \le j \le J_i$, where $K = |\mathcal{C}|$, and $J_k = \widetilde{J}_i$ whenever $D(k,0) \subseteq \widetilde{D}(i,0)$.

We claim that this finer Kakutani–Rokhlin partition $D(i,j)$ refines $\mathcal{P}$. Indeed, take any level $D(i,j)$. By construction there are integers k and p such that $D(i,j) \subseteq \widetilde{D}(p,j) \cap P_k$ and therefore $D(i,j) \subseteq P_k$. □

We now give a formal definition.

Definition 1.17. By a *Kakutani–Rokhlin partition* we shall mean a family of sets $D(i,j)$ satisfying all the items of Proposition 1.16 (for the trivial partition $\mathcal{P} = \{X\}$ if no other partition is specified). We use the Greek capital letter chi Ξ to denote Kakutani–Rokhlin partitions. A *tower* of Ξ is the family $\{\, D(i,j) \mid 0 \le j < J_i \,\}$ for some fixed i. The ith tower will be denoted by T_i and $\mathcal{T}(\Xi)$ will denote the set of all towers. There are K towers in Ξ. The *height* of the tower T_i is the integer $J_i = |T_i|$. The set $D(i,0)$ is said to be the *base* of the tower T_i and $\phi^{J_i-1}(D(i,0)) = D(i, J_i - 1)$ is the *top* of T_i. The union D of all $D(i,0)$ is said to be the *base of* Ξ (see Figure 4).

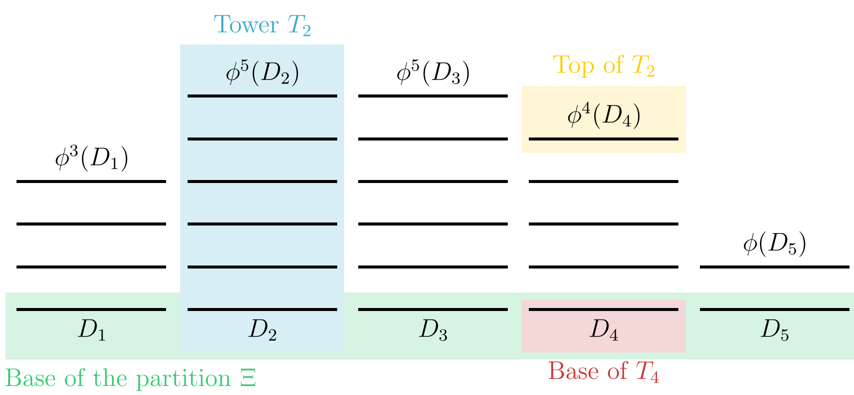

FIGURE 4. Elements of a Kakutani–Rokhlin partition.

LECTURE 2

Invariant measures

The set $\mathrm{M}(X)$ of countably additive Borel probability measures on X is separable, compact and metrizable in the weak-* topology, when viewed as a closed subset of the unit ball of the space $(C(X))^*$ — the dual to the space of continuous functions on X. The topology is given by the basis of neighbourhoods

$$U(\mu; f_1, \ldots, f_n, \epsilon) = \left\{ \nu \in \mathrm{M}(X) \,:\, \left| \int f_i \, d\mu - \int f_i \, d\nu \right| < \epsilon \text{ for } i \leq n \right\},$$

where $f_i \in C(X)$ are continuous real-valued functions on X. To generate the topology it is enough to take for f_i characteristic functions of clopen sets.

With a homeomorphism $\phi \in \mathrm{Homeo}(X)$ we associate the closed subspace of invariant measures $\mathrm{M}(\phi)$

$$\mathrm{M}(\phi) = \{ \mu \in \mathrm{M}(X) \mid \mu = \phi \circ \mu \},$$

where $(\phi \circ \mu)(A) = \mu\big(\phi^{-1}(A)\big)$. According to the Krylov–Bogoliubov Theorem this set is never empty.

Theorem 2.1 (Krylov–Bogoliubov)**.** *For any $\phi \in \mathrm{Homeo}(X)$ the set $\mathrm{M}(\phi)$ is non-empty.*

Proof. Pick an $x \in X$ and let δ_x be the Dirac measure concentrated at x. Set

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \phi^i \circ \delta_x.$$

Note that $\phi \circ \delta_x = \delta_{\phi(x)}$. Since $\mu_n \in \mathrm{M}(X)$ and since $\mathrm{M}(X)$ is compact, there is a subsequence (n_k) and a measure $\nu \in \mathrm{M}(X)$ such that $\mu_{n_k} \to \nu$. We claim that $\nu \in \mathrm{M}(\phi)$. Indeed, for any $f \in C(X)$

$$\int f \, d\mu_{n_k} = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} f\big(\phi^i(x)\big),$$

$$\int f \, d(\phi \circ \mu_{n_k}) = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} f\big(\phi^{i+1}(x)\big) = \int f \, d\mu_{n_k} + \frac{1}{n_k}\big(f(\phi^{n_k}(x)) - f(x)\big),$$

and therefore

$$\left| \int f \, d\big(\phi \circ \mu_{n_k}\big) - \int f \, d\mu_{n_k} \right| \leq \frac{2}{n_k} \, ||f||_\infty.$$

This implies that $\phi \circ \mu_{n_k} \to \nu$, but also $\phi \circ \mu_{n_k} \to \phi \circ \nu$, whence $\phi \circ \nu = \nu$. □

Proposition 2.2. *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism. For any non-empty clopen $A \subseteq X$ the infimum $\inf\{ \mu(A) \mid \mu \in \mathrm{M}(\phi) \} > 0$ is strictly positive.*

Proof. Let $c = \inf\{ \mu(A) \mid \mu \in \mathrm{M}(\phi) \}$. If $c = 0$, then we can find a sequence $\mu_n \in \mathrm{M}(\phi)$ such that $\mu_n(A) \leq 1/n$. By compactness of $\mathrm{M}(\phi)$ there is a measure $\mu \in \mathrm{M}(\phi)$ such that $\mu(A) = 0$, and thus $\mu(X) = \mu\big(\bigcup_{i \in \mathbb{Z}} \phi^i(A)\big) = 0$, which is impossible. □

Theorem 2.3 (Glasner–Weiss [**GW95**], Lemma 2.5)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism and $A, B \subseteq X$ be clopen subsets such that $\mu(B) < \mu(A)$ for all $\mu \in \mathrm{M}(\phi)$. There exists an element $g \in [\![\phi]\!]$ such that $g(B) \subset A$. Moreover one can find such a $g \in [\![\phi]\!]$ that also satisfies $g^2 = \mathrm{id}$ and $g|_{\sim(B \cup g(B))} = \mathrm{id}$.*

PROOF. Without loss of generality we may assume that $A \cap B = \varnothing$. Put $f = 1_A - 1_B$, and note that $\int f \, d\mu > 0$ for any $\mu \in \mathrm{M}(\phi)$. We claim that there is $c > 0$ such that

$$\inf_{\mu \in \mathrm{M}(\phi)} \int f \, d\mu > c > 0.$$

To see this we let

$$\epsilon_\mu = 1/2 \cdot \int f \, d\mu.$$

The family of neighbourhoods $\{ U(\mu; f, \epsilon_\mu) \mid \mu \in \mathrm{M}(\phi) \}$ covers $\mathrm{M}(\phi)$. By compactness there is a finite family $\mu_1, \ldots, \mu_n$ such that $\mathrm{M}(\phi) = \bigcup_i U(\mu_i; f, \epsilon_{\mu_i})$. One can now set $c = 1/2 \cdot \min\{ \epsilon_{m_i} \mid i \leq n \}$.

The next step is to show that there must be an $N_0 > 0$ such that for all $x \in X$ and all $N \geq N_0$

$$c \leq \frac{1}{N} \sum_{i=0}^{N-1} f\big(\phi^i(x)\big). \tag{1}$$

If this isn't so, then there is an increasing sequence n_k of natural numbers and a sequence of points $x_k \in X$ such that

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} f\big(\phi^i(x_k)\big) \in [-1, c].$$

As in the proof of the Krylov–Bogoliubov Theorem we set $\mu_k = \frac{1}{n_k} \sum_{i=0}^{n_k-1} \phi \circ \delta_{x_k}$, and after passing to a subsequence we may assume that $\mu_k \to \nu \in \mathrm{M}(\phi)$, hence

$$\int f \, d\nu \leq c,$$

contradicting the choice of c.

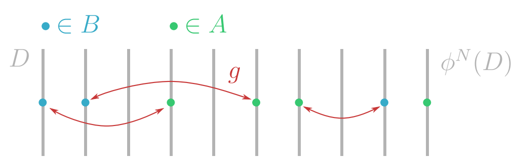

FIGURE 5. Construction of g.

We fix an $N_0 > 0$ such that (1) holds, and find a non-empty clopen $D \subseteq B$ such that $\phi^i(D) \cap D = \varnothing$ for all $i \leq N_0$. The inequality

$$c \leq \frac{1}{N} \sum_{i=0}^{N-1} f\big(\phi^i(x)\big)$$

implies that each column in the Kakutani–Rokhlin stack over D has more elements in A, than in B and we define g in a natural way (see Figure 5). □

Theorem 2.4 (Glasner–Weiss [**GW95**], Proposition 2.6)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism, and $A, B \subseteq X$ be clopen sets such that $\mu(A) = \mu(B)$ for all $\mu \in \mathrm{M}(\phi)$. There exists $g \in [\phi]$ such that $g(A) = B$, $g^2 = \mathrm{id}$, and $g|_{\sim(A \cup B)} = \mathrm{id}$. Moreover, g can be chosen such that the corresponding cocycle n_g has at most two points of discontinuity.*

PROOF. Without loss of generality we may assume that $A \cap B = \varnothing$. Pick an $x_0 \in A$ and n_0 such that $y_0 = \phi^{n_0}(x_0) \in B$. We fix a complete metric d on X. Find A_1 — a clopen neighbourhood of x_0 of diameter < 1 and such that $A_1' = A \setminus A_1$ satisfies

$$\mu(A)/2 < \mu(A_1') < \mu(A) \qquad \forall \mu \in \mathrm{M}(\phi).$$

Next we choose a clopen $V_1 \subseteq B$ a neighbourhood of y_0 such that

$$\mu(A_1') < \mu(B \setminus V_1) < \mu(B) \qquad \forall \mu \in \mathrm{M}(\phi).$$

By Theorem 2.3 we can find an element $g_1 \in [\![\phi]\!]$ with $g_1(A_1') = B_1' \subset B \setminus V_1$, $g_1(B_1') = A_1'$ and $g_1|_{\sim(A_1' \cup B_1')} = \mathrm{id}$. We set $B_1 = B \setminus B_1'$; note that $\mu(B_1) = \mu(A_1)$ for all $\mu \in \mathrm{M}(\phi)$.

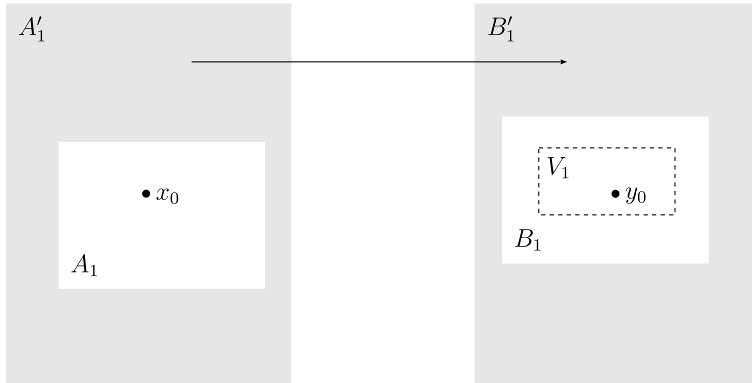

FIGURE 6. Construction of g_1

We can now repeat the process in the opposite direction: pick B_2 a clopen neighbourhood of y_0 such that $B_2' = B_1 \setminus B_2$ satisfies
$$\mu(B_1)/2 < \mu(B_2') < \mu(B_1) \qquad \forall \mu \in \mathrm{M}(\phi),$$
choose $V_2 \subset A_1$ a clopen neighbourhood of x_0 such that
$$\mu(B_2') < \mu(A_1 \setminus V_2) < \mu(A_1) \qquad \forall \mu \in \mathrm{M}(\phi),$$
and by Theorem 2.3 choose a $g_2 \in [\![\phi]\!]$ such that $g_2(B_2') = A_2'$, $g_2(A_2') = B_2'$ and g_2 is trivial on the complement of $A_2' \cup B_2'$. Set $A_2 = A \setminus A_2'$; note that $\mu(B_2) = A_2$ for all $\mu \in \mathrm{M}(\phi)$. Continuing in this fashion we obtain a decomposition of the space
$$X = \big(X \setminus (A \cup B)\big) \sqcup \Big(\bigcup A_n'\Big) \sqcup \Big(\bigcup B_n'\Big) \sqcup \{x_0, y_0\},$$
and define $g \in [\phi]$ by
$$g(x) = \begin{cases} x & \text{if } x \in X \setminus (A \cup B), \\ g_n(x) & \text{if } x \in A_n' \cup B_n', \\ y_0 & \text{if } x = x_0, \\ x_0 & \text{if } x = y_0. \end{cases}$$
The cocycle n_g may have discontinuities at points x_0 and y_0 only. □

Exercise 2.5. Let $A_1, \ldots, A_n$ be disjoint clopen subsets of X such that $\mu(A_i) = \mu(A_j)$ for all $\mu \in \mathrm{M}(\phi)$ and let σ be a permutation of $\{1, \ldots, n\}$. Show that there exists $h \in [\phi]$ such that $h(A_i) = A_{\sigma(i)}$ for all $i \leq n$.

LECTURE 3

Spatial realization

Let for brevity Γ denote the topological full group $[\![\phi]\!]$ of a minimal homeomorphism.

Proposition 3.1. *For every non-empty clopen $A \subseteq X$, every $x \in A$, and every $n > 0$ there is an $h \in \Gamma$ such that* $\mathrm{supp}(h) \subseteq A$, $x \in \mathrm{supp}(h)$ *and* $h|_{\mathrm{supp}(h)}$ *has period n.*

PROOF. By the minimality of ϕ we can find $0 = k_0 < k_1 < \ldots < k_{n-1}$ such that $\phi^{k_i}(x) \in A$. Let U be a sufficiently small neighbourhood of x such that $\phi^{k_i}(U) \cap \phi^{k_j}(U) = \varnothing$ for $i \neq j$, and set

$$h|_{\phi^{k_i}(U)} = \phi^{k_{i+1}-k_i}|_{\phi^{k_i}(U)}, \text{ for } i < n \text{ and } \quad h|_{\phi^{k_{n-1}}(U)} = \phi^{-\sum_i k_i}|_{\phi^{k_{n-1}}(U)}. \qquad \square$$

For a clopen subset A define

$$\Gamma_A = \{\, g \in \Gamma \mid \mathrm{supp}(g) \subseteq A \,\}.$$

Note that Γ_A is a subgroup of Γ.

For a subset $F \subseteq \Gamma$, the *centralizer* of F is denoted by F' and is defined to be the set of elements in Γ that commute with all elements from F:

$$F' = \{\, g \in \Gamma \mid \forall f \in F \ gf = fg \,\}.$$

Note that $F \subseteq F''$ and $(F_1 \cup F_2)' = F_1' \cap F_2'$.

Lemma 3.2. *Let $A_1, \ldots, A_n$ be clopen subsets of X.*

(i) *If* $\Gamma_{A_1} = \Gamma_{A_2}$, *then* $A_1 = A_2$.
(ii) $\left(\Gamma_{A_1} \cup \cdots \cup \Gamma_{A_2}\right)' = \Gamma_{\sim \bigcup A_i}$.
(iii) $\Gamma_{A_1} \cap \Gamma_{A_2} = \Gamma_{A_1 \cap A_2}$.

PROOF. (i) We show the contrapositive. Suppose that $A_1 \setminus A_2 \neq \varnothing$. By Proposition 3.1 one can find an involution $g \in \Gamma$ such that $\mathrm{supp}(g) \subseteq A_1 \setminus A_2$, and therefore $g \in \Gamma_{A_1} \setminus \Gamma_{A_2}$.

(ii) Suppose $g \in \left(\Gamma_{A_1} \cup \ldots \cup \Gamma_{A_n}\right)'$ and assume towards a contradiction that $g \notin \Gamma_{\sim \bigcup_i A_i}$, i.e., there are $i \leq n$ and $B \subseteq A_i$ such that $g(B) \cap B = \varnothing$. We can find an $h \in \Gamma_{A_i}$ such that $\mathrm{supp}(h) \subseteq B$ and $C \subseteq B$ is such that $h(C) \cap C = \varnothing$. Therefore $gh(C) \neq hg(C) = g(C)$. Hence $g \notin \Gamma'_{A_i}$, which is a contradiction. The other inclusion is obvious.

(iii) The equality follows immediately from the definitions. $\square$

Let $\pi \in \Gamma$ be an involutions: $\gamma^2 = \mathrm{id}$. Note that the support $\mathrm{supp}(\pi)$ is a clopen subset of X. We construct the following subsets of Γ:

$$\begin{aligned}
C_\pi &= \{\, g \in \Gamma &&\mid gh\pi = \pi g &&\}, \\
U_\pi &= \{\, g \in C_\pi &&\mid g^2 = \mathrm{id}, \text{ and } g(hgh^{-1}) = (hgh^{-1})g \text{ for all } h \in C_\pi &&\}, \\
V_\pi &= \{\, g \in \Gamma &&\mid gh = hg \text{ for all } h \in U_\pi &&\}, \\
S_\pi &= \{\, g^2 &&\mid g \in V_\pi &&\}, \\
W_\pi &= \{\, g \in \Gamma &&\mid gh = hg \text{ for all } h \in S_\pi &&\}.
\end{aligned}$$

Lemma 3.3 (Bezuglyi–Medynets [**BM08**], Lemma 5.10)**.** $W_\pi = \Gamma_{\mathrm{supp}(\pi)}$.

PROOF. We prove a series of claims each clarifying some properties of the sets constructed above. The proof of the lemma will then follow from these claims.

(1) $\qquad g(\mathrm{supp}(\pi)) = \mathrm{supp}(\pi)$ for all $g \in C_\pi$.
It is easy to verify that $\mathrm{supp}(g\pi g^{-1}) = g(\mathrm{supp}(\pi))$. Since $g\pi g^{-1} = \pi$, we get $g(\mathrm{supp}(\pi)) \subseteq \mathrm{supp}(\pi)$.

(2-i) $\operatorname{supp}(g) \subseteq \operatorname{supp}(\pi)$ for all $g \in U_\pi$. Suppose this is false and there are a clopen $A \subseteq \sim\operatorname{supp}(\pi)$ such that $g(A) \cap A = \varnothing$. By Proposition 3.1 we can find an $h \in \Gamma$ with support in A such that for some $V \subseteq A$ one has $h^i(V) \cap V = \varnothing$ for $i = 1, 2$. Note that $h \in C_\pi$, but

$$\begin{aligned} g(hgh^{-1})(V) &= g^2h^{-1}(V) &&= h^{-1}(V), \\ (hgh^{-1})g(V) &= hg^2(V) &&= h(V). \end{aligned}$$

Since $h^{-1}(V) \neq h(V)$, we get $g \notin U_\pi$.

(2-ii) If a clopen set A is π-invariant, then $\pi_A \in U_\pi$.
Obviously $\pi_A^2 = 1$. Since for $x \in A$ we have $\pi \circ \pi_A(x) = \pi \circ \pi(x) = x = \pi_A \circ \pi(x)$, and for $x \in \sim A$ we have $\pi \circ \pi_A(x) = \pi(x) = \pi_A \circ \pi(x)$, it follows that $\pi_A \in C_\pi$. Finally one checks that

$$\pi_A(h\pi_A h^{-1})(x) = (h\pi_A h^{-1})\pi_A(x) = \begin{cases} x & \text{if } x \in (\sim A \cap h(\sim A)) \cup (A \cap h(A)), \\ \pi(x) & \text{if } x \in (\sim A \cap h(A)) \cup (A \cap h(\sim A)). \end{cases}$$

(3-i) $V_\pi \subseteq C_\pi$.
For this we show that $\pi \in U_\pi$. Indeed $\pi \in C_\pi$, $\pi^2 = \mathrm{id}$, and $\pi(h\pi h^{-1}) = \mathrm{id} = (h\pi h^{-1})\pi$ for all $h \in C_\pi$.

(3-ii) If $g \in V_\pi$, then $g(B) \subseteq B \cup \pi(B)$ for all $B \subseteq \operatorname{supp}(\pi)$. Suppose this is false and let B be such that $g(B) \nsubseteq B \cup \pi(B)$. Set $B_0 = B \cup \pi(B)$, and $C = g(B_0) \setminus B_0$. Note that $\pi(B_0) = B_0$ and $C \neq \varnothing$. By (3-i) we know that $\pi g(B_0) = g\pi_{B_0} = g(B_0)$ and therefore

$$\pi(C) = \pi(g_{B_0} \setminus B_0) = \pi g(B_0) \setminus \pi(B_0) = g_{B_0} \setminus B_0 = C.$$

Using (1) and (3-i) we see that $g(\operatorname{supp}(\pi)) = \operatorname{supp}(\pi)$. Since $B \subseteq \operatorname{supp}(\pi)$, this implies $B_0 \subseteq \operatorname{supp}(\pi)$. We therefore can write $C = C_1 \sqcup C_2$ such that $\pi(C_1) = C_2$. Note that by construction $g(C) \cap C = \varnothing$. By (2-ii) $\pi_C \in U_\pi$, but also

$$\pi_C g(C_1) = g(C_1) \neq g(C_2) = g\pi_C(C_1).$$

Whence $g \notin V_\pi$.

(3-iii) If $g \in V_\pi$, then $g^2(B) = B$ for any clopen $B \subseteq \operatorname{supp}(\pi)$.
Suppose there is $B \subseteq \operatorname{supp}(\pi)$ such that $g^2(B) \neq B$. By shrinking B we may assume that

$$g(B) \cap B = \varnothing = g^2(B) \cap B.$$

By (3-ii) $g(B) \subseteq B \cup \pi(B)$ and

$$g^2(B) \subseteq g(B) \cup g\pi(B) = g(B) \cup \pi g(B).$$

But since $g(B) \cap B = \varnothing$, we conclude $g(B) \subseteq \pi(B)$ and $g^2(B) \subseteq \pi g(B) \subseteq \pi^2(B) = B$. Note that $\mu(B \setminus g^2(B)) = 0$ for all $\mu \in \mathrm{M}(\phi)$. Therefore the minimality of ϕ implies $B \setminus g^2(B) = \varnothing$.

(4-i) If $g \in S_\pi$, then $\operatorname{supp}(g) \subseteq \sim\operatorname{supp}(\pi)$.
Follows immediately from (3-iii).

(4-ii) For any clopen $C \subseteq \sim\operatorname{supp}(\pi)$ there is an involution $h \in S_\pi$ supported on C.
By Proposition 3.1 there exists a periodic homeomorphism g of order 4 with support in C. By (2-i) $g \in V_\pi$ and therefore $g^2 \in S_\pi$.

(5) $W_\pi = \Gamma_{\operatorname{supp}(\pi)}$.
It follows from (4-i) that $\Gamma_{\operatorname{supp}(\pi)} \subseteq W_\pi$. If $g \in W_\pi$ and for some $B \subseteq \sim\operatorname{supp}(\pi)$ we have $g(B) \cap B = \varnothing$, then take by (4-ii) any involution $h \in S_\pi$ supported on B, let C be such that $h(C) \cap C = \varnothing$. It now follows that $hg(C) = g(C) \neq gh(C)$. Hence $gh \neq hg$, contradicting the choice of g. □

Lemma 3.4. *If $\pi_1, \ldots, \pi_n \in \Gamma$ and $\rho_1, \ldots, \rho_m \in \Gamma$ are involutions, then $\bigcup_i \operatorname{supp}(\pi_i) = \bigcup_j \operatorname{supp}(\rho_j)$ if and only if $\left(W_{\pi_1} \cup \ldots \cup W_{\pi_n}\right)' = \left(W_{\rho_1} \cup \ldots \cup W_{\rho_m}\right)'$.*

Proof. Follows from Lemma 3.3 and Lemma 3.2. □

Theorem 3.5 (Stone)**.** *Homeomorphisms of the Cantor space X are in one-to-one correspondence with the automorphisms of the Boolean algebra $CO(X)$ of clopen subsets of X. In other words any automorphisms $\hat{\alpha}$ of $CO(X)$ has a unique realization $\psi \in \mathrm{Homeo}(X)$ such that $\psi(A) = \hat{\alpha}(A)$ for all clopen $A \subseteq X$.*

Exercise 3.6. Prove Stone's Theorem.

Theorem 3.7 (Giordano–Putnam–Skau [**GPS99**], Theorem 4.2)**.** *Let ϕ_1 and ϕ_2 be minimal homeomorphisms, and let $\Gamma^1 = [\![\phi_1]\!]$, $\Gamma^2 = [\![\phi_2]\!]$. If $\alpha : \Gamma^1 \to \Gamma^2$ is a group isomorphism, then α is necessarily spatial: there is a homeomorphism $\Lambda : X \to X$ such that $\alpha(g) = \Lambda g \Lambda^{-1}$ for all $g \in \Gamma^1$.*

Proof. By Stone's Theorem it is enough to define Λ on the clopen subsets of X. By Proposition 3.1 for any clopen $A \subseteq X$ we can find a finite family of involutions $\pi_1, \dots, \pi_n \in \Gamma^1$ such that $\bigcup_i \operatorname{supp}(\pi_i) = \sim A$. By Lemma 3.3 there exists a clopen subset $\Lambda(A)$ such that

$$(W_{\alpha(\pi_1)} \cup \ldots \cup W_{\alpha(\pi_n)})' = \Gamma^2_{\Lambda(A)}.$$

By Lemma 3.4 the map $A \mapsto \Lambda(A)$ is well-defined.

We claim that Λ is an automorphism of the boolean algebra of clopen subsets of X. First of all we show that $\Lambda(A_1 \cap A_2) = \Lambda(A_1) \cap \Lambda(A_2)$. If $\pi_1, \dots, \pi_n \in \Gamma^1$ and $\rho_1, \dots, \rho_m \in \Gamma^1$ are involutions such that $\sim A_1 = \bigcup_i \operatorname{supp}(\pi_i)$ and $\sim A_2 = \bigcup_j \operatorname{supp}(\rho_j)$, then

$$\sim(A_1 \cap A_2) = (\sim A_1) \cup (\sim A_2) = \Big(\bigcup_i \operatorname{supp}(\pi_i)\Big) \cup \Big(\bigcup_j \operatorname{supp}(\rho_j)\Big)$$

and hence

$$\begin{aligned}\Gamma^2_{\Lambda(A_1 \cap A_2)} &= (W_{\alpha(\pi_1)} \cup \cdots \cup W_{\alpha(\pi_n)} \cup W_{\alpha(\rho_1)} \cdots W_{\alpha(\rho_m)})' \\ &= (W_{\alpha(\pi_1)} \cup \cdots \cup W_{\alpha(\pi_n)})' \cap (W_{\alpha(\rho_1)} \cdots W_{\alpha(\rho_m)})' \\ &= \Gamma^2_{\Lambda(A_1)} \cap \Gamma^2_{\Lambda(A_2)} = \Gamma^2_{\Lambda(A_1) \cap \Lambda(A_2)}.\end{aligned}$$

It now follows that $\Lambda(A_1 \cap A_2) = \Lambda(A_1) \cap \Lambda(A_2)$.

The next step is to show that $\Lambda(\sim A) = \sim \Lambda(A)$. Let $\pi_1, \dots, \pi_n \in \Gamma^1$ and $\rho_1, \dots, \rho_m \in \Gamma^1$ be involutions such that $\sim A = \bigcup_i \operatorname{supp}(\pi_i)$ and $A = \bigcup_j \operatorname{supp}(\rho_j)$. Since $(\Gamma^1_A)' = \Gamma^1_{\sim A}$, we get

$$(W_{\pi_1} \cup \cdots \cup W_{\pi_n})'' = (W_{\rho_1} \cup \cdots \cup W_{\rho_m})'$$

and therefore also

$$(W_{\alpha(\pi_1)} \cup \cdots \cup W_{\alpha(\pi_n)})'' = (W_{\alpha(\rho_1)} \cup \cdots \cup W_{\alpha(\rho_m)})',$$

which implies

$$\begin{aligned}\Gamma^2_{\Lambda(\sim A)} &= \big(W_{\alpha(\rho_1)} \cup \cdots \cup W_{\alpha(\rho_m)}\big)' \\ &= \big(W_{\alpha(\pi_1)} \cup \cdots \cup W_{\alpha(\pi_n)}\big)'' \\ &= \big(\Gamma^2_{\Lambda(A)}\big)' = \Gamma^2_{\sim \Lambda(A)},\end{aligned}$$

and therefore $\Lambda(\sim A) = \sim \Lambda(A)$.

Since $\varnothing = \operatorname{supp}(\mathrm{id})$, we see that $\Lambda(X) = X$ and $\Lambda(\varnothing) = \varnothing$. And we have proved that Λ is an endomorphism of $CO(X)$. It is easy to see that Λ is bijective, since its inverse is defined by: if B is clopen and $\pi_1, \dots, \pi_n \in \Gamma^2$ are such that $\sim B = \bigcup_i \operatorname{supp}(\pi_i)$, then $\Lambda^{-1}(B)$ is defined to be such that

$$\Gamma^1_{\Lambda^{-1}(B)} = (W_{\alpha^{-1}(\pi_1)} \cup \cdots \cup W_{\alpha^{-1}(\pi_1)})'.$$

So Λ is an automorphism of $CO(X)$.

Claim. If $\pi \in \Gamma^1$ is an involution, then $\Lambda(\operatorname{supp}(\pi)) = \operatorname{supp}(\alpha(\pi))$. Indeed

$$\sim \Lambda\big(\operatorname{supp}(\pi)\big) = \Lambda\big(\sim \operatorname{supp}(\pi)\big) = \sim \operatorname{supp}(\alpha(\pi)),$$

whence $\Lambda(\operatorname{supp}(\pi)) = \operatorname{supp}(\alpha(\pi))$.

We finally show that for any clopen set B we have $\alpha(g)(B) = \Lambda g \Lambda^{-1}(B)$. Suppose this is not the case. Let V be a non-empty clopen set such that $V \cap \alpha(g^{-1})\Lambda g \Lambda^{-1}(V) = \varnothing$. Pick an involution $\pi \in \Gamma^2$ such that $\operatorname{supp}(\pi) \subseteq V$. Note that by the claim $\alpha^{-1}(\pi)$ is supported by $\Lambda^{-1}(V)$, and therefore $g\alpha^{-1}(\pi)g^{-1}$ is supported by $g\Lambda^{-1}(\pi)$. This implies $\alpha\big(g\alpha^{-1}(\pi)g\big) = \alpha(g)\pi\alpha(g^{-1})$ is supported by $\Lambda g \Lambda^{-1}(V)$. But on the other hand $\alpha(g)\pi\alpha(g^{-1})$ is supported by $\alpha(g)(V)$. This shows that $\alpha(g)V \cap \Lambda g \Lambda^{-1}(V) \neq \varnothing$, contradicting the choice of V. ◻

LECTURE 4

Boyle's Theorem and Flip conjugacy

Definition 4.1. We say that two homeomorphisms $\phi, \psi \in \operatorname{Homeo}(X)$ are *flip conjugated* if there is an $\alpha \in \operatorname{Homeo}(X)$ such that either $\phi = \alpha\psi\alpha^{-1}$ or $\phi^{-1} = \alpha\psi\alpha^{-1}$. This is an equivalence relation.

Theorem 4.2 (Boyle–Tomiyama [**BT98**]). *Let ϕ and ψ be minimal homeomorphisms. If $\alpha \in \operatorname{Homeo}(X)$ is such that*

$$[\![\phi]\!] \ni g \mapsto \alpha g \alpha^{-1} \in [\![\psi]\!]$$

is an isomorphism, then ϕ and ψ are flip conjugated.

Proof. By switching from ϕ to $\alpha\phi\alpha^{-1}$ we may assume that $\alpha = \mathrm{id}$ and that $[\![\phi]\!] = [\![\psi]\!]$. Let $n : X \to \mathbb{Z}$ be the cocycle $\psi(x) = \phi^{n(x)}(x)$, and define

$$f(k,x) = \begin{cases} -\big(n(\psi^{-1}(x)) + \cdots + n(\psi^{k}(x))\big) & \text{for } k < 0, \\ 0 & \text{for } k = 0, \\ n(x) + \cdots + n(\psi^{k-1}(x)) & \text{for } k > 0. \end{cases}$$

This function satisfies $\psi^k(x) = \phi^{f(k,x)}(x)$ for all $k \in \mathbb{Z}$ and the following *cocycle identity:*

$$f(k+l,x) = f\big(k,\psi^l(x)\big) + f(l,x).$$

Fix an N such that $|n(x)| \leq N$ for all $x \in X$. The cocycle identity implies

$$\big|f(k \pm 1,x) - f(k,x)\big| \leq N,$$

and also

$$\big|f\big(k,\psi(x)\big) - f(k,x)\big| \leq \big|f(k+1,x) - f(k,x)\big| + |f(-1,\psi(x))| \leq 2N.$$

From $\psi^k(x) = \phi^{f(k,x)}(x)$ we see that the map $k \mapsto f(k,x_0)$ is a bijection for any fixed $x_0 \in X$, and therefore for any $x_0 \in X$ there is an $\bar{N} > 0$ such that

$$[-N,N] \subseteq \big\{\, f(k,x_0) \;\big|\; k \in [-\bar{N},\bar{N}] \,\big\}.$$

By continuity of the cocycle n, the function f is locally constant, hence for any x_0 there is a neighbourhood U_{x_0} of x_0 such that

$$[-N,N] \subseteq \big\{\, f(k,y) \;\big|\; k \in [-\bar{N},\bar{N}] \,\big\}$$

holds for all $y \in U_{x_0}$. By compactness we can take $\bar{N}$ to be large enough to work for all $x \in X$.

Note that $f(\bar{N},x) \neq 0$ for all $x \in X$. Moreover $f(\bar{N},x) > 0$ if and only if $f(n,x) > 0$ and $f(-n,x) < 0$ for all $n \geq \bar{N}$. Similarly, $f(\bar{N},x) < 0$ if and only if $f(n,x) < 0$ and $f(-n,x) > 0$ for all $n \geq \bar{N}$. We define sets

$$A = \big\{\, x \in X \;\big|\; f(\bar{N},x) > 0 \,\big\},$$
$$B = \big\{\, x \in X \;\big|\; f(\bar{N},x) < 0 \,\big\}.$$

These sets are clopen, ψ-invariant, and $X = A \sqcup B$. Therefore either $A = \varnothing$, or $B = \varnothing$. By taking ψ^{-1} for ψ we may assume without loss of generality that $A = X$. Define a function $c : X \to \mathbb{N}$ as follows.

$$\begin{aligned} c(x) &= \#\,[-N\bar{N},\infty) \cap \{\, f(i,x) \mid i \leq 0 \,\} \\ &= \#\,[-N\bar{N},\infty) \cap \{\, f\big(i-1,\psi(x)\big) + n(x) \mid i \leq 0 \,\} \\ &= \#\,[-N\bar{N},\infty) \cap \{\, f\big(i,\psi(x)\big) + n(x) \mid i \leq 0 \,\} - 1 \\ &= \#\,[-N\bar{N} - n(x),\infty) \cap \{\, f\big(i,\psi(x)\big) \mid i \leq 0 \,\} - 1 \\ &= \#\,[-N\bar{N},\infty) \cap \{\, f\big(i,\psi(x)\big) \mid i \leq 0 \,\} + n(x) - 1 \\ &= c(\psi(x)) + n(x) - 1. \end{aligned}$$

Therefore $1 + c(x) = c(\psi(x)) + n(x)$.

Finally we define $g(x) = \phi^{c(x)}x$. Note that

$$\phi g(x) = \phi^{1+c(x)}x = \phi^{n(x)+c(\psi(x))}(x) = \phi^{c(\psi(x))}\psi(x) = g\psi(x).$$

This implies $\phi^k g = g\psi^k$ for all k, and hence g is surjective. Also if $g(x) = g\psi^k(x)$, then $\phi^k g(x) = g(x)$, hence $Orb_\phi(g(x))$ is finite, which is impossible. This shows that g is bijective. Since c is continuous, g is in fact a homeomorphism of X such that $\phi = g\psi g^{-1}$. □

Combining Theorem 3.7 and Theorem 4.2 we get

Theorem 4.3 (Giordano–Putnam–Skau [**GPS99**], Corollary 4.4)**.** *Two minimal homeomorphisms have isomorphic full groups if and only if they are flip conjugated.*

Simplicity of commutator subgroups

Recall that for a group Γ its *commutator subgroup* is the subgroup $\mathcal{D}(\Gamma)$ generated by all the elements of the form $[g,h] = ghg^{-1}h^{-1}$. In this section we shall prove that the commutator subgroup of the topological full group of a minimal homeomorphism is simple. In our exposition we follow Section 3 of [**BM08**].

Lemma 5.1 (Bezuglyi–Medynets [**BM08**], Lemma 3.2)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism. For any $g \in [\![\phi]\!]$ and $\delta > 0$ there exist $g_1, \ldots, g_m \in [\![\phi]\!]$ such that $g = g_1 \cdots g_m$ and $\mu\big(\mathrm{supp}(g_i)\big) < \delta$ for all $\mu \in \mathrm{M}(\phi)$.*

Proof. Let $g \in [\![\phi]\!]$ be given and suppose first that g is periodic. Since g is an element of the topological full group, by Propositions 1.13 and 1.14 we can find non-empty clopen sets $\{A_k\}_{k\in I}$, where $I \subset \mathbb{Z}$ is finite such that the space X decomposes into disjoint clopen sets

$$X = \bigsqcup_{k\in I} \bigsqcup_{i=0}^{k-1} g^i(A_k),$$

and $g^k(x) = x$ for all $x \in A_k$.

We now can decompose each A_k into non-empty clopen subsets

$$A_k = \bigsqcup_{j=1}^{n_k} B_j^{(k)}$$

such that for each k and each $1 \le j \le n_k$ we have $\mu\big(B_j^{(k)}\big) < \delta/k$ for all $\mu \in \mathrm{M}(\phi)$. We set

$$C_{k,j} = \bigsqcup_{i=0}^{k-1} g^i\big(B_j^{(k)}\big)$$

and $g_{k,j} = g|_{C_{k,j}}$. It is easy to see that all the elements $g_{k,j} \in [\![\phi]\!]$, and $g = \prod_{k,j} g_{k,j}$.

We have proved the lemma for periodic homeomorphisms. We consider the case of a non-periodic $g \in [\![\phi]\!]$. Fix $k \in \mathbb{N}$ such that $1/k < \delta$ and put

$$X_{\ge k} = \{\, x \in X \mid \mathrm{Orb}_g(x) \text{ has at least } k \text{ elements} \,\}.$$

Since $g \in [\![\phi]\!]$, by Proposition 1.13 the set $X_{\ge k}$ is clopen.

For any $x \in X_{\ge k}$ we can find a clopen neighbourhood U_x such that $g^i(U_x) \cap U_x = \varnothing$ for all $1 \le i < k$. By compactness of $X_{\ge k}$ we can find finitely many $x_1, \ldots, x_n \in X_{\ge k}$ such that $X_{\ge k}$ is covered by $U_{x_1}, \ldots, U_{x_n}$. We now set $B_1 = U_{x_1}$ and

$$B_{l+1} = B_l \sqcup \left(U_{x_{l+1}} \setminus \bigcup_{i=-k+1}^{k-1} g^i(B_l) \right).$$

Set $B = B_n$. Note that B is a maximal k-discrete set; in particular, the set B meets every orbit of g in $X_{\ge k}$, and $g^i(B) \cap B = \varnothing$ for all $1 \le i < k$. This shows that $\mu(B) \le 1/k < \delta$ for all $\mu \in \mathrm{M}(\phi)$. Define

$$g_B(x) = \begin{cases} g^k(x) & \text{if } x \in B \text{ and } k = \min\{\, l \ge 1 \mid g^l(x) \in B \,\}, \\ x & \text{if } x \notin B. \end{cases}$$

It is easy to see that $g_B \in [\![\phi]\!]$, $\mu\big(\mathrm{supp}(g_B)\big) < \delta$ and $g_B^{-1} \circ g$ is periodic. The lemma is proved by appealing to the earlier case of a periodic g. ∎

Lemma 5.2 (Bezuglyi–Medynets [**BM08**], Lemma 3.3)**.** *Let H be a normal subgroup of a group G. If $g_1, \dots, g_n \in G$ and $h_1, \dots, h_m \in G$ are such that $[g_i, h_j]$ belong to H for any i, j, then the element $[g_1 \cdots g_n, h_1 \cdots h_m]$ also belongs to H. Moreover, the following identity holds:*

$$[g_1 \cdots g_n, h_1 \cdots h_m] = \prod_{p=n}^{1} \prod_{q=1}^{m} g_1 \cdots g_{p-1} h_1 \cdots h_{q-1} [g_p, h_q] h_{q-1}^{-1} \cdots h_1^{-1} g_{p-1}^{-1} \cdots g_1^{-1}.$$

Proof. It is straightforward to check that

$$[g_1 g_2, h_i] = g_1 [g_2, h_i] g_1^{-1} [g_1, h_i],$$
$$[g_j, h_1 h_2] = [g_j, h_1] h_1 [g_j, h_2] h_1^{-1}.$$

The general form now follows by induction from these identities. □

Lemma 5.3 (Bezuglyi–Medynets [**BM08**], Lemma 3.2)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism. For any $f \in \mathcal{D}([\![\phi]\!])$ and $\delta > 0$ there exist $g_1', \dots, g_N' \in [\![\phi]\!]$, $h_1', \dots, h_N' \in [\![\phi]\!]$ such that $f = [g_1', h_1'] \cdots [g_N', h_N']$ and $\mu\big(\mathrm{supp}(g_i') \cup \mathrm{supp}(h_i')\big) < \delta$ for all $\mu \in \mathrm{M}(\phi)$.*

Proof. Since $\mathcal{D}([\![\phi]\!])$ is generated by commutators $[g, h]$, it is enough to prove the lemma for elements of the form $[g, h]$. Fix a $\delta > 0$ and using Lemma 5.1 we can find $g_1, \dots, g_n \in [\![\phi]\!]$ and $h_1, \dots, h_m \in [\![\phi]\!]$ such that $g = g_1 \cdots g_n$, $h = h_1 \cdots h_m$ and $\mathrm{supp}(g_i) < \delta/2$, $\mathrm{supp}(h_j) < \delta/2$. By Lemma 5.2 we know that

$$[g_1 \cdots g_n, h_1 \cdots h_m] = \prod_{p=n}^{1} \prod_{q=1}^{m} g_1 \cdots g_{p-1} h_1 \cdots h_{q-1} [g_p, h_q] h_{q-1}^{-1} \cdots h_1^{-1} g_{p-1}^{-1} \cdots g_1^{-1}.$$

Note that $\mathrm{supp}([g_i, h_j]) \subseteq \mathrm{supp}(g_i) \cup \mathrm{supp}(h_j)$ and therefore $\mu\big(\mathrm{supp}([g_i, h_j])\big) < \delta$. Finally since any $f \in [\![\phi]\!]$ is μ-preserving for all $\mu \in \mathrm{M}(\phi)$, and since $\mathrm{supp}(f \alpha f^{-1}) = f(\mathrm{supp}(\alpha))$, we see that

$$\mathrm{supp}(g_1 \cdots g_{p-1} h_1 \cdots h_{q-1} [g_p, h_q] h_{q-1}^{-1} \cdots h_1^{-1} g_{p-1}^{-1} \cdots g_1^{-1}) < \delta,$$

and also $g_1 \cdots g_{p-1} h_1 \cdots h_{q-1} [g_p, h_q] h_{q-1}^{-1} \cdots h_1^{-1} g_{p-1}^{-1} \cdots g_1^{-1} \in \mathcal{D}([\![\phi]\!])$, because $\mathcal{D}([\![\phi]\!])$ is normal in $[\![\phi]\!]$. □

Lemma 5.4. *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism. If A and B are clopen subsets of X such that $2\mu(B) < \mu(A)$ for all $\mu \in \mathrm{M}(\phi)$, then there exists an $\alpha \in \mathcal{D}([\![\phi]\!])$ such that $\alpha(B) \subset A$.*

Proof. By setting α to be id on $A \cap B$ we may assume that $A \cap B = \varnothing$. Applying Theorem 2.3 we can find α_1 and α_2 in $[\![\phi]\!]$ such that $\alpha_1(B) \subseteq A$ and $\alpha_2(\alpha_1(B)) \subseteq A \setminus \alpha_1(B)$. Set $\alpha = \alpha_1 a_2$. Therefore $\alpha(B) = \alpha_1(B) \subseteq A$. Since $\alpha_2 = \alpha \alpha_1^{-1} \alpha^{-1}$, we get that $\alpha = \alpha_1 \alpha_2 = [\alpha_1, \alpha]$. □

Theorem 5.5 (Bezuglyi–Medynets [**BM08**], Theorem 3.4)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism. Let Γ be either $\mathcal{D}([\![\phi]\!])$ or $[\![\phi]\!]$. If H is a non-trivial normal subgroup of Γ, then $\mathcal{D}(\Gamma) \subseteq H$.*

Proof. We show that for all $g, h \in \Gamma$ their commutator $[g, h]$ is in H. Pick any non-trivial element $f \in H$ and a non-empty clopen set E such that $f(E) \cap E = \varnothing$. By compactness of the set $\mathrm{M}(\phi)$ we see that $2\delta = \inf\{\, \mu(E) \mid \mu \in \mathrm{M}(\phi) \,\} > 0$.

Using Lemma 5.1 and Lemma 5.3 we may find elements $g_i, h_j \in \Gamma$ such that $g = g_1 \cdots g_n$, $h = h_1 \cdots h_m$ and $\mu(\mathrm{supp}(g_i)) < \delta/2$, $\mu(\mathrm{supp}(h_j)) < \delta/2$ for all $\mu \in \mathrm{M}(\phi)$. In the view of Lemma 5.2 the proof would be over if we could show that for all $g, h \in \Gamma$ such that $\mu(\mathrm{supp}(g) \cup \mathrm{supp}(h)) < \delta$ for all $\mu \in \mathrm{M}(\phi)$ we have $[g, h] \in H$.

Put $F = \mathrm{supp}(g) \cup \mathrm{supp}(h)$ and find by Lemma 5.4 an element $\alpha \in \mathcal{D}([\![\phi]\!])$ such that $\alpha(F) \subseteq E$. By normality $q = \alpha^{-1} f \alpha \in H$. Therefore $\hat{h} = [h, q] = h q h^{-1} q^{-1} \in H$, and $[g, \hat{h}] \in H$. Since $q(F) \cap F = \varnothing$, the elements g^{-1} and $q h^{-1} q^{-1}$ commute. Whence

$$[g, \hat{h}] = g(hgh^{-1}g^{-1})g^{-1}(qhq^{-1}h^{-1}) = ghg^{-1}qh^{-1}q^{-1}qhq^{-1}h^{-1} = [g, h] \in H.$$

And so $\mathcal{D}(\Gamma) \leq H$. □

Corollary 5.6 (Matui [**Mat06**], Theorem 4.9)**.** *If $\phi \in \mathrm{Homeo}(X)$ is minimal, then $\mathcal{D}(\mathcal{D}([\![\phi]\!])) = \mathcal{D}([\![\phi]\!])$ and $\mathcal{D}([\![\phi]\!])$ is simple.*

PROOF. Since $\mathcal{D}(\mathcal{D}([\![\phi]\!]))$ is a normal subgroup of $[\![\phi]\!]$, we may apply Theorem 5.5 with $H = \mathcal{D}(\mathcal{D}([\![\phi]\!]))$ and $\Gamma = [\![\phi]\!]$. This shows that $\mathcal{D}([\![\phi]\!]) \leq \mathcal{D}(\mathcal{D}([\![\phi]\!]))$, and therefore $\mathcal{D}(\mathcal{D}([\![\phi]\!])) = \mathcal{D}([\![\phi]\!])$.

To show the simplicity of $\mathcal{D}([\![\phi]\!])$ let H be any non-trivial normal subgroup of $\mathcal{D}([\![\phi]\!])$. By another application of Theorem 5.5 we obtain $\mathcal{D}(\mathcal{D}([\![\phi]\!])) \leq H$, and therefore $\mathcal{D}([\![\phi]\!]) = H$. $\square$

LECTURE 6

Finite generation of commutator subgroups

Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism and let U be a clopen subset of X such that $\phi^{-1}(U)$, U, and $\phi(U)$ are pairwise disjoint. We define γ_U to be the homeomorphism

$$\gamma_U(x) = \begin{cases} \phi(x) & \text{if } x \in \phi^{-1}(U) \cup U, \\ \phi^{-2}(x) & \text{if } x \in \phi(U), \\ x & \text{otherwise.} \end{cases}$$

Lemma 6.1. *Elements γ_U are in the commutator subgroup $\mathcal{D}(\llbracket\phi\rrbracket)$.*

Proof. Define an involution $g \in \llbracket\phi\rrbracket$ by

$$g(x) = \begin{cases} \phi(x) & \text{if } x \in \phi^{-1}(U), \\ \phi^{-1}(x) & \text{if } x \in U. \end{cases}$$

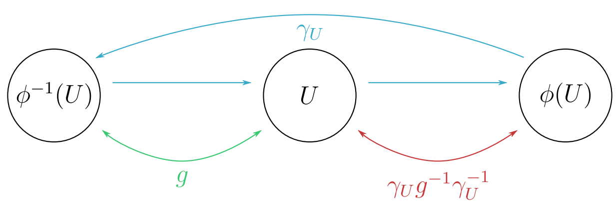

Figure 7. Homeomorphisms γ_U, g, and $\gamma_U g^{-1} \gamma_U^{-1}$ showing $\gamma_U = [g, \gamma_U]$.

The equality $\gamma_U = [g, \gamma_U]$ corresponds to the following identity within the symmetric group on three elements:

$$(01)(012)(01)(021) = (012)$$

□

Let $H = \langle \gamma_U \rangle$ be the subgroup of $\llbracket\phi\rrbracket$, where U ranges over clopen subsets such that $\phi^{-1}(U)$, U, and $\phi(U)$ are pairwise disjoint. We shall show that H is a normal subgroup of $\mathcal{D}(\llbracket\phi\rrbracket)$, and conclude using Corollary 5.6 that $H = \mathcal{D}(\llbracket\phi\rrbracket)$.

Lemma 6.2. *If $g \in \llbracket\phi\rrbracket$ has order 3, then $g \in H$.*

Proof. Let $g \in \llbracket\phi\rrbracket$ be an element of order 3. By Propositions 1.13 and 1.14 we can find a clopen subset $A \subseteq X$ such that A, $g(A)$, and $g^2(A)$ are pairwise disjoint, and $\mathrm{supp}(g) = A \sqcup g(A) \sqcup g^2(A)$. Since $g \in \llbracket\phi\rrbracket$, we can find a partition $B_1, \dots, B_m$ of X and integers r_i such that $g|_{B_i} = \phi^{r_i}|_{B_i}$. Let $\mathcal{P}_0$, $\mathcal{P}_1$, and $\mathcal{P}_2$ be partitions of A defined by

$$\mathcal{P}_0 = \{B_i \cap A\}_{i \le m},$$
$$\mathcal{P}_1 = g^{-1}\{B_i \cap g(A)\}_{i \le m},$$
$$\mathcal{P}_2 = g^{-2}\{B_i \cap g^2(A)\}_{i \le m}.$$

The common refinement of partitions $\mathcal{P}_j$ is a partition $A_1, \dots, A_n$ of A such that for any $i \le n$ there are integers k_i and l_i such that $g|_{A_i} = \phi^{k_i}|_{A_i}$, $g|_{g(A_i)} = \phi^{l_i}|_{g(A_i)}$, $g|_{g^2(A_i)} = \phi^{-k_i - l_i}|_{g^2(A_i)}$. Let g_i be the restriction of g onto $A_i \cup g(A_i) \cup g^2(A_i)$. Elements g_i commute and $g = g_1 \cdots g_n$.

It is therefore enough to prove the lemma for elements $g \in \llbracket\phi\rrbracket$, $g^3 = \mathrm{id}$, for which there is a clopen set A and two integers k, l such that A, $g(A)$, and $g^2(A)$ partition the support of g, and $g|_A = \phi^k|_A$, $g|_{g(A)} = \phi^l|_{g(A)}$. Fix such a g. For any $x \in A$ there is a clopen neighbourhood $x \in U \subseteq A$ such that $\phi^i(U) \cap \phi^j(U) = \varnothing$ for all $0 \le i, j \le k + l$, $i \ne j$. By compactness, we may find a finite family of these neighbourhoods U_j, $j \le N$, that covers all of A. Let $C_1, \dots, C_p$ be the partition of A generated by U_j. Let g_i be the restriction of g onto the set $C_i \cup g(C_i) \cup g^2(C_i)$. Elements g_i commute and $g = g_1 \cdots g_p$.

It is therefore enough to prove the lemma for elements $g \in [\![\phi]\!]$, $g^3 = \mathrm{id}$, for which there is a clopen set A and two integers k, l such that A, $g(A)$, and $g^2(A)$ partition the support of g, $g|_A = \phi^k|_A$, $g|_{g(A)} = \phi^l|_{g(A)}$, and $\phi^i(A) \cap \phi^j(A) = \varnothing$ for all $0 \le i, j \le k + l$, $i \ne l$. Such an element can naturally be regarded as an element in S_{k+l+1} and $\gamma_{\phi^i(A)}$ corresponds to a cyclic permutation $(i-1\ i\ i+1)$, which generate the alternate subgroup $A_{k+l+1} \lhd S_{k+l+1}$. It remains to note that since g has an odd order, its signature is 0, whence $g \in A_{k+l+1}$. $\square$

Exercise 6.3. Prove that for any $n \ge 3$ the group $A_n \lhd S_n$ is generated by elements $(i-1\ i\ i+1)$ for $2 \le i < n$.

Lemma 6.4. *The subgroup $H \le \mathcal{D}([\![\phi]\!])$ is normal. Since $\mathcal{D}([\![\phi]\!])$ is simple, it follows that $H = \mathcal{D}([\![\phi]\!])$.*

Proof. It is enough to show that for $\gamma_U \in H$, and any $f \in \mathcal{D}([\![\phi]\!])$ (or even $f \in [\![\phi]\!]$), we have $f\gamma_U f^{-1} \in H$. Since $f\gamma_U f^{-1}$ has order 3, this follows from Lemma 6.2. $\square$

If $U \subseteq X$ is clopen and $\phi^{-2}(U)$, $\phi^{-1}(U)$, U, $\phi(U)$, and $\phi^2(U)$ are pairwise disjoint, we set $\tau_U = \gamma_{\phi^{-1}(U)}\gamma_{\phi(U)}$.

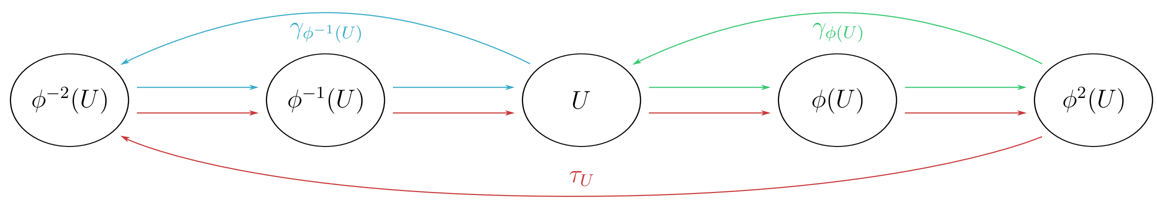

Figure 8. Homeomorphism $\tau_U = \gamma_{\phi^{-1}(U)}\gamma_{\phi(U)}$.

Lemma 6.5. *Let U and V be clopen subsets of X.*

(i) If $\phi^{-2}(V)$, $\phi^{-1}(V)$, V, $\phi(V)$, and $\phi^2(V)$ are pairwise disjoint and $U \subseteq V$, then $\tau_V\gamma_U\tau_V^{-1} = \gamma_{\phi(U)}$ and $\tau_V^{-1}\gamma_U\tau_V = \gamma_{\phi^{-1}(U)}$; see Figure 9.

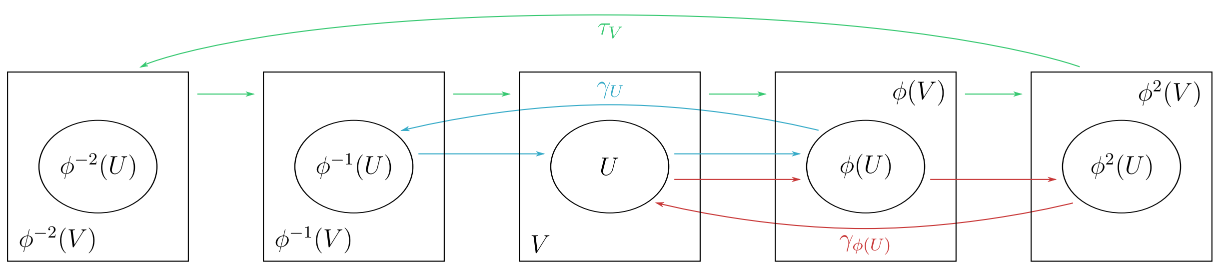

Figure 9. $\tau_V\gamma_U\tau_V^{-1} = \gamma_{\phi(U)}$.

(ii) If $\phi^{-1}(U)$, U, $\phi(U) \cup \phi^{-1}(V)$, V, and $\phi(V)$ are pairwise disjoint, then $[\gamma_V, \gamma_U^{-1}] = \gamma_{\phi(U)\cap\phi^{-1}(V)}$; see Figure 10.

Proof. (i) We may write $\tau_V = \tau_U\tau_{V\setminus U}$, and using that the support of $\tau_{V\setminus U}$ is disjoint from supports of other homeomorphisms, we get

$$\tau_V\gamma_U\tau_V^{-1} = \tau_U\gamma_U\tau_U^{-1} = \gamma_{\phi(U)},$$

where the last identity is a consequence of the following identity on permutations

$$(01234)(123)(04321) = (012).$$

Equality $\tau_V^{-1}\gamma_U\tau_V = \gamma_{\phi^{-1}(U)}$ is checked similarly.

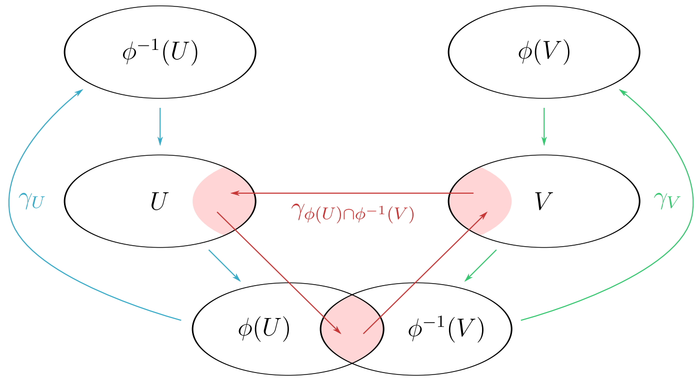

FIGURE 10. $[\gamma_V, \gamma_U^{-1}] = \gamma_{\phi(U)\cap\phi^{-1}(V)}$.

(ii) Let $C = \phi(U) \cap \phi^{-1}(V)$. We may decompose $\gamma_U = \gamma_{\phi^{-1}(C)}\gamma_{U\setminus\phi^{-1}(C)}$ and $\gamma_V = \gamma_{\phi(C)}\gamma_{V\setminus\phi(C)}$. Using the disjointness of support argument as in the previous item, one sees that

$$[\gamma_V, \gamma_U^{-1}] = [\gamma_{\phi(C)}, \gamma_{\phi^{-1}(C)}^{-1}] = \gamma_{\phi(C)}\gamma_{\phi^{-1}(C)}^{-1}\gamma_{\phi(C)}^{-1}\gamma_{\phi^{-1}(C)} = \phi_C,$$

where the last equality is equivalent to

$$(234)(021)(243)(012) = (123).$$

□

Theorem 6.6 (Matui [**Mat06**], Theorem 5.4)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be minimal. The commutator subgroup $\mathcal{D}([\![\phi]\!])$ is finitely generated if and only if (X, ϕ) is conjugate to a minimal subshift.*

PROOF. $\Longrightarrow$ Suppose $\mathcal{D}([\![\phi]\!])$ is finitely generated, and let $g_1, \dots, g_m \in \mathcal{D}([\![\phi]\!])$ be a finite set of generators, n_i be the corresponding cocycles $g_i(x) = \phi^{n_i(x)}(x)$, and $\mathcal{P}$ be the common refinement of partitions $\{n_i^{-1}(k)\}_{k\in\mathbb{Z}}$. Let $s : \mathcal{P}^{\mathbb{Z}} \to \mathcal{P}^{\mathbb{Z}}$ be the shift map. We define a continuous map $\pi : X \to \mathcal{P}^{\mathbb{Z}}$ by $\phi^k(x) \in \pi(x)(k)$. Note that π is a factor map from (X, ϕ) to $(\pi(X), s)$. Define homeomorphisms $f_i \in \mathrm{Homeo}(\pi(X))$ by $f_i(z) = s^k(z)$ when $z(0) \subseteq n_i^{-1}(k)$. It is easy to see that $f_i \in [\![s]\!]$ and $\pi g_i = f_i\pi$. It remains to show that π is injective.

Suppose $x, y \in X$ are distinct and $\pi(x) = \pi(y)$, pick $g \in \mathcal{D}([\![\phi]\!])$ such that $g(x) \neq x$ and $g(y) = y$. Write g as $g_{i_1}^{r_1} \cdots g_{i_l}^{r_l}$. Since $\pi g_i = f_i\pi$, we get

$$\begin{aligned}\pi g(x) &= \pi g_{i_1}^{r_1} \cdots g_{i_l}^{r_l}(x)\\ &= f_{i_1}^{r_1} \cdots f_{i_l}^{r_l}\pi(x)\\ &= f_{i_1}^{r_1} \cdots f_{i_l}^{r_l}\pi(y)\\ &= \pi g_{i_1}^{r_1} \cdots g_{i_l}^{r_l}(y)\\ &= \pi g(y) = \pi(y) = \pi(x),\end{aligned}$$

whence $s^k\pi(x) = \pi\phi^k(x) = \pi(x)$ for some $k \in \mathbb{Z}$, contradicting the minimality of s.

$\Longleftarrow$ Suppose (X, ϕ) is conjugate to a minimal subshift. Without loss of generality we may assume that X is a shift invariant closed subset of $A^{\mathbb{Z}}$, where A is finite. Moreover, we may assume that $x(i) \neq x(j)$ for all $x \in X$ and $i, j \in \mathbb{Z}$ with $|i - j| \leq 4$. We define cylinder sets by

$$\langle\!\langle a_{-m} \cdots a_{-1}\underline{a_0}a_1 \cdots a_n \rangle\!\rangle = \{\, x \in X \mid x(i) = a_i,\ -m \leq i \leq n \,\},$$

for $m, n \in \mathbb{N}$, and $a_i \in A$. Because of our assumptions, sets $\phi^{-2}(U)$, $\phi^{-1}(U)$, U, $\phi^2(U)$ are disjoint for any cylinder set U. Let H be the subgroup of $\mathcal{D}([\![\phi]\!])$ generated by the finite set of elements

$$\{\, \gamma_U \mid U = \langle\!\langle a\underline{b}c \rangle\!\rangle,\ a, b, c \in A \,\}.$$

We claim that $H = \mathcal{D}([\![\phi]\!])$, and for this it is enough to show that $\gamma_U \in H$ for any cylinder set U. From

$$\gamma_{\phi(\langle\!\langle \underline{a} \rangle\!\rangle)} = \prod_{b\in A} \gamma_{\langle\!\langle a\underline{b} \rangle\!\rangle}, \quad \gamma_{\phi^{-1}(\langle\!\langle \underline{a} \rangle\!\rangle)} = \prod_{b\in A} \gamma_{\langle\!\langle \underline{b}a \rangle\!\rangle}$$

we conclude $\gamma_{\phi(\langle\langle \underline{a} \rangle\rangle)} \in H$ and $\gamma_{\phi^{-1}(\langle\langle \underline{a} \rangle\rangle)} \in H$, and therefore also $\tau_{\langle\langle \underline{a} \rangle\rangle}$. For a cylindrical set

$$U = \langle\langle a_{-m} \cdots a_{-1} \underline{a_0} a_1 \cdots a_n \rangle\rangle \subseteq \langle\langle \underline{a_0} \rangle\rangle = V$$

an application of Lemma 6.5 implies

$$\tau_{\langle\langle \underline{a_0} \rangle\rangle} \gamma_U \tau^{-1}_{\langle\langle \underline{a_0} \rangle\rangle} = \gamma_{\phi(U)}, \quad \tau^{-1}_{\langle\langle \underline{a_0} \rangle\rangle} \gamma_U \tau_{\langle\langle \underline{a_0} \rangle\rangle} = \gamma_{\phi^{-1}(U)},$$

whence it suffices to show that γ_U can be generated for every cylinder set $U = \langle\langle a_{-m} \cdots a_{-1} \underline{a_0} a_1 \rangle\rangle$. The latter follows by induction from the second item of Lemma 6.5 with $U = \langle\langle a_{-m} \cdots \underline{a_0} a_1 \rangle\rangle$ and $V = \langle\langle a_1 \underline{a_2} \rangle\rangle$. □

LECTURE 7

Bratteli diagrams and Vershik maps

1. Bratteli diagrams

Our main reference for this lecture is the work of R. Herman, I. Putnam, and C. Skau [**HPS92**].

A *Bratteli diagram* consists of a vertex set V graded as a disjoint union of non-empty finite sets $V = \bigsqcup_{n=0}^{\infty} V_n$ and an edge set $E = \bigsqcup_{n=1}^{\infty} E_n$, where the sets E_n are all non-empty and finite, together with *source maps* $s : E_n \to V_{n-1}$ and *range maps* $r : E_n \to V_n$ which are both assumed to be surjective. We also require that V_0 consists of a single element $V_0 = \{\varnothing\}$.

An *ordered Bratteli diagram* is a Bratteli diagram (V, E) together with a partial ordering $\leq$ on the edge set E such that $e_1, e_2 \in E$ are comparable if and only if $r(e_1) = r(e_2)$. In other words, an ordered Bratteli diagram is a Bratteli diagram such that for any vertex all the edges coming into this vertex are linearly ordered.

Let $(V, E, \leq)$ be an ordered Bratteli diagram. An edge $e \in E$ is said to be *minimal* (resp. *maximal*) if it is the minimal (resp. the maximal) element of the set $r^{-1}\big(r(e)\big)$. The sets of minimal and maximal elements in E are denoted by $E_{\min}$ and $E_{\max}$ respectively.

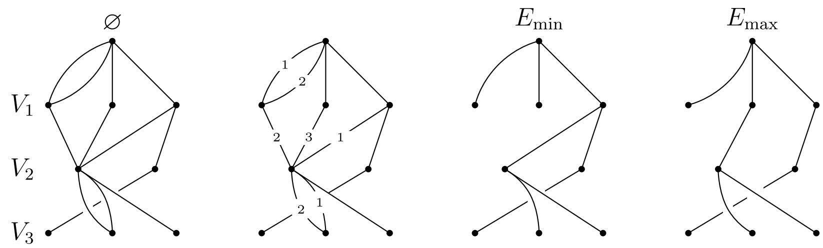

FIGURE 11. A Bratteli diagram, an ordered Bratteli diagram, $E_{\min}$, and $E_{\max}$.

We recall that a rooted tree is an acyclic connected graph with a distinguished vertex—*the root* of the tree.

Proposition 7.1. *The graphs $(V, E_{\max})$ and $(V, E_{\min})$ are rooted trees with $\varnothing$ being their root.*

PROOF. Pick a vertex $v \in V$. Let k be such that $v \in V_k$ and put $v_k = v$. Since the set $r^{-1}(v)$ is linearly ordered, there is a unique maximal element $e_k \in r^{-1}(v_k)$; put $v_{k-1} = s(e_k)$. Similarly, there is a unique $e_{k-1} \in E_{\max}$ such that $r(e_{k-1}) = v_{k-1}$. Continuing this argument we construct a sequence $e_k, \ldots, e_1$ such that $e_i \in E_{\max}$ and $s(e_1) = \varnothing$. This proves that every vertex $v \in V$ is connected within $E_{\max}$ to the root $\varnothing$, and so $(V, E_{\max})$ is a connected graph. To show that $(V, E_{\max})$ is acyclic let $e_1, \ldots, e_m \in E_{\max}$ and $e'_1, \ldots, e'_n \in E_{\max}$ be two simple paths from $\varnothing$ to a vertex $v \in V$; note that $s(e_1) = \varnothing = s(e'_1)$. Since $e_i \in E_{\max}$, we cannot have $r(e_i) = r(e_{i+1})$, therefore we must necessarily have $r(e_i) = s(e_{i+1})$ and therefore also $m = n$, $r(e_m) = r(e'_n)$. But this implies $e_m = e'_n$, and therefore inductively $e_i = e'_i$ for all i. This proves that $(V, E_{\max})$ is a tree. The proof for $(V, E_{\min})$ is similar. □

Note that $E_{\max}$ and $E_{\min}$ are trees with *finite splitting*, and therefore by König's Lemma there are infinite branches $e_{\max}$ in $E_{\max}$ and $e_{\min}$ in $E_{\min}$. Note that it is possible that $e_{\min} = e_{\max}$.

Definition 7.2. An ordered Bratteli diagram $(V, E, \leq)$ is called *essentially simple* if the trees $E_{\min}$ and $E_{\max}$ have *unique* infinite branches $e_{\min}$ and $e_{\max}$.

Up to now we used the word "path" in the sense of graph theory. Since Bratteli diagrams are graded, it will be convenient to modify the notion of path. Let (V, E) be a Bratteli diagram. A *path* from a vertex

$v \in V_k$ to a vertex $u \in V_l$, $k < l$, is a sequence of edges $e_{k+1}, \dots, e_l$ such that $e_i \in V_i$, $s(e_{k+1}) = v$, $r(e_l) = u$ and $s(e_{i+1}) = r(e_i)$ for all $k+1 \le i < l$. We use $P(v,u)$ to denote the set of all paths between v and u and

$$P(V_k, V_l) = \bigsqcup_{\substack{v \in V_k \\ u \in V_l}} P(v,u).$$

An *infinite path* in a Bratteli diagram is a sequence of edges $e_1, e_2, \dots$ such that $e_i \in E_i$ and $r(e_i) = s(e_{i+1})$.

With any Bratteli diagram $B = (V,E)$ we associate the *Bratteli compactum:* the space X_B of all infinite paths in B. By definition $X_B \subseteq \prod_{n=1}^{\infty} E_n$ and we endow X_B with the induced product topology. This makes X_B into a compact metrizable zero-dimensional space. Note that X_B is a Cantor space if and only if it has no isolated points.

2. Vershik maps

Let $B = (V, E, \le)$ be an essentially simple Bratteli diagram. The *Vershik map* $\phi_B : X_B \to X_B$ is defined as follows. First of all we define $\phi_B(e_{\max}) = e_{\min}$. If $x \in X_B$ is a non-maximal infinite path, let n be the smallest such that $x(n) \notin E_{\max}$. Let $e_n > x(n)$ be the successor of $x(n)$ in $r^{-1}\big(r(x(n))\big)$. Let $e_1, \dots, e_{n-1}$ be the path from $\varnothing$ to $s(e_n)$ within $E_{\min}$. We set

$$\phi_B(x)(m) = \begin{cases} e_m & \text{if } m \le n, \\ x(m) & \text{if } m > n. \end{cases}$$

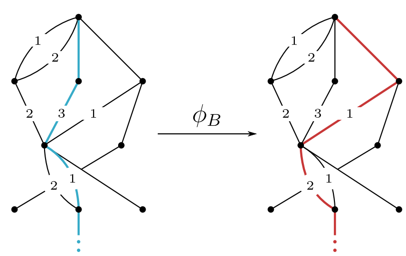

FIGURE 12. Vershik map acting on an ordered Bratteli diagram.

Proposition 7.3. *Let $B = (V, E, \le)$ be an essentially simple ordered Bratteli diagram. The Vershik map $\phi_B : X_B \to X_B$ is a homeomorphism.*

PROOF. We first show that ϕ_B is a bijection. Define the map $\psi_B : X_B \to X_B$ by $\psi_B(e_{\min}) = e_{\max}$ and for a non-minimal path x we take n to be minimal such that $x(n) \notin E_{\min}$. Let $e_n < x(n)$ be the predecessor of $x(n)$ in $r^{-1}\big(r(x(n))\big)$ and let $e_1, \dots, e_{n-1}$ be the path from $\varnothing$ to $s(e_n)$ within $E_{\max}$. We set

$$\psi_B(x)(m) = \begin{cases} e_m & \text{if } m \le n, \\ x(m) & \text{if } m > n. \end{cases}$$

Is is straightforward to check that $\phi_B \circ \psi_B = \mathrm{id} = \psi_B \circ \phi_B$, and therefore ϕ_B is a bijection. Since the definition of ϕ_B is local, it is obviously continuous as a map $\phi_B : X_B \setminus \{e_{\max}\} \to X_B \setminus \{e_{\min}\}$. The continuity at the point $e_{\max}$ is also straightforward to check. □

Proposition 7.4. *Let $B = (V, E, \le)$ be an essentially simple Bratteli diagram, and $\phi_B : X_B \to X_B$ be the Vershik map. Pick an $x \in X_B$ and a natural number M.*

(i) There exists $k_1 \ge 0$ such that $\phi_B^{-k_1}(x)(i) \in E_{\min}$ for all $i \le M$.
(ii) There exists $k_2 \ge 0$ such that $\phi_B^{k_2}(x)(i) \in E_{\max}$ for all $i \le M$.
(iii) With k_1 and k_2 defined as above, $\phi_B^{-k_1+j}(x)|_M$, $0 \le j \le k_2 + k_1$, is an enumeration of all the paths $P\big(\varnothing, r(x(M))\big)$.

PROOF. (i) We prove the statement by induction on M. If $M = 1$, the statement is obvious from the definition of ψ_B—the inverse of ϕ_B. For the induction step let $x \in X_B$ and M be given. By inductive hypothesis there is l_1 such that $\phi^{-l_1}(x)(i) \in E_{\min}$ for all $i \leq M-1$. Therefore $\phi^{-l_1-1}(x)(i) \in E_{\max}$ for all $i \leq M-1$ and $\phi^{-l_1-1}(x)(M)$ is the predecessor of $x(M)$. We therefore may continue and find l_2 such that $\phi^{-l_1-1-l_2}(x)(i) \in E_{\min}$ for all $i \leq M-1$, hence $\phi^{-l_1-1-l_2-1}(x)(M)$ is the predecessor of $\phi^{-l_1-1}(x)(M)$, etc. For some $p \geq 1$ and

$$-k_1 = -l_1 - 1 - l_2 - 1 - \cdots - l_{p-1} - 1 - l_p$$

we have $\phi^{-k_1}(x)(i) \in E_{\min}$ for all $i \leq M$.

Item (ii) is a statement symmetric to item (i), and (iii) is proved similarly by induction on M. □

Definition 7.5. A Bratteli diagram (V, E) is called *simple* if for every m there is $n > m$ such that from any vertex in V_m there is path to any vertex in V_n. An ordered Bratteli diagram $B = (V, E, \leq)$ is called *simple* if it is essentially simple as an ordered diagram, and simple in the above sense as an unordered diagram (V, E).

Note that if $B = (V, E)$ is simple, then X_B is a Cantor space.

Proposition 7.6. *Let $B = (V, E, \leq)$ be an essentially simple ordered Bratteli diagram. The Vershik map $\phi_B : X_B \to X_B$ is minimal if and only if B is simple.*

PROOF. Suppose B is simple. In order to prove the minimality of ϕ_B it is enough to show that for any $x \in X_B$, any $y \in X_B$, and any M there exists $n \in \mathbb{Z}$ such that $\phi^n(x)(i) = y(i)$ for all $i \leq M$. Since the diagram is assumed to be minimal, we may find an N such that any vertex in V_M is connected to any vertex in V_N. Let $v = r(x(N))$ and $u = r(y(M))$. By the choice of N we can find a path from u to v, and hence we can find some $z \in X_B$ (see Figure 13) such that

$$z(i) = \begin{cases} y(i) & \text{if } i \leq M, \\ x(i) & \text{if } i > N. \end{cases}$$

By item (iii) of Proposition 7.4, there is an $n \in \mathbb{Z}$ such that $\phi_B^n(x) = z$. Therefore also $\phi^n(x)(i) = y(i)$ for all $i \leq M$, hence ϕ_B is minimal.

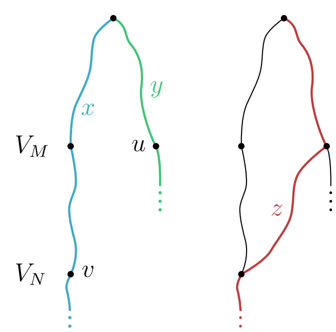

FIGURE 13. Paths x, y, and z.

For the inverse implication we prove the contrapositive. Suppose B is not simple: there is m such that for any $n > m$ there are $u_n \in V_m$ and $v_n \in V_n$ such that $P(u_n, v_n)$ is empty. Since V_m is finite, there is $u \in V_m$, an increasing sequence n_k, and $v_k \in V_{n_k}$ such that $P(u, v_k)$ is empty. Let $y_k \in X_B$ be such that $r\big(y_k(n_k)\big) = v_k$. By compactness of X_B we may find a converging subsequence; let $y \in X_B$ be a limit point of $(y_k)_{k \in \mathbb{N}}$. Note that $P\big(u, r(y(i))\big)$ is empty for all $i > m$, because if there were a path from u to $r(y(i_0))$ for some i_0, then we would find a big enough k such that $n_k \geq i_0$, and y would agree with y_k up to index i_0, hence there would be a path from u to v_k contrary to the assumption.

Pick $x \in X_B$ such that $r(x(m)) = u$. Suppose towards the contradiction that ϕ_B is minimal. Then we can find $k \in \mathbb{Z}$ such that $\phi_B^k(y)(i) = x(i)$ for all $i \leq m$. Without loss of generality we may assume that $\phi_B^k(y)$ is tail equivalent to y (this is because by minimality we may find both a negative and a positive such $k \in \mathbb{Z}$) and therefore $\phi_B^k(y)(N) = y(N)$ for all large enough N. This implies $P\big(u, r(y(N))\big)$ is non-empty, contradicting the construction of y. □

LECTURE 8

Minimal homeomorphisms as Vershik maps

1. Realization of homeomorphisms

Theorem 8.1 (Herman–Putnam–Skau [**HPS92**], Theorem 4.6)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism and $x \in X$, then there is a simple Bratteli diagram $B = (V, E, \leq)$ such that (ϕ, X, x) and $(\phi_B, X_B, e_{\min})$ are conjugated.*

Proof. Using Proposition 1.16 we can find a sequence of Kakutani–Rokhlin partitions

$$\Xi_n = \left\{ D^{(n)}(i, j) \mid 1 \leq i \leq K^{(n)},\, 0 \leq j < J_i^{(n)} \right\}$$

with bases $D^{(n)} = \bigsqcup_i D^{(n)}(i, 0)$ such that

(i) $\Xi_0 = \{X\}$;
(ii) $D^{(n+1)} \subseteq D^{(n)}$ for all n;
(iii) Ξ_{n+1} refines Ξ_n;
(iv) $\bigcap_n D^{(n)} = \{x\}$;
(v) $\bigcup_n \Xi_n$ generates the topology of X.

The Bratteli diagram $B = (V, E, \leq)$ is constructed out of this sequence as follows. Vertices of V_n are the towers of Ξ_n: $V_n = \mathcal{T}(\Xi_n)$. For each inclusion $D^{(n+1)}(i, j) \subset D^{(n)}(k, 0)$ we put an edge between $T_k^{(n)}$ and $T_i^{(n+1)}$. Edges are ordered in a natural way: if e_1 corresponds to an inclusion $D^{(n+1)}(i, j_1) \subset D^{(n)}(k, 0)$ and e_2 to $D^{(n+1)}(i, j_2) \subset D^{(n)}(k, 0)$, then $e_1 \leq e_2$ whenever $j_1 \leq j_2$. Figure 14 gives an instructive example. Note that B is essentially simple with $e_{\min}$ corresponding to inclusions $D^{(n+1)}(i, 0) \subseteq D^{(n)}(j, 0)$, and $e_{\max}$ corresponding to inclusions $D^{(n+1)}(i, J_i^{(n+1)} - J_j^{(n)}) \subseteq D^{(n)}(j, 0)$. Indeed, if there were two minimal paths corresponding to inclusions $D^{(n+1)}(i_{n+1}, 0) \subseteq D^{(n)}(i_n, 0)$ and $D^{(n+1)}(j_{n+1}, 0) \subseteq D^{(n)}(j_n, 0)$, then we would have

$$\bigcap_n D^{(n)}(i_n, 0) = \{x\} = \bigcap_n D^{(n)}(j_n, 0),$$

which is impossible if $i_n \neq j_n$ for some n. Note also that we can always reorder the towers in Ξ_n to assure that $e_{\min}$ corresponds to inclusions $D^{(n+1)}(1, 0) \subseteq D^{(n)}(1, 0)$, and $e_{\max}$ to $D^{(n+1)}\big(K^{(n)}, J_{K^{(n+1)}}^{(n+1)} - J_{K^{(n)}}^{(n)}\big) \subseteq D^{(n)}\big(K^{(n+1)}, 0\big)$.

Our goal is to show that (ϕ, X, x) is conjugated to $(\phi_B, X_B, e_{\min})$. The conjugation map $\xi : X \to X_B$ is defined as follows. Pick an $x \in X$ and $n \geq 1$. Let $D^{(n-1)}(i_{n-1}, j_{n-1})$ and $D^{(n)}(i_n, j_n)$ be the elements of partitions Ξ_{n-1} and Ξ_n that contain x. Therefore $j_{n-1} \leq j_n$ and $D^{(n)}(i_n, j_n - j_{n-1}) \subseteq D^{(n-1)}(i_{n-1}, 0)$ and we let $\xi(x)(n)$ to be the edge e that corresponds to this inclusion. In particular, $r(e) = T_{i_n}^{(n)}$ and $s(e) = T_{i_{n-1}}^{(n-1)}$. An example is shown in Figure 15.

We claim that for any $x \in X$ the initial path of $\xi(x)$ of length n determines precisely the element $D^{(n)}(i, j)$ such that $x \in D^{(n)}(i, j)$ (see Figure 15). More formally,

$$\forall i \leq n \; \xi(x)(i) = \xi(y)(i) \iff x \text{ and } y \text{ are in the same atom of } \Xi_n.$$

$\Leftarrow$ is obvious. We prove $\Rightarrow$ by induction on n. For the base of induction we note that $\Xi_0 = \{X\}$ implies that $\xi(x)(1)$ are in one-to-one correspondence with elements of Ξ_1. Suppose $\xi(x)(i) = \xi(y)(i)$ for all $i \leq n$. The edge $\xi(x)(n)$ corresponds to an inclusion $D^{(n)}(i_n, k) \subseteq D^{(n-1)}(i_{n-1}, 0)$. By inductive assumption x and y are in the same atom $D^{(n-1)}(i_{n-1}, j_{n-1})$ of $D^{(n-1)}$, therefore $x, y \in D^n(i_n, k + j_{n-1})$.

From the above claim properties of ξ are almost obvious. It is easy to see that ξ is continuous and bijective (injectivity follows from item (v)), hence ξ is a homeomorphisms. It is straightforward to check that $\xi \circ \phi = \phi_B \circ \xi$. □

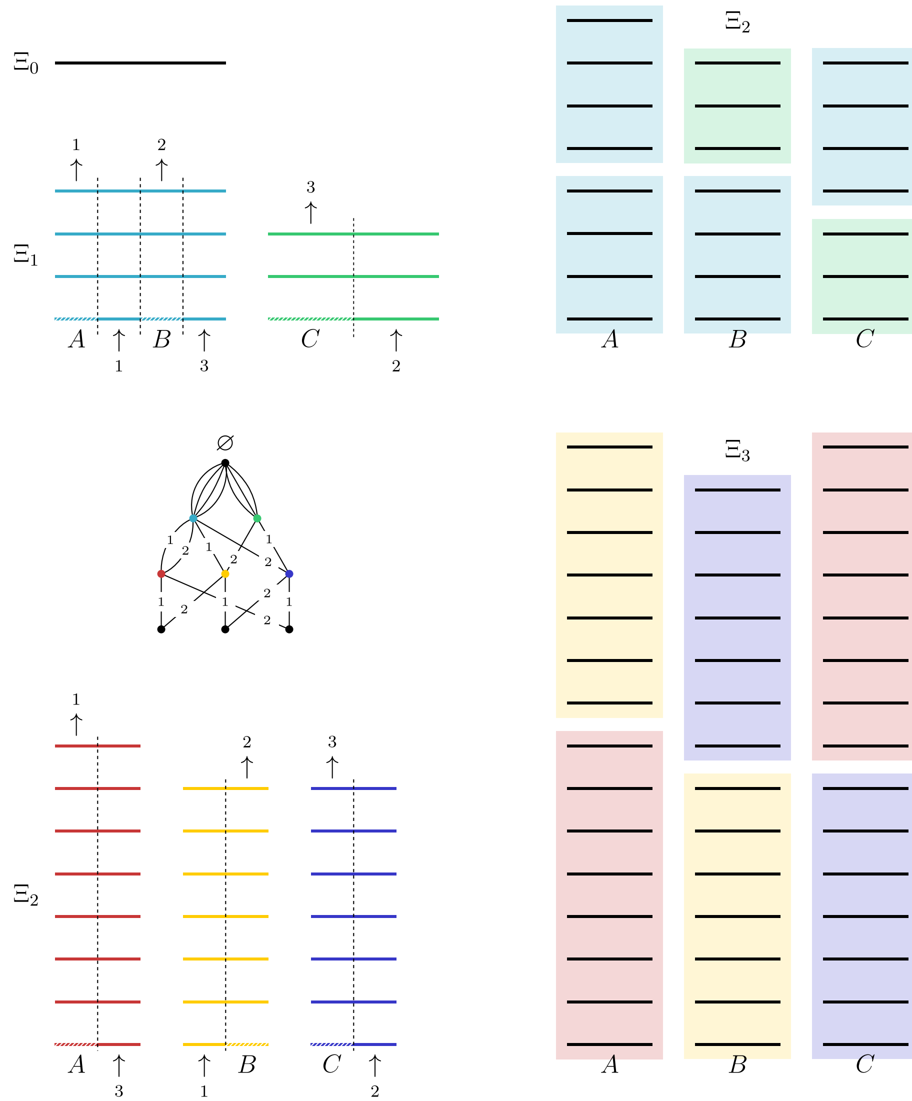

FIGURE 14. Construction of a Bratteli diagram out of Kakutani–Rokhlin partitions.

Remark 8.2. Note that given a Bratteli diagram $B = (V, E, \leq)$ we can reconstruct a sequence of Kakutani–Rokhlin partitions: for a path p from $\varnothing$ to $u \in V_n$ we set

$$C(p) = \{ x \in X_B \mid x(i) = p(i) \ \forall i \leq n \}$$

and $\Xi_n = \{ C(p) \mid p \in P(V_0, V_n) \}$. Therefore any Vershik map ϕ_B that realizes a minimal homeomorphism ϕ is constructed as in Theorem 8.1.

2. Telescoping diagrams

In view of Remark 8.2 it is natural to ask: *When does two simple ordered Bratteli diagrams give rise to isomorphic Vershik maps?* In this section we give a complete answer to this question.

Definition 8.3. Let $B = (V, E)$ be a Bratteli diagram and let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers with $n_0 = 0$. A *telescope* of B with respect to (n_k) is a Bratteli diagram $B' = (V', E')$ defined by $V'_k = V_{n_k}$ and $E'_k = P(V_{n_{k-1}}, V_{n_k})$. More precisely, for each path $e_{n_{k-1}+1}, \dots, e_{n_k}$ in B with $s(e_{n_{k-1}+1}) = u \in V_{n_{k-1}}$, $r(e_{n_k}) = v \in V_{n_k}$ we have an edge $e' \in E'_k$ with $s'(e') = u$ and $r'(e') = v$ (see Figure 16).

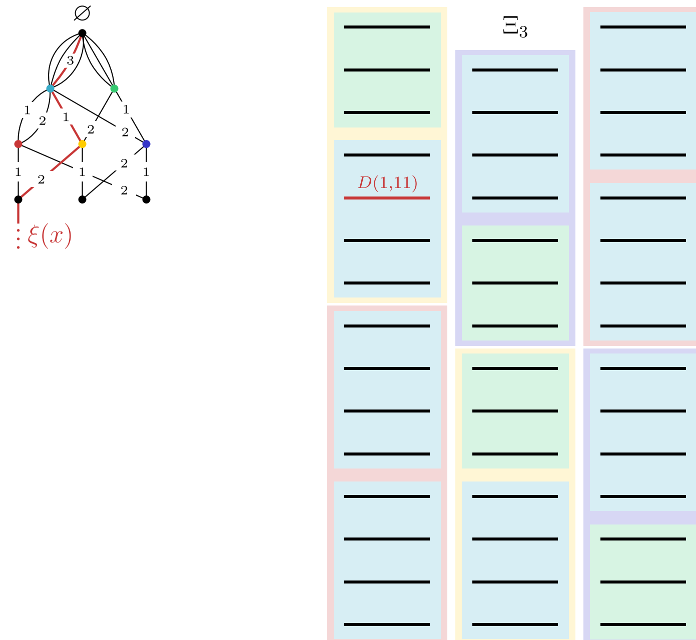

FIGURE 15. A point $x \in D(1,11)$ will have an image $\xi(x)$.

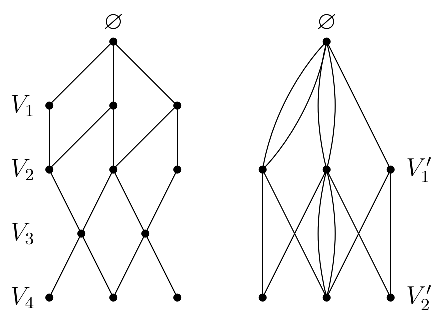

FIGURE 16. Four levels of a Bratteli diagram B and two levels of B' with $n_1 = 2$ and $n_2 = 4$.

If $B = (V, E, \leq)$ is an ordered Bratteli diagram to begin with, then for any two levels $k < l$ and $v \in V_l$ we have a natural ordering on $P(V_k, v)$: a path $e_{k+1}, \ldots, e_l$ is less than a path $f_{k+1}, \ldots, f_l$, where $r(e_l) = v = r(f_l)$ and $s(e_{k+1}), s(f_{k+1}) \in V_k$, if for the largest $k < m \leq l$ with $e_m \neq f_m$ we have $e_m < f_m$.

If now B is an ordered Bratteli diagram and (n_k) is an increasing sequence with $n_0 = 0$, then the telescope B' of B is also an ordered Bratteli diagram, when edges are endowed with this ordering. If B is essentially simple, then so is B'.

An increasing sequence of integers (n_k) with $n_0 = 0$ will be called a *telescoping sequence*.

Proposition 8.4. *Let B be an essentially simple Bratteli diagram and (n_k) be a telescoping sequence; let B' the telescope of B with respect to (n_k). Homeomorphisms $(X_B, \phi_B, e_{\min})$ and $(X_{B'}, \phi_{B'}, e'_{\min})$ are conjugated.*

PROOF. The conjugation $\xi : X_B \to X_{B'}$ is defined as follows. For $x \in X_B$, $\xi(x)(k)$ is defined to be the edge that corresponds to the path $x(n_{k-1}+1), \ldots, x(n_k)$. It is obvious that $\xi : X_B \to X_{B'}$ is a homeomorphism, and $\xi \circ \phi_B = \phi_{B'} \circ \xi$. □

Remark 8.5. In the context of Theorem 8.1, telescoping of Bratteli diagrams corresponds to taking subsequences of Kakutani–Rokhlin partitions.

Definition 8.6. We say that two ordered Bratteli diagrams B and B' are *equivalent*, if there is a sequence of ordered Bratteli diagrams $B_1, \ldots, B_n$ such that $B_1 = B$, $B_n = B'$ and for each $1 \leq i < n$ one of the three possibilities hold: either B_i is isomorphic to B_{i+1}, or B_{i+1} is a telescope of B_i, or B_i is a telescope of B_{i+1}. In other words, equivalence of ordered Bratteli diagrams is the finest equivalence relations that preserves isomorphisms and telescoping.

Theorem 8.7 (Herman–Putnam–Skau [**HPS92**], Theorem 4.5)**.** *Let B_1 and B_2 be simple ordered Bratteli diagrams. Two Vershik maps $\phi_1 = \phi_{B_1}$ and $\phi_2 = \phi_{B_2}$ are conjugated if and only if B_1 and B_2 are equivalent.*

Proof. $\Leftarrow$ follows from Proposition 8.4. We show $\Rightarrow$. There is no loss in generality to assume that B_1 and B_2 are constructed from sequences of Kakutani–Rokhlin partitions $\Xi_n^{(1)}$ and $\Xi_n^{(2)}$ respectively. By passing to subsequences we may assume that $\Xi_{n+1}^{(1)}$ refines $\Xi_n^{(2)}$ and $\Xi_{n+1}^{(2)}$ refines $\Xi_n^{(1)}$ for each n. We define $\Xi_n^{(3)}$ by

$$\Xi_n^{(3)} = \begin{cases} \Xi_n^{(1)} & \text{if } n \text{ is even;} \\ \Xi_n^{(2)} & \text{if } n \text{ is odd.} \end{cases}$$

The sequence $\Xi_n^{(3)}$ satisfies all the items in the construction from Theorem 8.1, and we let B_3 be the diagram obtained from $\Xi_n^{(3)}$. Since B_3 is equivalent to the telescope of B_1 with respect to $(2k)_{k\in\mathbb{N}}$ and also to the telescope of B_2 with respect to $(2k+1)_{k\in\mathbb{N}}$, we see that B_1 and B_2 are equivalent. $\square$

LECTURE 9

Invariant means

1. Basic theory

Let G be a discrete group acting on a countable set X. A *mean* is a linear functional $m \in \ell^\infty(X)^*$ such that $m(f) \geq 0$ for all $f \geq 0$, and $m(\mathbb{1}) = 1$. Means are in one-to-one correspondence with finitely additive probability measures on X. We shall let the context to explain whether we refer to a linear function or to a finitely additive measure. The set of means on X is denoted by $\mathbb{M}(X)$. A mean $m \in \mathbb{M}(X)$ is said to be G-invariant if $m(g \circ f) = m(f)$ for all $f \in \ell^\infty(X)$ and all $g \in G$. Let $\mathbb{P}(X)$ be the set of all countably additive probability measures on X:

$$\mathbb{P}(X) = \{ \mu \in \ell^1(X) \mid \mu \geq 0, ||\mu||_1 = 1 \}.$$

We can naturally view $\mathbb{P}(X)$ as a subset of $\mathbb{M}(X)$.

Exercise 9.1. If m is a mean on X, then for any $f \in \ell^\infty(X)$

$$\inf f \leq m(f) \leq \sup f.$$

Lemma 9.2. $\overline{\mathbb{P}(X)}^{w*} = \mathbb{M}(X)$.

Proof. Since $\overline{\mathbb{P}(X)}^{w*}$ is a convex closed subsets of $\ell^\infty(X)^*$, if $m_0 \in \mathbb{M}(X) \setminus \overline{\mathbb{P}(X)}^{w*}$, then by separation theorem we can find $f \in \ell^\infty(X)$ and $c > 0$ such that $m_0(f) \geq c + m(f)$ for all $m \in \overline{\mathbb{P}(X)}^{w*}$. Since $\overline{\mathbb{P}(X)}^{w*}$ includes all Dirac measures, we obtain

$$m_0(f) > \sup\{ m(f) \mid m \in \overline{\mathbb{P}(X)}^{w*} \} \geq \sup\{ f(x) \mid x \in X \},$$

whence m_0 is not a mean. □

Corollary 9.3. *Let $m \in \mathbb{M}(X)$ be a G-invariant mean. There exists a net $\mu_n \in \mathbb{P}(X)$ such that $\mu_n \xrightarrow{w*} m$ and $g \circ \mu_n - \mu_n \xrightarrow{w*} 0$ for all $g \in G$.*

Lemma 9.4. *Let $m \in \mathbb{M}(X)$ be a G-invariant mean. There exists a net $\mu_n \in \mathbb{P}(X)$ such that $\mu_n \xrightarrow{w*} m$ and $g \circ \mu_n - \mu_n \xrightarrow{||\cdot||_1} 0$ for all $g \in G$.*

Proof. Let $\nu_n \in \mathbb{P}(X)$ be such that $\nu_n \xrightarrow{w*} m$ and $g \circ \nu_n - \nu_n \xrightarrow{w*} 0$ for all $g \in G$. For each $g \in G$ we take a copy of $\ell^1(X)$, and form a locally convex topological vector space

$$E = \prod_{g \in G} \ell^1(X).$$

We have a map $T : \ell^1(X) \to E$ given by $T(\mu)(g) = g \circ \mu - \mu$. The weak topology on E coincides with the product of weak topologies on factors. Since $g \circ \nu_n - \nu_n \xrightarrow{w*} 0$ for each $g \in G$, zero lies in the weak closure $\overline{T(\mathbb{P}(X))}$. Since E is locally convex and $T(\mathbb{P}(X))$ is convex, the weak and strong closures coincide, hence there is some net $(\mu_n) \subseteq \mathbb{P}(X)$ such that $T(\mu_n) \to 0$ in E, which is equivalent to saying $||g \circ \mu_n - \mu_n||_1 \to 0$ for all $g \in G$. □

Definition 9.5. A group G is said to be *amenable* if the action $G \curvearrowright G$ by left multiplication has an invariant mean.

Fact 9.6 (see, for example, Juschenko–Monod [**JM12**], Lemma 3.2)**.** *If $G \curvearrowright X$ has an invariant mean and if stabilizers of all points are amenable subgroups of G, then G itself is amenable.*

2. Actions on finite subsets

If G acts on a set X, then it also acts on $\mathcal{P}_f(X)$—the group of finite subsets of X with symmetric difference as the group operation. Hence we get an action $\mathcal{P}_f(X) \rtimes G \curvearrowright \mathcal{P}_f(X)$. Fix a point $x_0 \in X$ and let

$$S_{x_0} = \{\, F \in \mathcal{P}_f(X) \mid x_0 \in F \,\}.$$

For $E \in \mathcal{P}_f(X)$ let $\mathbb{1}_E \in L^2\big(\{0,1\}^X\big)$ be the function defined by

$$\mathbb{1}_E(w) = \begin{cases} 1 & \text{if } w(x) = 0 \text{ for all } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We write $\mathbb{1}_{x_0}$ for $\mathbb{1}_{\{x_0\}}$. If $\mu \in \mathbb{P}(\mathcal{P}_f(X))$ and $E \in \mathcal{P}_f(X)$, we also write $\mu(E)$ instead of $\mu(\{E\})$.

Lemma 9.7 (Juschenko–Monod [**JM12**], Lemma 3.1)**.** *Suppose that the action $G \curvearrowright X$ is transitive. In the above notations the following conditions are equivalent.*

(i) There exists a G-almost invariant net $\{f_n\} \in L^2\big(\{0,1\}^X\big)$ such that

$$\frac{||f_n \cdot \mathbb{1}_{x_0}||_2}{||f_n||_2} \to 1.$$

(ii) The action $\mathcal{P}_f(X) \rtimes G \curvearrowright \mathcal{P}_f(X)$ admits an invariant mean.
(iii) The action $G \curvearrowright \mathcal{P}_f(X)$ admits an invariant mean m such that $m(S_{x_0}) = 1/2$.
(iv) The action $G \curvearrowright \mathcal{P}_f(X)$ admits an invariant mean m such that $m(S_{x_0}) = 1$.

Proof. (i) $\Longrightarrow$ (ii) Let f_n be a G-almost invariant net with $\frac{||f_n \cdot \mathbb{1}_{x_0}||_2}{||f_n||_2} \to 1$. Without loss of generality we may assume that $||f_n||_2 = 1$. Recall that a Fourier transform $\widehat{f_n} \in \ell^2(\mathcal{P}_f(X))$ of $f_n \in L^2\big(\{0,1\}^X\big)$ is given by

$$\widehat{f_n}(E) = \int\limits_{\{0,1\}^X} f_n(w)(-w, E)\; d\lambda,$$

where

$$(w, E) = \exp\big(i\pi \sum_{x \in E} w(x)\big).$$

Note that every element in $\{0,1\}^X$ has order two, therefore $(-w, E) = (w, E)$. The Fourier transform $\widehat{f_n}$ gives G-almost invariant vectors in $\ell^2(\mathcal{P}_f(X))$, since

$$||g \circ \widehat{f_n} - \widehat{f_n}||_2 = \big|\big|\widehat{(g \circ f_n - f_n)}\big|\big|_2 = ||g \circ f_n - f_n||_2.$$

We claim that $\widehat{f_n}$ are also $\{x_0\}$-almost invariant. Since $||f_n||_2 = 1$ and

$$\frac{||f_n \cdot \mathbb{1}_{x_0}||_2}{||f_n||_2} \to 1$$

we get $||f_n \cdot (\mathbb{1} - \mathbb{1}_{x_0})||_2 \to 0$. Therefore

$$\begin{aligned} ||\{x_0\} \circ \widehat{f_n} - \widehat{f_n}||_2^2 &= \sum_{E \in \mathcal{P}_f(X)} \Big| \int\limits_{\{0,1\}^X} f_n(w)(w, E)(e^{i\pi w(x_0)} - 1)\; d\lambda \Big|^2 \\ &= 4 \sum_E \Big| \int\limits_{\{0,1\}^X} f_n(w)(\mathbb{1} - \mathbb{1}_{x_0})(w)(w, E)\; d\lambda \Big|^2 \\ &= 4 \sum_E \Big| \widehat{\big(f_n \cdot (\mathbb{1} - \mathbb{1}_{x_0})\big)}(E) \Big|^2 \\ &= 4 \,\big|\big| \widehat{\big(f_n \cdot (\mathbb{1} - \mathbb{1}_{x_0})\big)} \big|\big|_2^2 = 4\, ||f_n \cdot (\mathbb{1} - \mathbb{1}_{x_0})||_2^2 \to 0. \end{aligned}$$

Thus $\widehat{f}_n$ is $\{x_0\}$-almost invariant. Since G acts transitively on X, for any $y \in X$ there is $g \in G$ such that $gx_0 = y$, hence $\widehat{f}_n$ is also $\{y\}$-almost invariant. Whence the net $\widehat{f}_n$ is actually $\mathcal{P}_f(X) \rtimes G$-almost invariant. By the Cauchy–Schwarz inequality

$$\begin{aligned} ||g \circ \widehat{f}_n^{\,2} - \widehat{f}_n^{\,2}||_1 &= ||(g \circ \widehat{f}_n - \widehat{f}_n)(g \circ \widehat{f}_n + \widehat{f}_n)||_1 \\ &\leq ||g \circ \widehat{f}_n - \widehat{f}_n||_2 \cdot ||g \circ \widehat{f}_n + \widehat{f}_n||_2 \\ &\leq 2\,||g \circ \widehat{f}_n - \widehat{f}_n||_2 \end{aligned}$$

Thus the net $\widehat{f}_n^{\,2} \in \mathbb{P}(X)$ is G-almost invariant, and any of its w^*-limit points in $\mathbb{M}(X)$ is a G-invariant mean on X.

(ii) $\implies$ (iii) Let m be a $\mathcal{P}_f(X) \rtimes G$-invariant mean. Since $\{x_0\} \cdot S_{x_0} = \sim S_{x_0}$, we get

$$m(S_{x_0}) = m(\{x_0\} \cdot S_{x_0}) = m(\sim S_{x_0}) = 1/2.$$

(iii) $\implies$ (iv) Let m be a G-invariant mean such that $m(S_{x_0}) = 1/2$. Repeating arguments of Lemmata 9.2 and 9.4 one shows that there exists a net $\mu_n \in \mathbb{P}(\mathcal{P}_f(X))$ such that $\mu_n \xrightarrow{w*} m$, $\mu_n(S_{x_0}) = 1/2$, and $||g \circ \mu_n - \mu_n||_1 \to 0$ for all $g \in G$.

Fix $k \geq 1$. Let $U : \mathcal{P}_f(X)^k \to \mathcal{P}_f(X)$ be the "union function:"

$$U(F_1, \ldots, F_k) = \bigcup_i F_i.$$

Let $\mu_n^{(k)} = U_* \mu_n^{\times k}$ be the push-forward of $\mu_n^{\times k}$ to a measure on $\mathcal{P}_f(X)$:

$$\mu_n^{(k)}(A) = \mu_n^{\times k}(U^{-1}(A)).$$

We have

$$\begin{aligned} \mu_n^{(k)}(S_{x_0}) &= \mu_n^{\times k}\{\, (F_1, \ldots, F_k) \mid \exists i\ x_0 \in F_i \,\} \\ &= 1 - \mu_n^{\times k}\{\, (F_1, \ldots, F_k) \mid \forall i\ x_0 \notin F_i \,\} \\ &= 1 - \mu_n^{\times k}(\sim S_{x_0} \times \cdots \times \sim S_{x_0}) = 1 - 2^{-k}. \end{aligned}$$

The net $\mu_n^{(k)}$ is G-almost invariant, since

$$\begin{aligned} ||g \circ \mu_n^{(k)} - \mu_n^{(k)}||_1 &= \sum_{E \in \mathcal{P}_f(X)} \left| \mu_n^{(k)}(gE) - \mu_n^{(k)}(E) \right| \\ &= \sum_E \left| \mu_n^{\times k}\{\, (F_1, \ldots, F_k) \mid \bigcup_i F_i = gE \,\} - \mu_n^{\times k}\{\, (F_1, \ldots, F_k) \mid \bigcup_i F_i = E \,\} \right| \\ &= \sum_E \Big| \sum_{\substack{(F_1,\ldots,F_k) \\ \bigcup F_i = gE}} \prod_{j=1}^k \mu_n(F_j) - \sum_{\substack{(F_1,\ldots,F_k) \\ \bigcup F_i = E}} \prod_{j=1}^k \mu_n(F_j) \Big| \\ &= \sum_E \Big| \sum_{\substack{(F_1,\ldots,F_k) \\ \bigcup F_i = E}} \prod_{j=1}^k \mu_n(gF_j) - \sum_{\substack{(F_1,\ldots,F_k) \\ \bigcup F_i = E}} \prod_{j=1}^k \mu_n(F_j) \Big| \\ &\leq \sum_E \sum_{\substack{(F_1,\ldots,F_k) \\ \bigcup F_i = E}} \left| \prod_{j=1}^k \mu_n(gF_j) - \prod_{j=1}^k \mu_n(F_j) \right| \\ &= \sum_{(F_1,\ldots,F_k)} \left| \prod_{j=1}^k \mu_n(gF_j) - \prod_{j=1}^k \mu_n(F_j) \right| \\ &\leq \sum_{j=1}^k \sum_{(F_1,\ldots,F_k)} \mu_n(gF_1) \cdots \mu_n(gF_{j-1}) \big| \mu_n(gF_j) - \mu_n(F_j) \big| \mu_n(F_{j+1}) \cdots \mu_n(F_k) \\ &= k||g \circ \mu_n - \mu_n||_1 \end{aligned}$$

Let $m_k \in \mathbb{M}(\mathcal{P}_f(X))$ be a limit point of the net $(\mu_n^{(k)})$. The mean m_k is G-invariant and $m_k(S_{x_0}) = 1 - 2^{-k}$. Let finally $\widetilde{m} \in \mathbb{M}(\mathcal{P}_f(X))$ be any limit point of the sequence m_k. It is G-invariant and $\widetilde{m}(S_{x_0}) = 1$.

(iv) $\implies$ (i) Let m be a G-invariant mean with $m(S_{x_0}) = 1$. There exists a net $\mu_n \in \mathbb{P}(\mathcal{P}_f(X))$ such that $\mu_n \xrightarrow{w*} m$, $||g \circ \mu_n - \mu_n||_1 \to 0$ for all $g \in G$, and $\mu_n(S_{x_0}) = 1$. Set

$$f_n = \sum_{F \in \mathcal{P}_f(X)} \mu_n(F) 2^{|F|} \mathbb{1}_F.$$

Since μ_n is supported on S_{x_0}, $f_n \cdot \mathbb{1}_{x_0} = f_n$. The norm $||f_n||_1 = 1$, since

$$\begin{aligned}
||f_n||_1 &= \int \Big| \sum_{F \in \mathcal{P}_f(X)} \mu_n(F) 2^{|F|} \mathbb{1}_F \Big| d\lambda \\
&= \int \sum_F \mu_n(F) 2^{|F|} \mathbb{1}_F \; d\lambda \\
&= \sum_F 2^{|F|} \mu_n(F) \int \mathbb{1}_F \; d\lambda \\
&= \sum_F 2^{|F|} \mu_n(F) 2^{-|F|} \; d\lambda = 1.
\end{aligned}$$

We claim that $||g \circ f_n - f_n||_1 \leq ||g \circ \mu_n - \mu_n||_1$. Indeed,

$$\begin{aligned}
||g \circ f_n - f_n||_1 &= \int \Big| \sum_{F \in \mathcal{P}_f(X)} \mu_n(F) 2^{|F|} \mathbb{1}_{g^{-1}F} - \sum_{F \in \mathcal{P}_f(X)} \mu_n(F) 2^{|F|} \mathbb{1}_F \Big| d\lambda \\
&= \int \Big| \sum_F \mu_n(gF) 2^{|F|} \mathbb{1}_F - \sum_F \mu_n(F) 2^{|F|} \mathbb{1}_F \Big| d\lambda \\
&= \int \Big| \sum_F 2^{|F|} \mathbb{1}_F (\mu_n(gF) - \mu_n(F)) \Big| d\lambda \\
&\leq \sum_F |g \circ \mu_n - \mu_n| = ||g \circ \mu_n - \mu_n||_1.
\end{aligned}$$

Therefore $f_n^{1/2} \in L^2\big(\{0,1\}^X\big)$ are as required, since

$$||g \circ f_n^{1/2} - f_n^{1/2}||_2 = \Big(\int \big| g \circ f_n^{1/2} - f_n^{1/2} \big|^2 d\lambda \Big)^{1/2} \leq \Big(\int \big| g \circ f_n - f_n \big| d\lambda \Big)^{1/2} = ||g \circ f_n - f_n||_1^{1/2}. \qquad \square$$

Amenability of topological full groups

Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism. Fix some $x \in X$. The orbit $\mathrm{Orb}_\phi(x)$ can naturally be identifies with the set of integers $\mathbb{Z}$, where x corresponds to $0 \in \mathbb{Z}$. Via this identification we get an action of $[\![\phi]\!]$ on $\mathbb{Z}$. In other words, for any $x \in X$ we have a homomorphism $\pi_x : [\![\phi]\!] \to S(\mathbb{Z})$, where $S(\mathbb{Z})$ is the group of permutations of the integers. The images $\pi_x(g)$ are quite special, since they have *bounded displacement.* Let for $g \in S(\mathbb{Z})$

$$|\, g\,|_w = \sup_{n\in\mathbb{Z}} |g(n) - n| \in \mathbb{N} \cup \{\infty\}.$$

We say that $g \in S(\mathbb{Z})$ has bounded displacement if $|\, g\,|_w < \infty$. Such elements form a subgroup of $S(\mathbb{Z})$, which we denote by $W(\mathbb{Z})$. For any $x \in X$, $\pi_x([\![\phi]\!]) < W(\mathbb{Z})$.

A subgroup $G < S(\mathbb{Z})$ is said to have *ubiquitous pattern property* if for every finite set $F \subseteq G$ and every $n \geq 1$ there exists $k = k(n, F)$ such that for every $j \in \mathbb{Z}$ there exists $t \in \mathbb{Z}$,

$$[t - n, t + n] \subseteq [j - k, j + k],$$

and $g(i) + t = g(i + t)$ for every $g \in F$ and every $i \in [-n, n]$.

Lemma 10.1 (Juschenko–Monod [**JM12**], Lemma 4.2)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be a minimal homeomorphism and $x \in X$. The group $\pi_x([\![\phi]\!])$ has ubiquitous pattern property.*

Proof. Suppose towards the contradiction that there exists a finite set $F \subseteq G$ and $n > 0$ such that for any $k > n$ there exists j_k such that for all t with $[t - n, t + n] \subseteq [j_k - k, j_k + k]$ the action of F on $[-n, n]$ is different from the its action on $[t - n, t + n]$. Let $\mathcal{P}$ be the common refinement of partitions $\{n_g^{-1}(k)\}_{k\in\mathbb{Z}}$ for $g \in F$. Given $y \in X$ and an interval of natural numbers $[t - n, t + n]$ let $\mathcal{Q}(y, [t - n, t + n])$ be the partition of $[-n, n]$ defined by identifying naturally $[-n, n]$ with $\{\phi^i(y)\}_{i\in[t-n,t+n]}$ and setting

$$\mathcal{Q}(y, [t - n, t + n]) = \mathcal{P} \cap \{\phi^i(y)\}_{i\in[t-n,t+n]}.$$

For any t with $[t - n, t + n] \subseteq [j_k - k, j_k + k]$ partitions $\mathcal{Q}(x, [-n, n])$ and $\mathcal{Q}(x, [t - n, t + n])$ are different. Define sets

$$M_k = \left\{\, y \in X \;\middle|\; \forall [t - n, t + n] \subseteq [-k, k]\ \mathcal{Q}(y, [t - n, t + n]) \neq \mathcal{Q}(x, [-n, n]) \,\right\}.$$

The sets M_k are non-empty, closed, and $M_{k+1} \subseteq M_k$, therefore $M = \bigcap_k M_k$ is a non-empty closed subset of X. Since $\phi(M_k) \subseteq M_{k-1}$, the set M is ϕ-invariant. But $x \notin M$, contradicting the minimality of ϕ. □

Lemma 10.2 (Juschenko–Monod [**JM12**], Lemma 4.1)**.** *If $G < W(\mathbb{Z})$ has ubiquitous patter property, then the stabilizer in G of $E\triangle\mathbb{N}$ is locally finite for every $E \in \mathcal{P}_f(X)$.*

Proof. Let $E \in \mathcal{P}_f(\mathbb{Z})$ and $F \subseteq \mathrm{Stab}_G(E\triangle\mathbb{N})$ be finite. Put $M = \max_{e\in E} |e|$ and $N = \max_{g\in F} |\, g\,|_w$. Let $k = k(M + 2N, F)$ be from the definition of the ubiquitous pattern property. Let for $n \in \mathbb{Z}$

$$I_n = [(2n - 1)k + n, (2n + 1)k + n].$$

The intervals I_n partition $\mathbb{Z}$. Let $E_0 = (E\triangle\mathbb{N}) \cap [-M - 2N, M + 2N]$ and by the choice of k we may find $E_n \subseteq I_n$ and t_n such that $E_n = E_0 + t_n$ and $g(s) + t_n = g(s + t_n)$ for all $g \in F$ and all $s \in E_0$ (see Figure 17). We define sets B_n by

$$B_n = E_n \cup \big([\max(E_n) + 1, \max(E_{n+1})] \setminus E_{n+1}\big).$$

Note that $\mathbb{Z} = \bigsqcup_{n\in\mathbb{Z}} B_n$, each B_n is finite and $|B_n| < 4k + 2$ for all n. We claim that sets B_n are g-invariant for all $g \in F$. Fix $g \in F$. Since $g(E\triangle\mathbb{N}) = E\triangle\mathbb{N}$, we get $gE_0 \subseteq E\triangle\mathbb{N}$, hence $\max E_0 < \min(gE_0 \setminus E_0)$ and therefore also

$$\max E_n < \min(gE_n \setminus E_n) \quad \forall n.$$

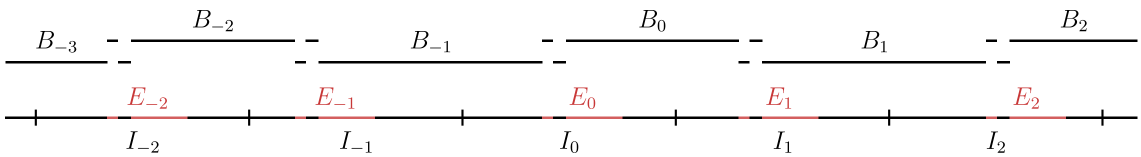

FIGURE 17. Construction of intervals I_n, sets E_n and B_n.

In other words, g "sends points from E_n to the right". Since $[\max E_n - |\,g\,|_w, \max E_n] \subseteq E_n$, it follows that B_n is g-invariant.

Since cardinalities $|B_n|$ are uniformly bounded by $4k+2$, we can view F as a subsets of a power of a finite group, hence F generates a finite group. □

Let $f_n : \{0,1\}^{\mathbb{Z}} \to [0,1]$ be the following sequence of functions:

$$f_n(w) = \exp\Big(-n \sum_{j \in \mathbb{Z}} w(j) e^{-|j|/n}\Big).$$

Fact 10.3 (Juschenko–Monod [**JM12**], Theorem 2.1)**.** *The sequence f_n satisfies conditions of item* (i) *of Lemma 9.7. Consequently, the action $W(\mathbb{Z}) \curvearrowright \mathbb{Z}$ has an invariant mean.*

Theorem 10.4 (Juschenko–Monod [**JM12**], Theorem A)**.** *Topological full groups of Cantor minimal systems are amenable.*

PROOF. Let ϕ be a minimal homeomorphism of a Cantor space X. For $x \in X$ we have an embedding $\pi_x : [\![\phi]\!] \to W(\mathbb{Z})$ and therefore by Fact 10.3 there is a $\mathcal{P}_f(\mathbb{Z}) \rtimes \pi_x([\![\phi]\!])$-invariant mean on $\mathcal{P}_f(\mathbb{Z})$. Consider the homomorphism $\xi : [\![\phi]\!] \to \mathcal{P}_f(\mathbb{Z}) \rtimes \pi_x([\![\phi]\!])$

$$\xi(g) = \big(\mathbb{N} \triangle \pi_x(g)(\mathbb{N}), \pi_x(g)\big).$$

The homomorphism ξ is injective and for any $E \in \mathcal{P}_f(X)$

$$\xi(g)(E) = E \iff \pi_x(g)(E \triangle \mathbb{N}) = E \triangle \mathbb{N}.$$

In other words, the stabilizer of E in $\xi([\![\phi]\!])$ is the stabilizer of $E \triangle \mathbb{N}$ in $\pi_x([\![\phi]\!])$. Thus the action $\xi([\![\phi]\!]) \curvearrowright \mathcal{P}_f(\mathbb{Z})$ has an invariant mean and by Lemma 10.2 stabilizers of all points are locally finite, hence amenable. Fact 9.6 finishes the proof. □

APPENDIX A

Topological full groups of $\mathbb{Z}^2$ actions

We present an example from [**EM13**] of a $\mathbb{Z}^2$ minimal action with a non-amenable topological full group.

Let Σ denote the space of all proper edge-colourings of the grid $\mathbb{Z}^2$ into six colours $\{a, b, c, d, e, f\}$. Denote by $\langle a \rangle$ the group with two elements $\{e, a\}$. Let $(w_i)_{i \in \mathbb{N}}$ be an enumeration of all the elements in the free product $\langle a \rangle * \langle b \rangle * \langle c \rangle$. Note that this free product contains a non-abelian free subgroup, hence is non-amenable. We pick a function $g : \mathbb{Z} \to \mathbb{N}$ satisfying the following: for any $i \in \mathbb{N}$ there is $L > 0$ such that any subinterval $I \subseteq \mathbb{Z}$ of length $\geq L$ contains $n \in I$ with $g(n) = i$. For example, we may take

$$g(n) = \begin{cases} i & |n| = 2^i m,\ m \text{ is odd}, \\ 0 & n = 0. \end{cases}$$

We construct an element $x \in \Sigma$ as follows. For $n \in \mathbb{Z}$ we take $w_{g(i)}$ and label edges with $w_{g(i)}^{-1} \frown d$ upward starting from the zero level (Figure 18). We continue this labelling periodically and colour horizontal edges with e and f in a proper way.

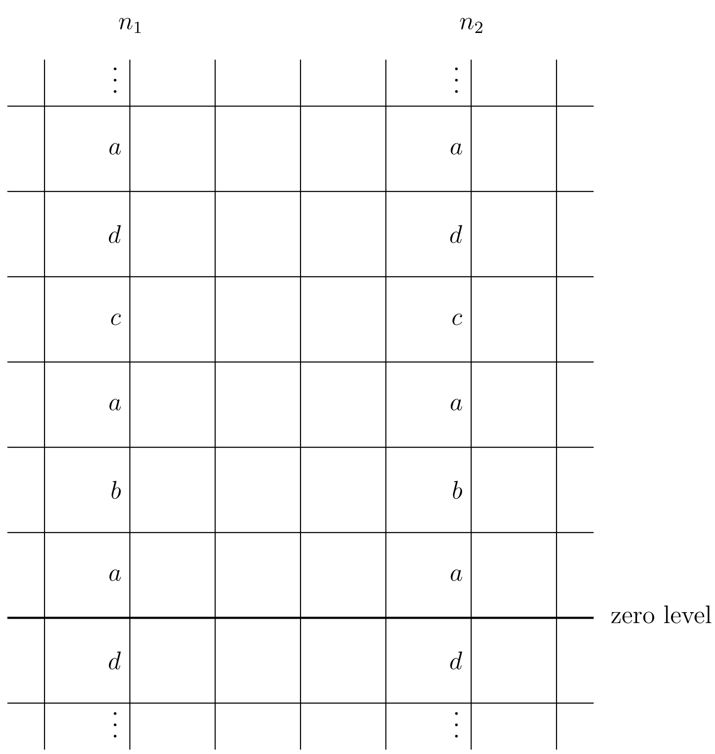

FIGURE 18. Construction of x, $w_{g(n_1)} = w_{g(n_2)} = caba$.

$\mathbb{Z}$ acts on Σ by shifting edges. With a letter a we associate a homeomorphism $a : \Sigma \to \Sigma$ defined as follows. Let $y \in \Sigma$. If there is $v \in \{(0, \pm 1), (\pm 1, 0)\}$ such that the edges starting from 0 in the direction of v is coloured with a, we let $a(y) = y + v$. Otherwise we set $a(y) = y$. The homeomorphisms a, b, c are in the topological full group of the shift. Let M be any minimal subshift of $\mathrm{Orb}_{\mathbb{Z}^2}(x)$. The action of $\langle a \rangle * \langle b \rangle * \langle c \rangle$ on M is faithful, hence the topological full group of the shift on M is non-amenable.

APPENDIX B

Dimension groups

We start by recalling the definition of the direct system of groups. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of groups with homomorphisms $\xi_n : G_{n-1} \to G_n$. For $i < n$ we let

$$\xi_{in} = \xi_n \circ \cdots \circ \xi_{i+1}.$$

The direct limit of (G_n, ξ_n) is the disjoint union $\bigsqcup_n G_n$ modulo the equivalence relation $x_m \in G_m, x_n \in G_n$, $x_m \sim x_n$ if there is $N > m, n$ such that $\xi_{mN}(x_m) = \xi_{nN}(x_n)$. Group operations are defined in the obvious way.

Given a Bratteli diagram $B = (V, E)$ with $k_n = |V_n|$, we consider integer valued matrices $M_n \in \mathcal{M}_{k_n \times k_{n-1}}$ defined by $M_n = (m_{ij})$, $m_{ij} = |P(v_j, v_i)|$, where $v_j \in V_{n-1}$ and $v_i \in V_n$. In other words, m_{ij} is the number of edges between the j^{th} vertex of V_{n-1} and the i^{th} vertex of V_n. For example, given the portion of Bratteli diagram in Figure 14, the corresponding matrices M_n are

$$M_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

A matrix M_n naturally defines a homomorphism $M_n : \mathbb{Z}^{k_{n-1}} \to \mathbb{Z}^{k_n}$ and therefore we have a direct system of Abelian groups

$$\mathbb{Z} \xrightarrow{M_1} \mathbb{Z}^{k_1} \xrightarrow{M_2} \mathbb{Z}^{k_2} \xrightarrow{M_3} \cdots \xrightarrow{M_n} \mathbb{Z}^{k_n} \xrightarrow{M_{n+1}} \cdots$$

The direct limit of this system is denoted by $\mathrm{K}(B)$. Each $\mathbb{Z}^{k_n}$ has a positive cone that consists of vectors with non-negative coordinates. The positive cones are preserved by homomorphisms M_n and the direct limit of these cones is the positive cone $\mathrm{K}^+(B)$ in $\mathrm{K}(B)$. The *dimension group* of the Bratteli diagram B is the triple $(\mathrm{K}(B), \mathrm{K}^+(B), \mathbb{1})$, where $\mathbb{1} \in \mathrm{K}(B)$ is the element that corresponds to $1 \in \mathbb{Z}$.

With a homeomorphism $\phi \in \mathrm{Homeo}(X)$ we associate the group $\mathrm{K}_0(\phi)$ that is defined to be the quotient of Abelian groups

$$\mathrm{K}_0(\phi) = C(X, \mathbb{Z}) / \partial_\phi C(X, \mathbb{Z}),$$

where $\partial_\phi C(X, \mathbb{Z}) = \{\, f - f \circ \phi \mid f \in C(X, \mathbb{Z}) \,\}$. This group also has a positive cone $\mathrm{K}_0^+(\phi)$, which is the image under the quotient map of the cone of non-negative functions. The *dimension group* of ϕ is the triple $(\mathrm{K}_0(\phi), \mathrm{K}_0^+(\phi), \mathbb{1})$, where $\mathbb{1}$ corresponds to the constant one function on X.

Theorem B.1 (Glasner–Weiss [**GW95**], Theorem 5.1)**.** *Let $\phi \in \mathrm{Homeo}(X)$ be minimal. If $B = (V, E, \leq)$ is a simple ordered Bratteli diagram such that ϕ_B is conjugated to ϕ, then $(\mathrm{K}(B), \mathrm{K}^+(B), \mathbb{1})$ is isomorphic to $(\mathrm{K}_0(\phi), \mathrm{K}_0^+(\phi), \mathbb{1})$.*

Proof. Define a map $\zeta : C(X, \mathbb{Z}) \to \mathrm{K}(B)$ as follows: given $f \in C(X, \mathbb{Z})$ choose an n such that V_n represents columns of a Kakutani-Rokhlin partition which is compactible with f, i.e., Ξ_n is finer than $\{f^{-1}(k)\}_{k\in\mathbb{Z}}$. Note that f is also compatible with all partitions Ξ_m, $m \geq n$. We define $\widetilde{f}_m \in \mathbb{Z}^{k_m}$ by setting $\widetilde{f}_m(j)$ to be the sum of values of f over all the levels of the j^{th} tower $\mathcal{T}_j$ in Ξ_n. Since

$$\widetilde{f}_{m+1}(j) = \sum_l (M_{m+1})_{j,l} \widetilde{f}_m(l) = (M_{m+1}\widetilde{f}_m)(j),$$

the sequence $(\widetilde{f}_m)$ defines an element $\zeta(f) \in \mathrm{K}(B)$. The map ζ is a homomorphism $\zeta : C(X, \mathbb{Z}) \to \mathrm{K}(B)$.

If $f = g \circ \phi - g$ for some $g \in C(X, \mathbb{Z})$, then $\widetilde{f}_m(j) = g \circ \phi^{J_j^{(m)}}(x) - g(x)$ for some $x \in D^{(m)}(j, 0)$ in the base of the tower, where $J_j^{(m)}$ is the height of the j^{th} tower in Ξ_m. If m is large enough, g is compatible with Ξ_m and is constant on its base. Since $\phi^{J_j^{(m)}}(x)$ is in the base, we get $\zeta(f) = 0$, hence $\partial_\phi C(X, \mathbb{Z}) \subseteq \ker \zeta$.

Conversely, if $\zeta(f) = 0$, there exists m such that $\widetilde{f}_m = 0$. We show that there is a function $g \in C(X, \mathbb{Z})$ such that $f = g \circ \phi - g$. We let g be equal 0 on $D^{(m)}(j, 0)$ and $f(x) + f(\phi(x)) + \cdots + f(\phi^{l-1}(x))$ on $D^{(m)}(j, l)$, where x is a point in $D^{(m)}(j, 0)$. Obviously $f = g \circ \phi - g$ everywhere, except possibly the top of the partition. For x in the top level the equality follows from $g(\phi^{J_j^{(m)}} x) = 0$ and $\widetilde{f}_m(j) = 0$. Whence $\zeta : \mathrm{K}_0(\phi) \to \mathrm{K}(B)$ is a monomorphism.

If d is an element in $\mathrm{K}(B)$, choose an m such that d can be represented as an element of $\mathbb{Z}^{k_m}$ and define f on the corresponding partition as follows. For $x \in D^{(m)}(j, 0)$ set $f(x) = d(m, j)$, and 0 elsewhere. Then $\widetilde{f}(j) = d(m, j)$ and ζ is onto. It is easy to check that $\zeta(\mathrm{K}_0^+(\phi)) = \mathrm{K}^+(B)$ and $\zeta(\mathbb{1}) = \mathbb{1}$. $\square$

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